Covering points by hyperplanes and related problems

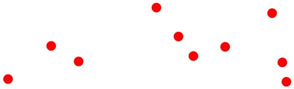
Zuzana Patáková Micha Sharir

Charles University

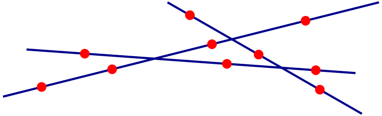
August 15, 2022

Motivation from computational geometry

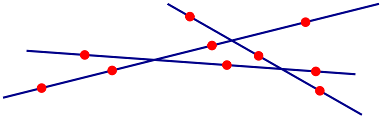
Hyperplane cover problem:
given a set P of n points in $\mathbb{R}^d$ and a number h, can we find h hyperplanes that cover all points from P?



Hyperplane cover problem:
given a set P of n points in $\mathbb{R}^d$ and a number h, can we find h hyperplanes that cover all points from P?

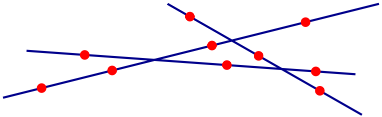


Hyperplane cover problem:
given a set P of n points in $\mathbb{R}^d$ and a number h, can we find h hyperplanes that cover all points from P?



- geometric variant of a set-cover problem
- NP-hard and APX-hard for $d = 2$ [Meggido-Tamir '82; Kumar-Arya-Ramesh '00]

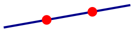
Hyperplane cover problem:
given a set P of n points in $\mathbb{R}^d$ and a number h, can we find h hyperplanes that cover all points from P?



- geometric variant of a set-cover problem
- NP-hard and APX-hard for $d = 2$ [Meggido-Tamir '82; Kumar-Arya-Ramesh '00]
- several FPT-algorithms known (fixed h) [e.g. Wang-Li-Chen '10]
  $d = 2, 3$ use of incidence bounds [Afshani-Berglin-van Duijn-Nielsen '16]

Motivation II: point-hyperplane incidences

- P . . . n points in $\mathbb{R}^d$ $\mathcal{H}$ . . . m hyperplanes in $\mathbb{R}^d$
- incidence . . . a pair (p, H) s.t. $p \in P, H \in \mathcal{H}$ and $p \in H$



Basic question: What is the max number of incidences between P and $\mathcal{H}$ in $\mathbb{R}^d$?

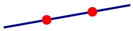
- P ... n points in $\mathbb{R}^d$ $\mathcal{H}$ ... m hyperplanes in $\mathbb{R}^d$
- incidence ... a pair (p, H) s.t. $p \in P, H \in \mathcal{H}$ and $p \in H$

Basic question: What is the max number of incidences between P and $\mathcal{H}$ in $\mathbb{R}^d$?

- $O(m^{2/3}n^{2/3} + m + n)$ for $d = 2$ tight! [Szemerédi-Trotter '83]

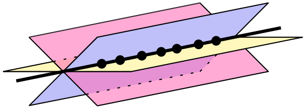
Motivation II: point-hyperplane incidences

- P … n points in $\mathbb{R}^d$ $\mathcal{H}$ … m hyperplanes in $\mathbb{R}^d$
- incidence … a pair (p, H) s.t. $p \in P, H \in \mathcal{H}$ and $p \in H$



Basic question: What is the max number of incidences between P and $\mathcal{H}$ in $\mathbb{R}^d$?

- $O(m^{2/3}n^{2/3} + m + n)$ for $d = 2$ tight! [Szemerédi-Trotter '83]



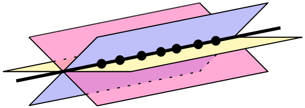
- mn for $d \geq 3$

Motivation II: point-hyperplane incidences

- P . . . n points in $\mathbb{R}^d$ $\mathcal{H}$ . . . m hyperplanes in $\mathbb{R}^d$
- incidence . . . a pair (p, H) s.t. $p \in P, H \in \mathcal{H}$ and $p \in H$

Basic question: What is the max number of incidences between P and $\mathcal{H}$ in $\mathbb{R}^d$?

- $O(m^{2/3}n^{2/3} + m + n)$ for $d = 2$ tight! [Szemerédi-Trotter '83]



- mn for $d \geq 3$

Improvements under further assumptions, e.g.:

- no lower-dim flat contains too many points
  or is contained in too many hyperplanes [Edelsbrunner-Guibas-Sharir '90]
- incidence graph between P and $\mathcal{H}$ doesn't contain $K_{r,r}$ [Braß-Knauer '03]
- P = vertices of the arrangement of $\mathcal{H}$ [Agarwal-Aronov '92]

Side remark: related problem from computational geometry

Hopcroft's problem (80's):
given a set P of n points and H a set of m hyperplanes, both in $\mathbb{R}^d$, is there a point-hyperplane incidence?

- special case of many other geometric problems
  (collision detection, ray shooting, range searching, ...)
- other variants: compute the number of incidences, report all of them
- prompted a strain of research in CG community, mainly in 2D
  [Chazelle '86, '93], [Edelsbrunner '87], [Edelsbrunner, Guibas, Sharir '90], [Agarwal '90], [Chazelle, Sharir, Welzl '92], [Matoušek '93], [Erickson '96]
- recent progress after cca 30 years [Chan, Zheng '21]

Setting:

- P ... n points in $\mathbb{R}^d$
- k-rich hyperplane wrt P ... contains $\geq k$ points from P

Problem (by Peyman Afshani)**:**
$\gtrsim \left(\frac{n^d}{k^{d+1}} + \frac{n}{k}\right)$ k-rich hyperplanes $\Rightarrow$ is there a low-dim flat with **"many"** points of P?

Setting:

- P ... n points in $\mathbb{R}^d$
- k-rich hyperplane wrt P ... contains $\geq k$ points from P

Problem (by Peyman Afshani)**:**

$\gtrsim \left(\frac{n^d}{k^{d+1}} + \frac{n}{k}\right)$ k-rich hyperplanes $\Rightarrow$ is there a low-dim flat with **"many"** points of P?

Answer: YES! [P., Sharir '22]

- $3 \leq d \leq k$ $\quad$ $d \leq \alpha < 2d - 1$
- $\gtrsim \left(\frac{n^d}{k^{\alpha}} + \frac{n}{k}\right)$ k-rich hyperplanes

$\Rightarrow$ there is a $(d-2)$-flat containing $\gtrsim k^{(2d-1-\alpha)/(d-1)}$ points of P

Note: Tight in some cases

Main result:

- P . . . n points in $\mathbb{R}^d$ $\quad 3 \le d \le k \quad d \le \alpha < 2d-1$
- $\gtrsim \left(\frac{n^d}{k^\alpha} + \frac{n}{k}\right)$ k-rich hyperplanes
  $\Rightarrow$ there is a $(d-2)$-flat containing $\gtrsim k^{(2d-1-\alpha)/(d-1)}$ points of P

- $\mathcal{H}$ . . . all k-rich hyperplanes determined by P
- $\mathcal{H}$ is finite
- $k|\mathcal{H}| \le I(P, \mathcal{H})$ . . . number of incidences between P and $\mathcal{H}$
- compute an upper bound on $I(P, \mathcal{H})$; compare
- we need *point-hyperplane duality*, *simplicial partitions*, *Cauchy-Schwartz*

Proof sketch – upper bound

- apply point-hyperplane duality
  - preserves incidences
  - each $(d-2)$-flat contains $\leq \ell$ points of P

    $\longleftrightarrow$ each line is contained in $\leq \ell$ hyperplanes of P^*

Proof sketch – upper bound

- apply point-hyperplane duality
  - preserves incidences
  - each $(d-2)$-flat contains $\leq \ell$ points of P
    $\longleftrightarrow$ each line is contained in $\leq \ell$ hyperplanes of P^*
- apply simplicial partitions

Proof sketch – upper bound

- apply point-hyperplane duality
  - preserves incidences
  - each $(d-2)$-flat contains $\leq \ell$ points of P

    $\longleftrightarrow$ each line is contained in $\leq \ell$ hyperplanes of P^*

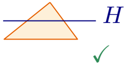
- apply simplicial partitions

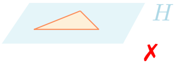
Simplicial partitions (Matoušek '92)

$Q \ldots m$ points in $\mathbb{R}^d$, $1 < r \leq m$, Q can be partitioned into $q \leq 2r$ sets $Q_1, \ldots Q_q$ s.t.

- $m/(2r) \leq |Q_i| \leq m/r$
- Q_i contained in the *relative interior* of a simplex Δ_i
- every hyperplane **crosses** $O(r^{1-1/d})$ of these simplices

H crosses S if $H \cap S \neq \emptyset$ but $S \not\subseteq H$





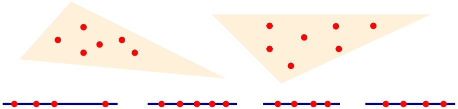
$Q \ldots m$ points in $\mathbb{R}^d$, $1 < r \leq m$, Q can be partitioned into $q \leq 2r$ sets $Q_1, \ldots Q_q$ s.t.

- $m/(2r) \leq |Q_i| \leq m/r$
- Q_i contained in the *relative interior* of a simplex Δ_i
- every hyperplane **crosses** $O(r^{1-1/d})$ of these simplices

$m = 30$
$r = 4$

$Q \ldots m$ points in $\mathbb{R}^d$, $1 < r \leq m$, Q can be partitioned into $q \leq 2r$ sets $Q_1, \ldots Q_q$ s.t.

- $m/(2r) \leq |Q_i| \leq m/r$
- Q_i contained in the *relative interior* of a simplex Δ_i
- every hyperplane **crosses** $O(r^{1-1/d})$ of these simplices

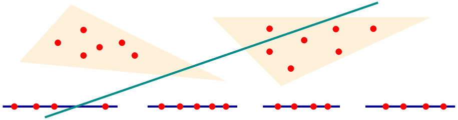


$m = 30$ q $= 6$

$r = 4$ $4 \leq |Q_i| \leq 7$

$Q \ldots m$ points in $\mathbb{R}^d$, $1 < r \leq m$, Q can be partitioned into $q \leq 2r$ sets $Q_1, \ldots Q_q$ s.t.

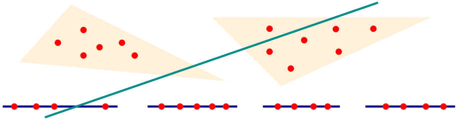
- $m/(2r) \leq |Q_i| \leq m/r$
- Q_i contained in the *relative interior* of a simplex Δ_i
- every hyperplane **crosses** $O(r^{1-1/d})$ of these simplices



$m = 30$ $\quad$ q $= 6$
$r = 4$ $\quad$ $4 \leq |Q_i| \leq 7$
each hyperplane crosses ≤ 3 simplices

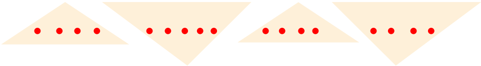
$Q \ldots m$ points in $\mathbb{R}^d$, $1 < r \leq m$, Q can be partitioned into $q \leq 2r$ sets $Q_1, \ldots Q_q$ s.t.

- $m/(2r) \leq |Q_i| \leq m/r$
- Q_i contained in the *relative interior* of a simplex Δ_i
- every hyperplane **crosses** $O(r^{1-1/d})$ of these simplices



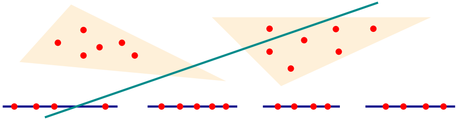
$m = 30$ q $= 6$
$r = 4$ $4 \leq |Q_i| \leq 7$
each hyperplane crosses ≤ 3 simplices

Note: $\dim \Delta_i$ can be $< d$ . . . necessary in degenerate cases



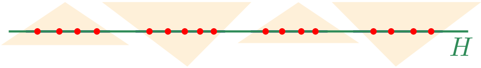
$Q \ldots m$ points in $\mathbb{R}^d$, $1 < r \leq m$, Q can be partitioned into $q \leq 2r$ sets $Q_1, \ldots Q_q$ s.t.

- $m/(2r) \leq |Q_i| \leq m/r$
- Q_i contained in the *relative interior* of a simplex Δ_i
- every hyperplane **crosses** $O(r^{1-1/d})$ of these simplices



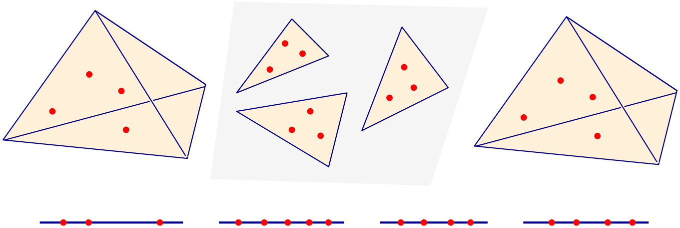
$m = 30$ $\quad$ $q = 6$
$r = 4$ $\quad$ $4 \leq |Q_i| \leq 7$
each hyperplane crosses ≤ 3 simplices

Note: $\dim \Delta_i$ can be $< d$ $\quad$ ... necessary in degenerate cases

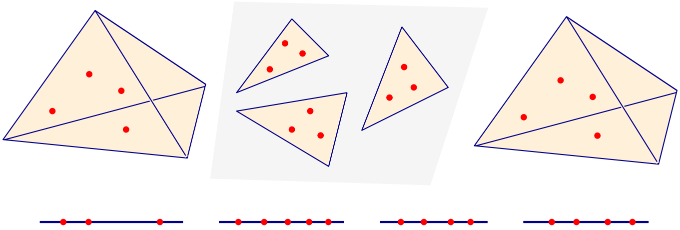


$q = O(r)$ $\quad$ H cross all the simplices

Proof sketch II – upper bound



- inside simplices use a *simple* bound $I(P_i, \mathcal{H}_i) \lesssim |\mathcal{H}_i||P_i|^{1/2}\ell^{1/2} + |P_i|$,
  where ℓ is the max number of points of P lying on a $(d-2)$-flat



- inside simplices use a *simple* bound $I(P_i, \mathcal{H}_i) \lesssim |\mathcal{H}_i||P_i|^{1/2}\ell^{1/2} + |P_i|$,
  where ℓ is the max number of points of P lying on a $(d-2)$-flat
- sum up over all simplices (Cauchy-Schwartz) $\lesssim |\mathcal{H}|\ell^{1/2}|P|^{1/2}r^{-1/(2d)} + r^{1-1/d}|P|$
- deal with low-dim simplices separately . . . adds another $\ell|\mathcal{H}|$ incidences
- specify the parameter r
- obtain upper bound on $I(P, \mathcal{H})$

Why it works?

Question: How come that using a *simple* bound gives something significantly better?

Why it works?

Question: How come that using a *simple* bound gives something significantly better?
Answer: power of divide & conquer

Question: How come that using a *simple* bound gives something significantly better?
Answer: power of divide & conquer

- $I(P,\mathcal{H}) \lesssim |P|^{1/2}\ell^{1/2}|\mathcal{H}| + |P|$ in general weak, but if $\ell^{1/2}|\mathcal{H}| \lesssim |P|^{1/2}$
  $\Rightarrow$ $I(P,\mathcal{H}) \lesssim |P|$, which is optimal

Why it works?

Question: How come that using a *simple* bound gives something significantly better?
Answer: power of divide & conquer

- $I(P, \mathcal{H}) \lesssim |P|^{1/2}\ell^{1/2}|\mathcal{H}| + |P|$ in general weak, but if $\ell^{1/2}|\mathcal{H}| \lesssim |P|^{1/2}$
  $\Rightarrow$ $I(P, \mathcal{H}) \lesssim |P|$, which is optimal
- simplicial partition guarantees we have much less hyperplanes than points
  (we did the partition in the dual)

Question: How come that using a *simple* bound gives something significantly better?
Answer: power of divide & conquer

- $I(P,\mathcal{H}) \lesssim |P|^{1/2}\ell^{1/2}|\mathcal{H}| + |P|$ in general weak, but if $\ell^{1/2}|\mathcal{H}| \lesssim |P|^{1/2}$
  $\Rightarrow$ $I(P,\mathcal{H}) \lesssim |P|$, which is optimal
- simplicial partition guarantees we have much less hyperplanes than points
  (we did the partition in the dual)

Moral: having a tight bound for unbalanced case can be helpful

make the setting unbalanced (divide the space) $\longrightarrow$ use the tight bound $\longrightarrow$ sum it up
$\longrightarrow$ optimize the dividing parameter & deal with "non-crossing" intersections

Tightness of our result: construction I

Setting: $\alpha = d + 1$ for simplicity $d = 3$, k is a square

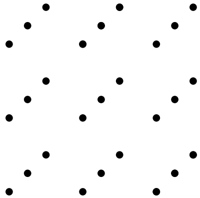
Thm: number of k-rich planes $\gtrsim n^3/k^4 + n/k \quad \Rightarrow \quad \exists$ a line with $\geq \sqrt{k}$ points of P

Setting: $\alpha = d + 1$ for simplicity $d = 3$, k is a square

Thm: number of k-rich planes $\gtrsim n^3/k^4 + n/k$ $\Rightarrow$ $\exists$ a line with $\geq \sqrt{k}$ points of P

Construction: P . . . set of vertices of $\sqrt{k} \times \sqrt{k} \times \sqrt{k}$ integer grid in $\mathbb{R}^3$

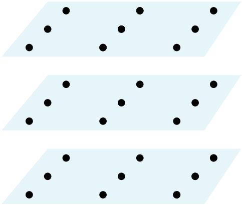
- $n = |P| = k^{3/2}$



Setting: $\alpha = d + 1$ for simplicity $d = 3$, k is a square

Thm: number of k-rich planes $\gtrsim n^3/k^4 + n/k \quad \Rightarrow \quad \exists$ a line with $\geq \sqrt{k}$ points of P

Construction: P . . . set of vertices of $\sqrt{k} \times \sqrt{k} \times \sqrt{k}$ integer grid in $\mathbb{R}^3$



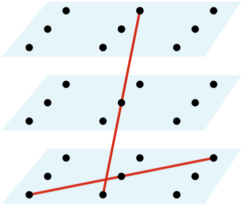
- $n = |P| = k^{3/2}$
- number of k-rich planes $\geq \sqrt{k}$

Setting: $\alpha = d + 1$ for simplicity $d = 3$, k is a square

Thm: number of k-rich planes $\gtrsim n^3/k^4 + n/k$ $\Rightarrow$ $\exists$ a line with $\geq \sqrt{k}$ points of P

Construction: P ... set of vertices of $\sqrt{k} \times \sqrt{k} \times \sqrt{k}$ integer grid in $\mathbb{R}^3$

- $n = |P| = k^{3/2}$
- number of k-rich planes $\geq \sqrt{k}$
- no line contains more than $\sqrt{k}$ points of P

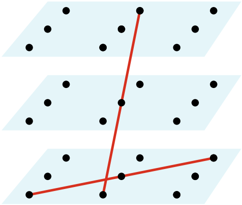


Tightness of our result: construction I

Setting: $\alpha = d + 1$ for simplicity $d = 3$, k is a square

Thm: number of k-rich planes $\gtrsim n^3/k^4 + n/k \quad \Rightarrow \quad \exists$ a line with $\geq \sqrt{k}$ points of P

Construction: P ... set of vertices of $\sqrt{k} \times \sqrt{k} \times \sqrt{k}$ integer grid in $\mathbb{R}^3$



- $n = |P| = k^{3/2}$
- number of k-rich planes $\geq \sqrt{k}$
- no line contains more than $\sqrt{k}$ points of P

Conclusion: Our bound is worst-case asymptotically tight when $k = \Theta(n^{1-1/d})$

Open problem: What happens for other values of k?

Tightness: construction II

Setting: $\alpha = d = 3$ $k, u \geq 2$ integers

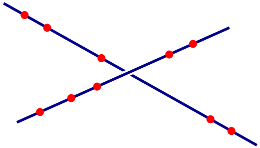
Thm: number of k-rich planes $\gtrsim n^3/k^3$ $\Rightarrow$ $\exists$ a line with $\geq k$ points of P

Setting: $\alpha = d = 3$ $k, u \geq 2$ integers

Thm: number of k-rich planes $\gtrsim n^3/k^3$ $\Rightarrow$ $\exists$ a line with $\geq k$ points of P

Construction: L . . . a set of u pairwise skew lines in $\mathbb{R}^3$
P . . . k distinguished points on each line

- $n := |P| = ku$

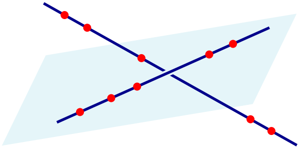


Setting: $\alpha = d = 3$ $k, u \geq 2$ integers

Thm: number of k-rich planes $\gtrsim n^3/k^3$ $\Rightarrow$ $\exists$ a line with $\geq k$ points of P

Construction: L . . . a set of u pairwise skew lines in $\mathbb{R}^3$
P . . . k distinguished points on each line

- $n := |P| = ku$
- infinitely many k-rich planes wrt P
- no line contains $> k$ points of P

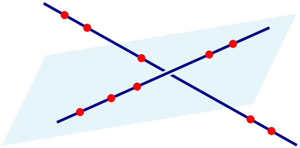


Setting: $\alpha = d = 3$ $\quad k, u \geq 2$ integers

Thm: number of k-rich planes $\gtrsim n^3/k^3$ $\Rightarrow$ $\exists$ a line with $\geq k$ points of P

Construction: L … a set of u pairwise skew lines in $\mathbb{R}^3$
P … k distinguished points on each line

- $n := |P| = ku$
- infinitely many k-rich planes wrt P
- no line contains $> k$ points of P



Conclusion: Our thm is tight for $\alpha = d = 3$

Open problem: What happens for other values of α?

- P ... n points in $\mathbb{R}^d$
- k-rich sphere wrt P ... contains $\geq k$ points from P

- $d \geq 3$ $\qquad k \geq d+1$ $\qquad d+1 \leq \alpha < 2d+1$
- $\gtrsim \left(\frac{n^{d+1}}{k^\alpha} + \frac{n}{k}\right)$ k-rich $(d-1)$-spheres

  $\Rightarrow$ there is a $(d-2)$-sphere containing $\gtrsim k^{(2d+1-\alpha)/d}$ points of P

- P ... n points in $\mathbb{R}^d$
- k-rich sphere wrt P ... contains $\geq k$ points from P

- $d \geq 3$ $\quad$ $k \geq d+1$ $\quad$ $d+1 \leq \alpha < 2d+1$
- $\gtrsim \left(\frac{n^{d+1}}{k^\alpha} + \frac{n}{k}\right)$ k-rich $(d-1)$-spheres
  $\Rightarrow$ there is a $(d-2)$-sphere containing $\gtrsim k^{(2d+1-\alpha)/d}$ points of P

Proof sketch:

- transform $(d-1)$-spheres in $\mathbb{R}^d$ to hyperplanes in $\mathbb{R}^{d+1}$

$$(x_1, \ldots, x_d) \mapsto (x_1, \ldots, x_d, x_1^2 + \cdots + x_d^2)$$

- observe it's the same problem as before, just in $\mathbb{R}^{d+1}$

Main result (P., Sharir)**:**

- P ... n points in $\mathbb{R}^d$ $3 \leq d \leq k$ $d \leq \alpha < 2d - 1$
- $\gtrsim \left(\frac{n^d}{k^\alpha} + \frac{n}{k}\right)$ k-rich hyperplanes

  $\Rightarrow$ there is a $(d-2)$-flat containing $\gtrsim k^{(2d-1-\alpha)/(d-1)}$ points of P

Open problems:

- tightness for various values of α tightness for spheres

Main result (P., Sharir)**:**

- P . . . n points in $\mathbb{R}^d$ $3 \leq d \leq k$ $d \leq \alpha < 2d - 1$
- $\gtrsim \left(\frac{n^d}{k^\alpha} + \frac{n}{k}\right)$ k-rich hyperplanes
  $\Rightarrow$ there is a $(d-2)$-flat containing $\gtrsim k^{(2d-1-\alpha)/(d-1)}$ points of P

Open problems:

- tightness for various values of α tightness for spheres
- What are further conditions, in $d \geq 4$, enforcing the existence of even lower-dim flat containing many points of P?

Summary & open problems

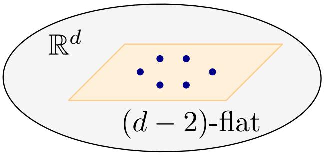
Main result (P., Sharir)**:**

- P . . . n points in $\mathbb{R}^d$ $3 \le d \le k$ $d \le \alpha < 2d - 1$
- $\gtrsim \left(\frac{n^d}{k^\alpha} + \frac{n}{k}\right)$ k-rich hyperplanes

  $\Rightarrow$ there is a $(d-2)$-flat containing $\gtrsim k^{(2d-1-\alpha)/(d-1)}$ points of P

Open problems:

- tightness for various values of α tightness for spheres
- What are further conditions, in $d \ge 4$, enforcing the existence of even lower-dim flat containing many points of P?



- P in general position
- ∞ many k-rich hyperplanes
- $\forall$ $(d-3)$-flat has $\le (d-2)$ points of P

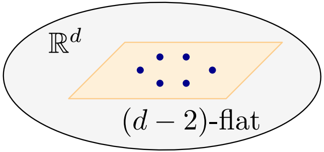
Main result (P., Sharir):

- P ... n points in $\mathbb{R}^d$ $3 \leq d \leq k$ $d \leq \alpha < 2d - 1$
- $\gtrsim \left(\frac{n^d}{k^\alpha} + \frac{n}{k}\right)$ k-rich hyperplanes

  $\Rightarrow$ there is a $(d-2)$-flat containing $\gtrsim k^{(2d-1-\alpha)/(d-1)}$ points of P

Open problems:

- tightness for various values of α tightness for spheres
- What are further conditions, in $d \geq 4$, enforcing the existence of even lower-dim flat containing many points of P?



- P in general position
- ∞ many k-rich hyperplanes
- $\forall$ $(d-3)$-flat has $\leq (d-2)$ points of P

Thank you!