Cluster structure on type A braid varieties from 3D plabic graphs

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joint work with Pavel Galashin
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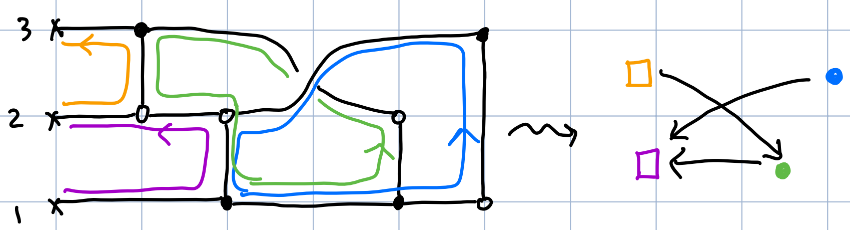

arXiv: 2210.04778

Slides: bit.ly/msherben "Talks"

Flag background

- $G = SL_n$, $B = \begin{bmatrix} \ddots & * \\ 0 & \end{bmatrix}$, $G/B = Fl_n$

  $= \{ F = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n : \dim V_i = i \}$

- Write $F \xrightarrow{s_i} F'$ if F, F' differ exactly in i^{th} subspace

  $F = gB$, $F' = g'B$

  $\Updownarrow$

  $g^{-1}g' \in B s_i B$

- Write $F \xrightarrow{w} F'$ if $g^{-1}g' \in BwB$.

  $F = gB$, $F' = g'B$

($\Leftrightarrow$ for every $\underline{w} = s_{i_1} \cdots s_{i_\ell}$ reduced $\exists$ (unique) seq. of flags

$F \xrightarrow{s_{i_1}} F_1 \xrightarrow{s_{i_2}} F_2 \xrightarrow{s_{i_3}} \cdots \xrightarrow{s_{i_\ell}} F'$.)

Type A braid varieties

- $G = SL_n$, $B = \begin{bmatrix} \ddots & * \\ 0 & \end{bmatrix}$, $G/B \cong Fl_n$, $F_- = w_0 B$
  = "antistandard flag"

- $\beta = s_{i_1} s_{i_2} \cdots s_{i_\ell}$, $u \in S_n$ s.t. β has red. expr. for u as subword.

$F \xrightarrow{s_i} F'$ means F, F' differ exactly in i^{th} subspace

$$X_{\beta,u} = \left\{ (F_1,\dots,F_{\ell+1}) : \begin{tikzcd} B=F_1 \arrow[r,"s_{i_1}"] & F_2 \arrow[r,"s_{i_2}"] & \cdots \arrow[r,"s_{i_\ell}"] & F_{\ell+1} \\ & & F_- \arrow[ull,"w_0"] \arrow[ur,"w_0 u"'] & \end{tikzcd} \right\}$$

Studied by Escobar, Mellit, CGGS, ...

$$\mu : \{B \xrightarrow{s_{i_1}} F_2 \to \cdots \xrightarrow{s_{i_\ell}} F_{\ell+1}\} \longrightarrow G/B$$
$$(F_1, \dots, F_{\ell+1}) \longmapsto F_{\ell+1}$$

"open Bott-Samelson"

$$X_{\beta,u} = \mu^{-1}\left(B^- u B/B\right)$$

Type A braid varieties

- $G = SL_n$, $B = \begin{bmatrix} \ddots & * \\ 0 & \end{bmatrix}$, $G/B \cong Fl_n$, $F_- = w_0 B$ = "antistandard flag"

- $\beta = s_{i_1} s_{i_2} \cdots s_{i_\ell}$, $u \in S_n$ s.t. β has red. expr. for u as subword.

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$F \xrightarrow{s_i} F'$ means F, F' differ exactly in i^{th} subspace

Studied by Escobar, Mellit, CGGS, ...

- β, γ related by braid moves $\Rightarrow X_{\beta,u} \cong X_{\gamma,u}$.
  - $\hookrightarrow$ Write $X_{\beta,u}$ where $\beta \in Br^+ = \langle s_1, \ldots, s_{n-1} \mid s_i s_j = s_j s_i,\ s_{i+1} s_i s_{i+1} = s_i s_{i+1} s_i \rangle$

- $X_{\beta,u}$ is smooth, irreducible, affine, $\dim = \ell(\beta) - \ell(u)$

Type A braid varieties

$$X_{\beta,u} = \left\{ \begin{tikzcd} B \arrow[r, "s_{i_1}"] & F_2 \arrow[r, "s_{i_2}"] & \cdots \arrow[r, "s_{i_\ell}"] & F_{\ell+1} \\ & F_- \arrow[ul, "w_0"] \arrow[urr, "w_0 u"'] & & \end{tikzcd} \right\}$$

e.g.

- If β is red. expr. for $v \in S_n$

$$X_{v,u} \cong \left\{ \begin{tikzcd} B \arrow[rr, "v"] & & F \\ & F_- \arrow[ul, "w_0"] \arrow[ur, "w_0 u"'] & \end{tikzcd} \right\} = \overbrace{B_v B \cap B_- u B}^{\substack{\text{open Richardson variety} \\ \parallel}} / B \subseteq Fl_n$$

↳ if v has ! descent, $X_{v,u} \cong$ open positroid variety [KLS], [P], [R]

Knutson Lam Speyer, Postnikov, Rietsch

↳ includes $Gr^{\circ}_{k,n}$

$$\parallel$$

$$Gr_{k,n} \setminus \{ P_{12\cdots k} P_{23\cdots k+1} \cdots P_{n12\cdots k-1} = 0 \}$$

Cluster structure on $X_{\beta,u}$

"3D plabic graph", generalize Postnikov's plabic graphs

Thm: [GLSBS] $(\beta, u) \rightsquigarrow G_{\beta,u} \rightsquigarrow \Sigma_{\beta,u}$.

$$\mathbb{C}[X_{\beta,u}] = \text{the cluster algebra } \mathcal{A}(\Sigma_{\beta,u})$$

(general type in progress)

- Related work: [Casals - Gorsky - Gorsky - Le - Shen - Simental '22].
- Resolves conj [Leclerc '14]: $\mathbb{C}[X_{v,u}]$ (open Richardson) is a cluster algebra.
  ↳ generalizes [GL], [BFZ], [SW].
  Galashin Lam, Berenstein Fomin Zelevinsky, Shen Weng
- Approach inspired by Ingermanson '19.

Cluster structure on $X_{\beta,u}$

"3D plabic graph", generalize Postnikov's plabic graphs

<u>Thm</u>: [GLSBS] $(\beta, u) \rightsquigarrow G_{\beta,u} \rightsquigarrow \Sigma_{\beta,u}$.

$\mathbb{C}[X_{\beta,u}]$ = the cluster algebra $\mathcal{A}(\Sigma_{\beta,u})$

(general type in progress)

<u>Cor</u>: [GLSBS]

$a = q^{-1/2}$ $\quad z = \frac{1}{a}(1 - q^{-1})$

$|X_{\beta,u}(\mathbb{F}_q)|$ = specialization of top a-deg. term of HOMFLY poly of $L_{u,\beta}$

<u>Hope</u>: $H^*(X_{\beta,u})$ related to Khovanov-Rozansky homology of $L_{u,\beta}$.

Known in Richardson [GL], $u = e$ cases [Trinh].

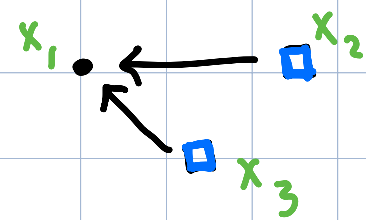
What is $\mathcal{A}(\Sigma)$?

[Fomin-Zelevinsky, 2000]

- $\Sigma = (\underline{x}, Q)$

  seed — cluster of cluster variables — quiver

  e.g.

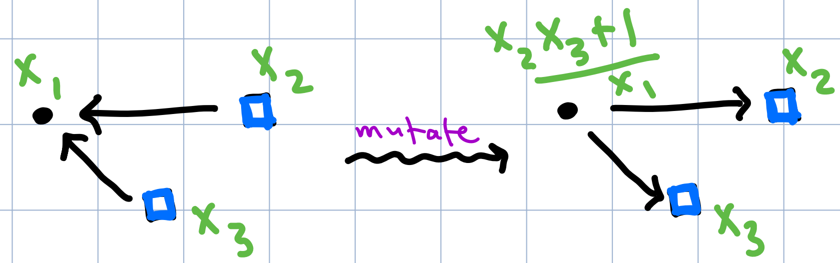


  x_1, x_2, x_3 alg. ind.

  frozen

- Can <u>mutate</u> at non-frozen vt of Q to get Σ':

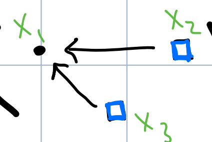
  e.g.



  (in general, get ∞'ly many seeds by repeated mutation)

- $\mathcal{A}(\Sigma) = \mathbb{C}[\text{frozen var}^{\pm 1}][\text{clust. var in } \Sigma']$

  runs over all seeds obtained from Σ by arb. mutation seq.

  e.g. $\mathcal{A}\Big($  $\Big) = \mathbb{C}\left[x_2^{\pm 1}, x_3^{\pm 1}, x_1, \frac{x_2x_3+1}{x_1}\right]$

  $= \mathbb{C}[SL_2^\circ]$

  $SL_2^\circ = \left\{ \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} : \begin{array}{l} x_1x_4 - x_2x_3 = 1 \\ x_2x_3 \neq 0 \end{array} \right\}$

- We say "SL_2° has a cluster structure".

Another perspective

- V ^affine^ has a <u>cluster structure</u> if $\mathbb{C}[V] \simeq \mathcal{A}(\Sigma)$ — cluster algebra, $\Sigma = (\underline{x}, Q)$

  ↳ V is covered up to codim 2 by union of <u>cluster tori</u> T_Σ

  ↳ Cluster $\underline{x}$ = distinguished basis of characters of T_Σ

  ↳ Quiver Q encodes relations, birational mutation maps btw nearby tori

e.g. $SL_2^\circ = \left\{ \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} : \begin{matrix} x_1x_4 - x_2x_3 = 1 \\ x_2x_3 \neq 0 \end{matrix} \right\}$

$T_{\Sigma_1} = \{x_1x_2x_3 \neq 0\}$

$$\begin{tikzcd} x_1 & \boxed{x_2} \arrow[l] \\ \boxed{x_3} \arrow[u] & \end{tikzcd}$$

$T_{\Sigma_2} = \{x_2x_3x_4 \neq 0\}$

$$\begin{tikzcd} x_4 = \dfrac{x_2x_3+1}{x_1} \arrow[r] \arrow[dr] & \boxed{x_2} \\ & \boxed{x_3} \end{tikzcd}$$

Another perspective

- V (affine) has a __cluster structure__ if $\mathbb{C}[V] \simeq \mathcal{A}(\Sigma)$ (cluster algebra, $\Sigma = (\underline{x}, Q)$)
  ↳ V is covered up to codim 2 by __cluster tori__ T_Σ

__Why care__: (under some assumptions on $\mathcal{A}$)

- endows $\mathbb{C}[V]$ with "canonical basis" w/ pos. struc. const. [GHKK] (Gross–Hacking–Keel–Kontsevich)
- seed $\Sigma \rightsquigarrow$ valuation $\nu_\Sigma \rightsquigarrow$ toric degen of $\overline{V}$
  ↳ [Bossinger], [BCMNC], ...
  ↳ [Fujita-Oya]: obtain toric degen of $\overline{BwB}/B$ via clust. struc on $U_- \cap BwB \hookrightarrow \overline{BwB}/B$
- [LS]: results on cohomology of V
- Natural notion of $V^{>0}$; $\bigsqcup_{u,v \in S_n} X_{v,u}^{>0}$ is regular CW-clx [GKL]

A Deodhar torus in $X_{\beta,u}$

$$= \left\{ \begin{tikzcd} B \arrow[r, "s_{i_1}"] & \bullet \arrow[r] & \cdots \arrow[r, "s_{i_\ell}"] & \bullet \\ & F_- \arrow[ul] \arrow[urr, "w_0u"'] & & \end{tikzcd} \right\}$$

- Get 1 torus for ea. braid word β ⟵ some coincide
- $v * s_i := \max\{vs_i, v\}$ ⟵ Demazure product

$$T_{\beta,u} := \left\{ \begin{tikzcd} B \arrow[r, "s_{i_1}"] & F_2 \arrow[r, "s_{i_2}"] & \cdots \quad \cdots \arrow[r] & F_\ell \arrow[r, "s_{i_\ell}"] & F_{\ell+1} \\ & & F_- \arrow[ull, "w_0"] \arrow[ul, "w * s_{i_\ell} * \cdots * s_{i_2}"'] \arrow[ur, "w * s_{i_\ell}"] \arrow[urr, "w = w_0 u"'] & & \end{tikzcd} \right\}$$

"Start at $F_{\ell+1}$ & greedily increase distance from F_-"

$$T_{\beta,w_0} = \left\{ \begin{tikzcd} \bullet \arrow[r, "s_2"] & \bullet \arrow[r, "s_1"] & \bullet \arrow[r, "s_2"] & \bullet \arrow[r, "s_1"] & \bullet \arrow[r, "s_1"] & \bullet \\ & & & F_- \arrow[ulll] \arrow[ull] \arrow[ul] \arrow[u] \arrow[ur] \arrow[urr, "id"'] & & \end{tikzcd} \right\}$$

A Deodhar torus in $X_{\beta,u}$

$$= \left\{ \begin{tikzcd} B \arrow[r, "s_{i_1}"] & \bullet \arrow[r] & \cdots \arrow[r, "s_{i_\ell}"] & \bullet \\ & F_- \arrow[ul] \arrow[urr, "w_0 u"'] & & \end{tikzcd} \right\}$$

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$v * s_i = \max\{v s_i, v\}$

"Start at $F_{\ell+1}$ & greedily increase distance from F_-"

- Terminology:

$$\begin{tikzcd} F_c \arrow[rr, "s_{i_c}"] & & F_{c+1} \\ & F_- \arrow[ul, "v"] \arrow[ur, "v"'] & \end{tikzcd} \qquad \begin{tikzcd} F_c \arrow[rr, "s_{i_c}"] & & F_{c+1} \\ & F_- \arrow[ul, "v s_{i_c}"] \arrow[ur, "v"'] & \end{tikzcd}$$

c solid $\qquad\qquad$ c hollow

$$T_{\beta,w_0} = \left\{ \begin{tikzcd} \bullet \arrow[r, "s_2"] & \bullet \arrow[r, "s_1"] & \bullet \arrow[r, "s_2"] & \bullet \arrow[r, "s_1"] & \bullet \arrow[r, "s_1"] & \bullet \\ & & & F_- \arrow[ulll, "w_0"] \arrow[ull, "w_0"] \arrow[ul, "s_1 s_2"] \arrow[u, "s_1"] \arrow[ur, "s_1"] \arrow[urr, "id"'] & & \end{tikzcd} \right\}$$

(Hollow crossings = rightmost subexpr. for u)

- $\ell(\beta) - \ell(u)$ solid crossings ⇜ will use to index cluster var's

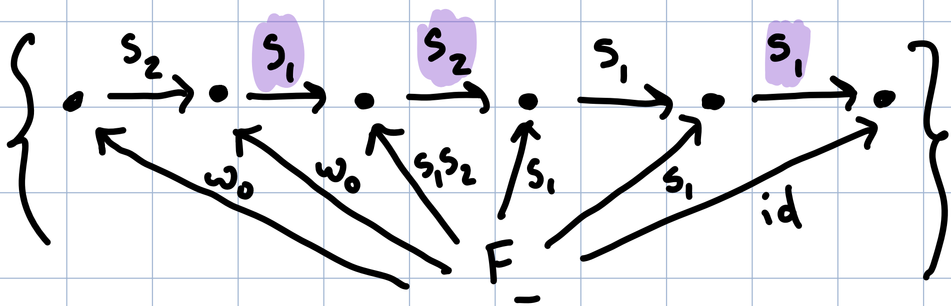
Cluster variables for $T_{\beta,u}$

- $X_{\beta,u} \setminus T_{\beta,u} = \bigcup_{c \text{ solid crossing}} V_c$

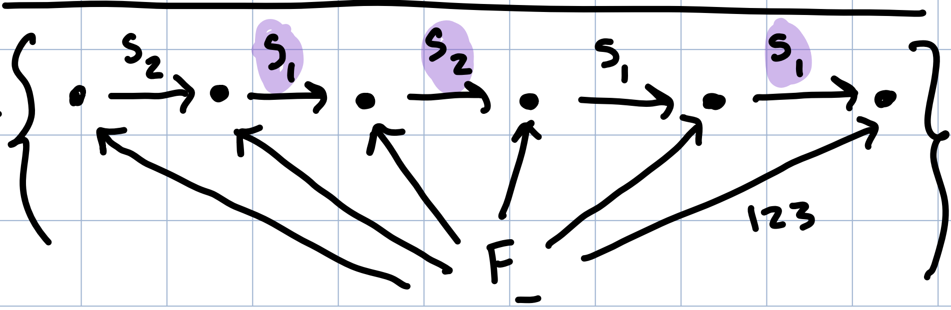
Deodhar hypersurface $V_c = \overline{\left\{ \begin{array}{c} B \to \cdots \to F_{\ell+1} \\ \nwarrow \quad \nearrow \\ F_- \end{array} : \text{make a "mistake" in greedy walk at crossing } c \right\}}$

irreducible
codim 1

e.g. $T_{\beta,w_0} =$



$V_4 =$



Cluster variables for $T_{\beta,u}$

- $X_{\beta,u} \setminus T_{\beta,u} = \bigcup_{c \text{ solid crossing}} V_c$

Deodhar hypersurface $V_c = \left\{ \begin{array}{c} B \to \cdots \to F_{\ell+1} \\ \nwarrow \; F_- \nearrow \end{array} : \overline{\text{make a "mistake" in greedy walk at crossing } c} \right\}$

- Cluster variable x_c = torus character vanishing to order 1 on V_c & order 0 on $V_{c'}$ for $c' \neq c$.

Warning: Some technicalities for frozen variables (must define some V_c in larger space)

$G_{\beta,u}$ & quiver

- $X_{\beta,u} \xrightarrow{m} X_{\beta',u}$. What are characters of $m^*(T_{\beta',u})$ in terms of characters of $T_{\beta,u}$?

  $T_{\beta,u} \subset X_{\beta,u}$, $T_{\beta',u} \subset X_{\beta',u}$

Plan:

$(\beta,u) \xrightarrow[\text{diagram}]{\text{wiring}}$ bicolored graph $G_{\beta,u}$ with n marked pts $\xrightarrow[\text{graph}]{\text{ribbon}}$ oriented surface $S_{\beta,u}$ with boundary

c solid crossing $\longrightarrow$ (relative) cycle C_c in $G_{\beta,u}$ $\longrightarrow$ (relative) cycle C_c in $S_{\beta,u}$

$$\boxed{\#\text{arrows } c \to d = \text{intersection } \# \text{ of } C_c \ \& \ C_d \text{ in } S_{\beta,u}}$$

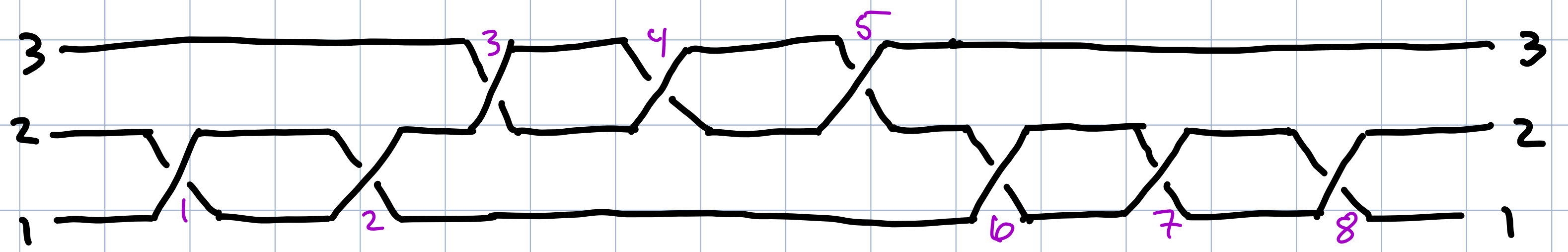
see [Goncharov-Kenyon '13]

$(\beta, u) \longrightarrow G_{\beta,u}$

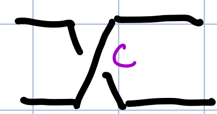
$n=3$

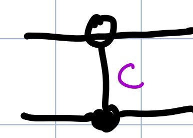
$\beta = 1\,1\,2\,2\,2\,1\,1\,1$ $\quad$ $u = w_0 = s_1 s_2 s_1$

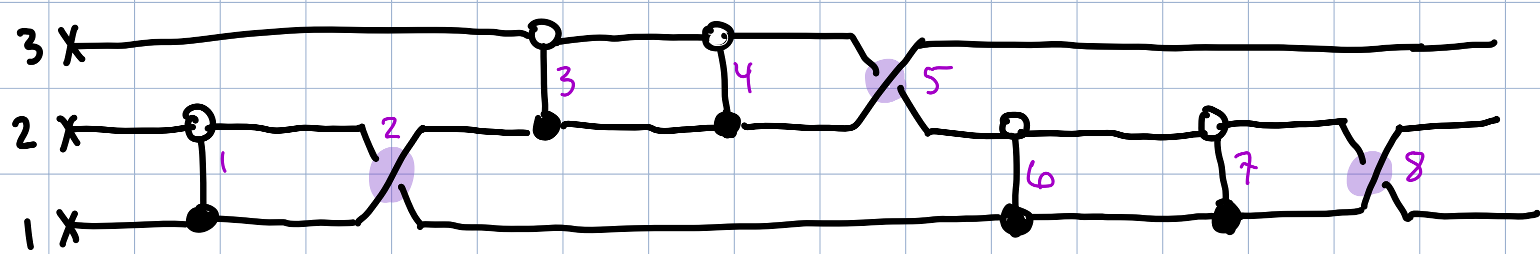
① Draw braid/wiring diagram of β



Highlight hollow crossings (= rightmost subexp. for u)

② Replace solid crossing 

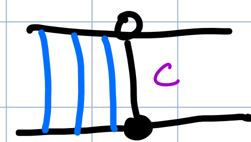
with bridge 



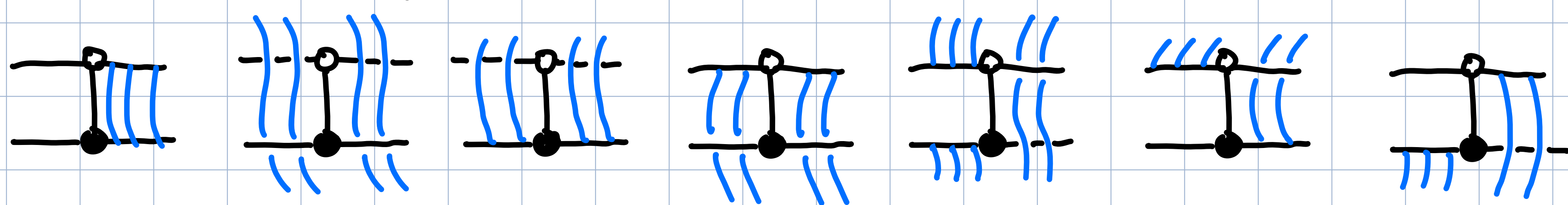
& add marked pts on left endpts

Solid crossing $c \longrightarrow$ cycle C_c

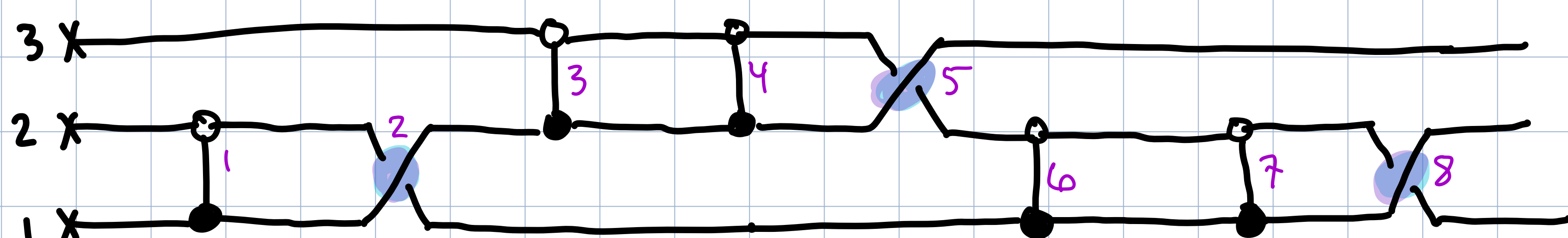
- First define a soap film (= disk) D_c bounded by edges of $G_{\beta,u}$; $C_c = \partial D_c$. ← DON'T glue D_c in!

- D_c begins at bridge c: 

and propagates according to



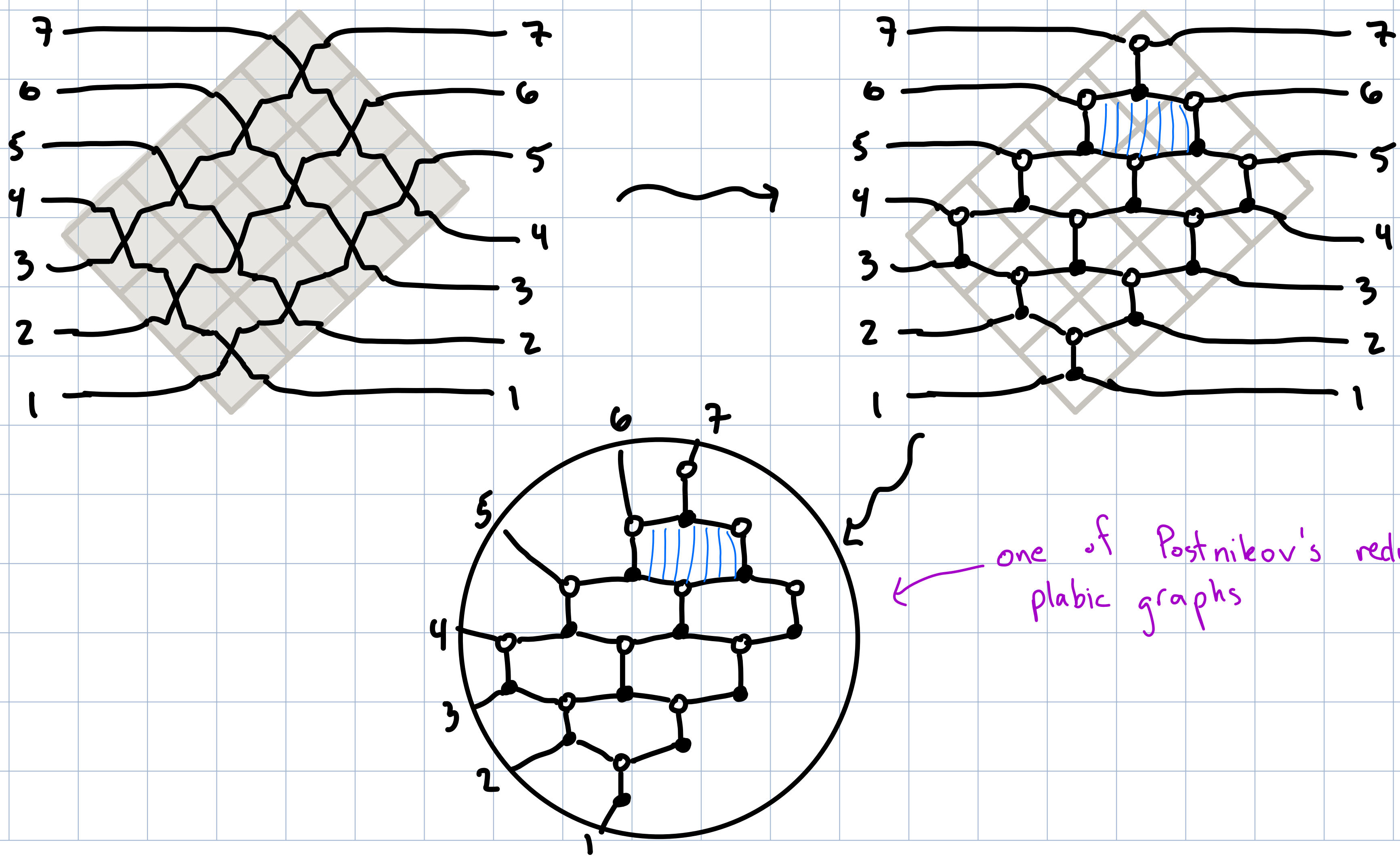
e.g.



- If C_c a cycle, c mutable; else, c frozen.

Examples for β reduced

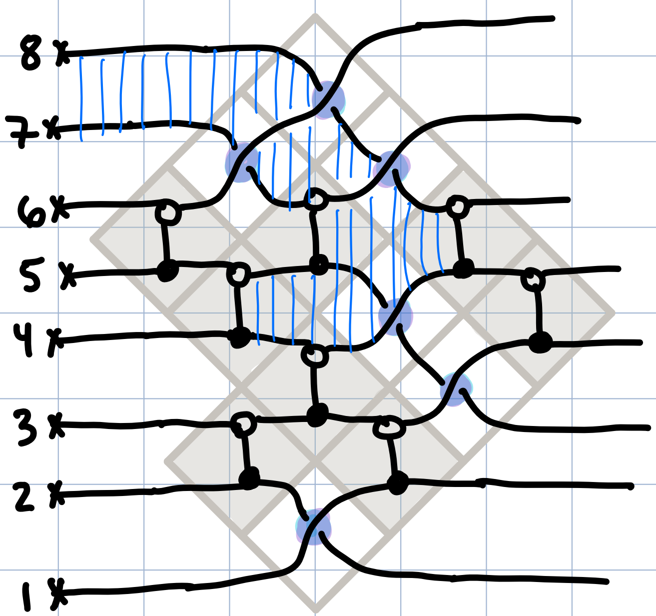
- $Gr^{o}_{3,7}$ case. β = reduced word for $v = 4567123 \in S_7$
  $u = id$



one of Postnikov's reduced plabic graphs

Examples for β reduced

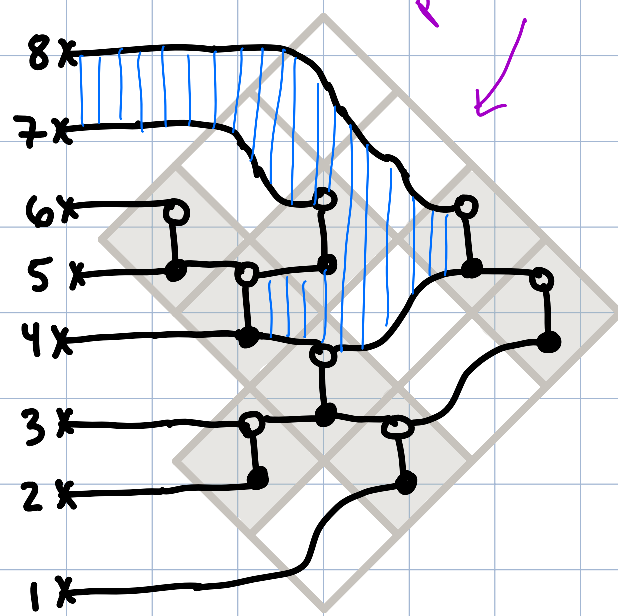
- Positroid case: $\beta = v \in S_n$, v has ! descent, $u \leq v$



delete "tails"

of wires past rightmost bridge

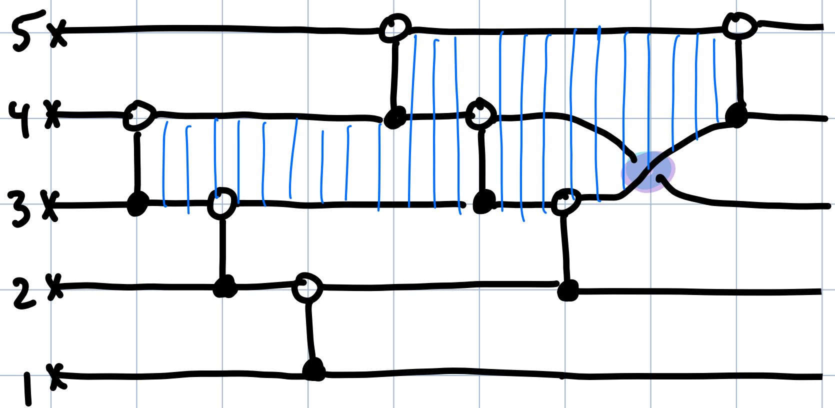
reduced plabic graph



- Always get planar graph after deleting tails in positroid case. Cycles bound faces.

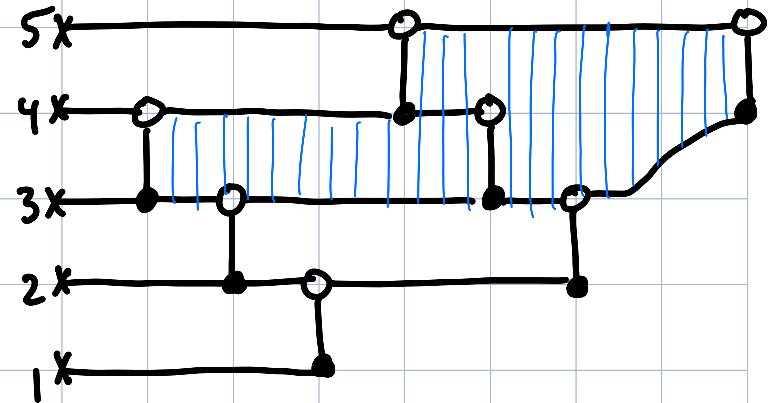
Examples for β reduced

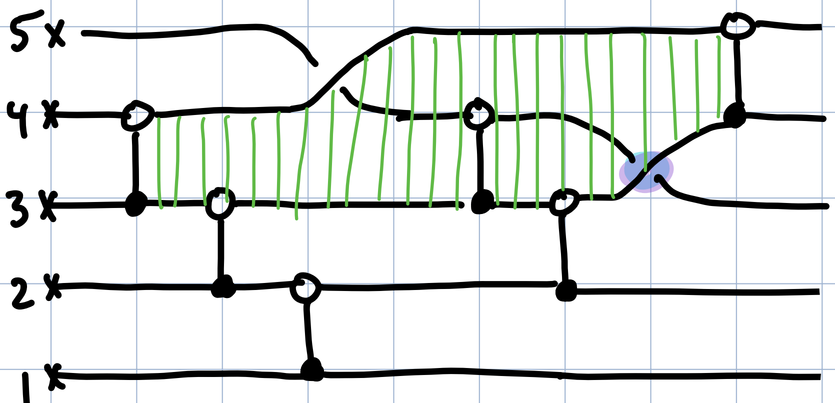
- Richardson case: $\beta = v \in S_n$, $u \leq v$.



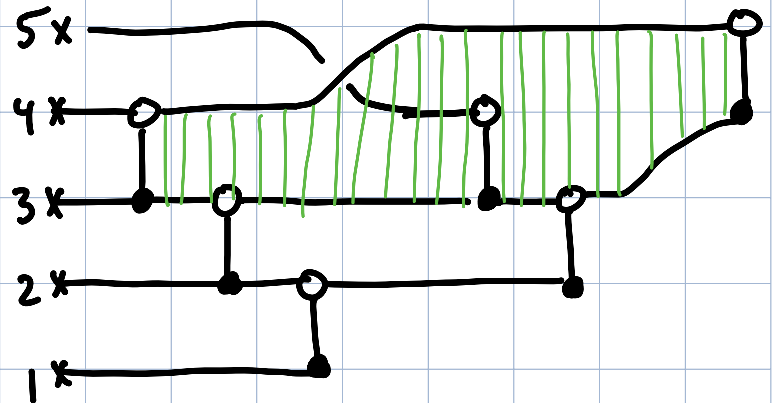
delete "tails" of wires past rightmost bridge

non-reduced plabic graph





delete "tails" of wires past rightmost bridge

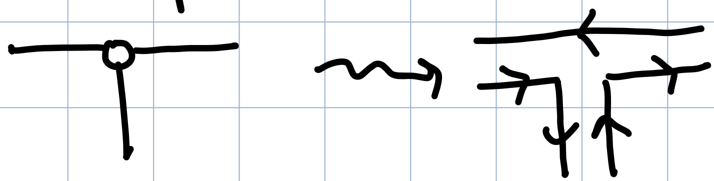


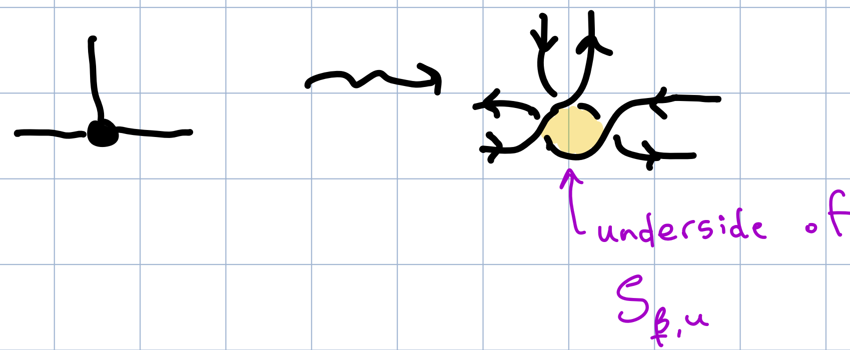
$\boxed{\text{WARNING:}}$ May have nonplanar graph after deleting tails.
Even if planar, cycles may bound >1 face.

Bicolored graph $G_{\beta,u} \longrightarrow$ surface $S_{\beta,u}$

- Make $G_{\beta,u}$ ribbon graph (○ oriented clockwise
  ● —"— counterclockwise)

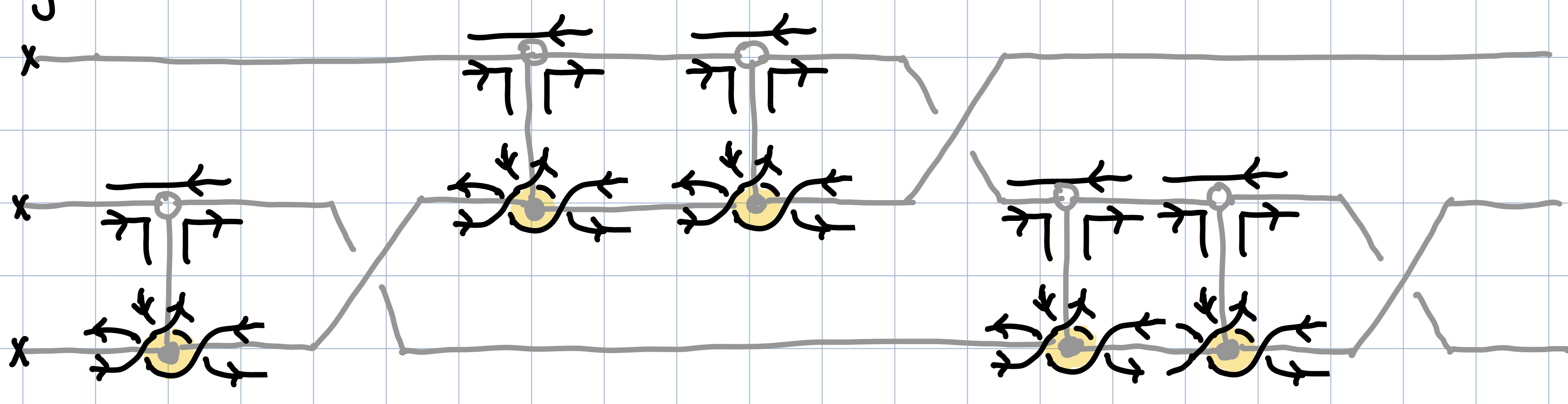
i.e. replace





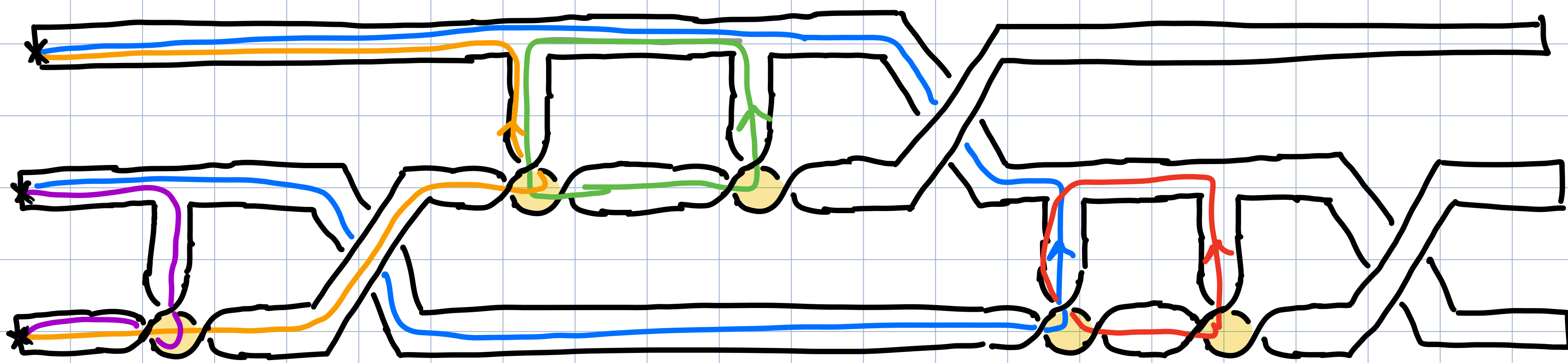
& connect ribbon fragments.

e.g.



$\# \, c \to d$ = intersection $\#$ of C_c & C_d in $S_{\beta,u}$.

Bicolored graph $G_{\beta,u} \longrightarrow$ surface $S_{\beta,u}$



$Q =$ $\square^1$ $\quad$ $\square^3 \longrightarrow \bullet^4$ $\quad$ $\square^6 \longrightarrow \bullet^7$

$\#\ c \to d$ = intersection # of C_c & C_d in $S_{\beta,u}$.

Summary

arXiv: 2210.04778

<u>Thm[GLSBS]</u>: $\mathbb{C}[X_{\beta,u}] = \mathcal{A}(\Sigma_{\beta,u})$.

- $T_{\beta,u}$ = Deodhar torus, defined by greedy procedure
- $\underline{x}_{\beta,u}$ = fcns defining Deodhar hypersurfaces (make 1 "mistake" in greedy procedure)

$\cdot (\beta,u) \rightsquigarrow$ graph $G_{\beta,u} \rightsquigarrow S_{\beta,u}$
$\underline{x}_{\beta,u} \rightsquigarrow$ cycles $\{C_c\}_{c \text{ solid crossing}}$
} Q records intersection #'s of $\{C_c\}$ on $S_{\beta,u}$

<u>Cor</u>: $|X_{\beta,u}(\mathbb{F}_q)|$ recovers some of HOMFLY for $L_{\beta,u}$.

<u>Cor</u>: Some symmetry for $H^*(X_{\beta,u}, \mathbb{C})$.

"curious Lefschetz"

Thanks for listening!