Growth rates for axisymmetric Euler flows

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Axisymmetric Euler flows

- fluid velocity: $u(x,t) : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d, \quad d \geq 3$
- Euler (incompressible, inviscid):
  $\partial_t u + (u \cdot \nabla) u = -\nabla p, \quad \nabla \cdot u = 0, \quad u|_{t=0} = u^0$
- axisymmetric, swirl-free: $u = u^r(r,z,t)\widehat{e}_r + u^z(r,z,t)\widehat{e}_z$
  $r^2 = x_1^2 + \cdots + x_{d-1}^2, \quad z = x_d, \quad \widehat{e}_r = \frac{(x_1,\ldots,x_{d-1},0)}{r}, \quad \widehat{e}_z = \widehat{e}_d$
- scalar vorticity formulation: $\omega(r,z,t) = \partial_r u^z - \partial_z u^r$
  $\boxed{(\partial_t + u \cdot \nabla) \frac{\omega}{r^{d-2}} = 0, \quad \omega|_{t=0} = \omega^0}$
- "anti-parallel vortex rings":
  - $\omega^0 \geq 0$ for $z \geq 0 \quad (\omega^0 \not\equiv 0)$
  - ω^0 odd in z
- further assume: $\omega^0, \ \frac{\omega^0}{r^{d-2}} \in L^\infty(\mathbb{R}^d); \ \ r\omega^0, \ z\frac{\omega^0}{r^{d-2}} \in L^1(\mathbb{R}^d)$

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Growth rates

- radial moment: $R(t) := \int_{z\geq 0} r\,\omega\, dx = \int_{z\geq 0} r^{d-1} \frac{\omega}{r^{d-2}}\, dx$

Thm [G-Miller-Tsai]:

$$d=3: \quad (1+t)^{\frac{9}{14}-\varepsilon} \lesssim_{\omega^0,\varepsilon} R(t) \lesssim_{\omega^0} (1+t)^4$$

$$d=4: \quad (1+t)^{\frac{2}{3}-\varepsilon} \lesssim_{\omega^0,\varepsilon} R(t) \lesssim_{\omega^0} e^{Ct}$$

$$d\geq 5: \quad (1+t)^{\frac{d}{d^2-2d-2}-\varepsilon} \lesssim_{\omega^0,\varepsilon} R(t) \lesssim_{\omega^0} (T-t)^{-\frac{2(d-2)}{d-4}}$$

for some $C=C(\omega^0)$, $T=T(\omega^0)$.

- [Choi-Jeong 21]: $R(t) \gtrsim (1+t)^{\frac{2}{15}-\varepsilon}$ for $d=3$
- $R(t)$ growth $\implies \|\omega\|_{L^p}$ growth for, eg, (smoothed) vortex patches
- rougher flows can blow up in $d=3$ ([Elgindi et al], ...)
- the upper bounds are relatively simple; we focus on the lower bounds

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Biot-Savart law

- $\begin{bmatrix} u^r \\ u^z \end{bmatrix} = \frac{1}{r^{d-2}} \begin{bmatrix} -\partial_z \\ \partial_r \end{bmatrix} \psi, \qquad \psi = \psi(r, z, t)$ stream function
- recover stream function from scalar vorticity:

$$\boxed{\psi(r, z, t) = \int_{[0,\infty)\times\mathbb{R}} \mathcal{F}(S)\,(r\bar{r})^{\frac{d}{2}-1}\,\overline{\omega}}$$

where $\quad \overline{\omega} = \omega(\bar{r}, \bar{z}, t)d\bar{r}d\bar{z}, \quad S = \frac{(r-\bar{r})^2+(z-\bar{z})^2}{r\bar{r}},$

$$\mathcal{F}(s) = \int_0^\pi \frac{\sin^{d-3}(\theta)\cos(\theta)d\theta}{[2(1-\cos(\theta)+s]^{\frac{d}{2}-1}}$$

Monotonicty of the horizontal moment

- following [Choi-Jeong], consider the horizontal moment:
  $Z(t) := \int_{z\geq 0} z \, \frac{\omega(r,z,t)}{r^{d-2}} \, dx = \int_{[0,\infty)^2} z \, \omega \, drdz$
- compute (by above formulas, integration by parts, and oddness in z)

  $$\boxed{-\dot{Z}(t) = c \int_{[0,\infty)^4} \left[\mathcal{H}(r,\overline{r},S) - \mathcal{H}(r,\overline{r},\overline{S})\right] (r\overline{r})^{\frac{d}{2}-1} \omega\overline{\omega}}$$

  $\omega = \omega(r,z,t)drdz, \quad \overline{\omega} = \omega(\overline{r},\overline{z},t)d\overline{r}d\overline{z},$

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  $\mathcal{H}(r,\overline{r},s) = 2(\overline{r}^{d-2} - r^{d-2})(r-\overline{r})\mathcal{F}'(s) + (r^{d-1} + \overline{r}^{d-1})\mathcal{F}^*(S)$
- can check:
  $-\mathcal{F}'(s)$ and $\mathcal{F}^*(s) = \left(\frac{d}{2}-1\right)\mathcal{F}(s) - s\mathcal{F}'(s)$ are decreasing in s
- conclusion: $\boxed{\dot{Z} < 0}$, so $\boxed{0 < Z(t) < Z(0) \lesssim 1}$

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- recall:
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  so in particular $\boxed{R(t) > R(0) \gtrsim 1}$
- finer estimate:
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Kinetic energy

- kinetic energy is conserved: $E = E(t) = \int_{\mathbb{R}^d} |u(x,t)|^2 dx = E(0)$
- compute: $E = c \int_{[0,\infty)^4} \left[\mathcal{F}(S) - \mathcal{F}(\overline{S})\right] (r\overline{r})^{\frac{d}{2}-1} \omega\overline{\omega}$
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$$\int_{\Sigma_\varepsilon} e \ \lesssim \varepsilon^a\, R^\gamma, \quad \gamma = \begin{cases} 0 & d = 3,4 \\ \frac{d-4}{d-2} & d \ge 5 \end{cases}, \quad a = \begin{cases} 1 & d = 3 \\ \frac{d}{d-2} & d \ge 4 \end{cases}$$

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