Some Advances in the Planar Directed Steiner Tree Problem

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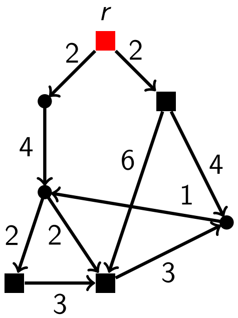
Definition

- Given a directed graph $G = (V, E)$, cost on edges, root node r, and a set of terminals $X \subseteq V - r$. The rest of vertices are called Steiner nodes
- Find a min cost subgraph such that every terminal is reachable from the root



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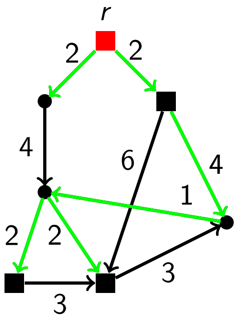

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- Unidrected version has ≈ 1.39-approx Byrka et al. - 2013

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Theorem (Friggstad-M.)

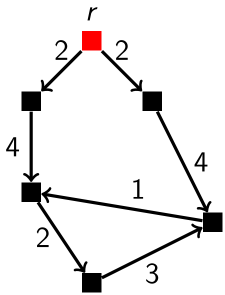
There is a 20-approx for DST on quasi-bipartite, planar instances. We can generalize it to any graph that excludes a fixed minor.

Toolbox: primal-dual

- Primal-dual algorithm is rare in the directed network design problems. One use of this is in Arborescence (more on this in the next slide).
- In contrast, primal-dual algorithm is used in the undirected network design abundantly, e.g. Guha et al. - 1999, Könemann et al. - 2013, Moldenhauer 2013, and Demaine et al. - 2014
- Why primal-dual algorithm is preferred? Can be viewed as combinatorial algorithm and usually fast and easy to implement!

Warm up (Arborescence)

Given a directed graph, edge cost and a root node r. Find a cheapest subgraph such that every node is reachable from r



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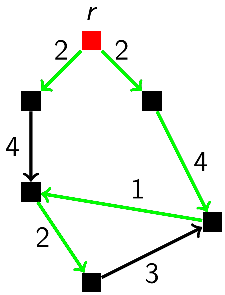

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$\delta^{in}(S)$ is the set of edges entering S and $x(\delta^{in}(S)) := \sum_{e:e\in\delta^{in}(S)} x_e$

Primal LP

$$\min \sum_{e} c_e \cdot x_e$$
$$x(\delta^{in}(S)) \geq 1, \ \forall S \subseteq V - r$$
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Dual LP

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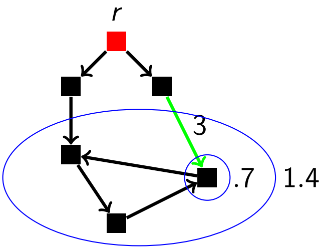

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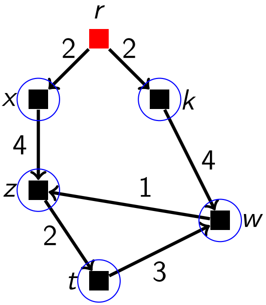
- $\mathrm{cost}(F) \leq \mathrm{cost}(\bar{y}) \leq$ Dual LP $\leq$ Primal LP $\leq \mathrm{OPT}$

Primal-dual in action!

Recall the dual constraint for edge e is $\sum_{S : e \in \delta^{in}(S)} y_S \leq c_e$



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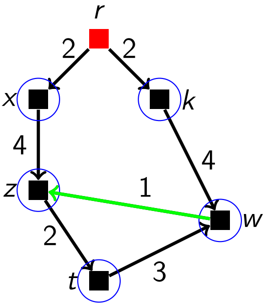

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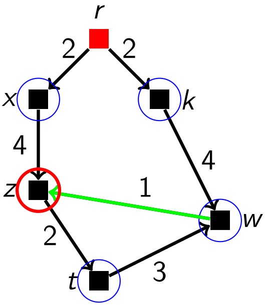

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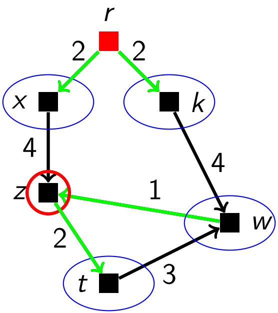

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- At time 2, rx, rk and zt are bought and the only active set is the blue one.

  $y_{\{x\}} = y_{\{k\}} = y_{\{w\}} = y_{\{t\}} = 2$, $y_{\{z\}} = 1$.

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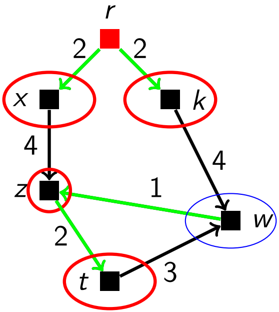

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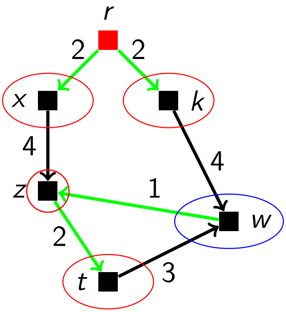

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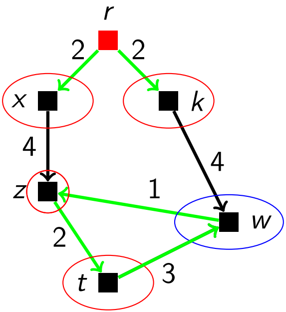

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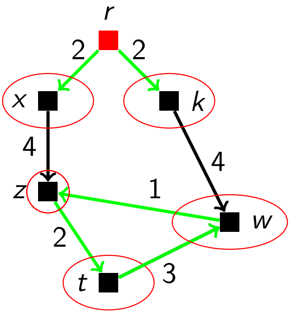

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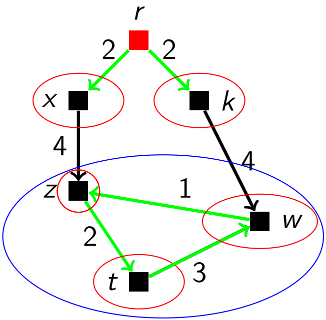

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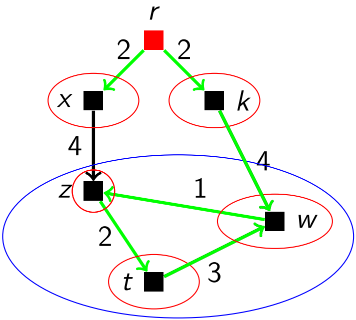

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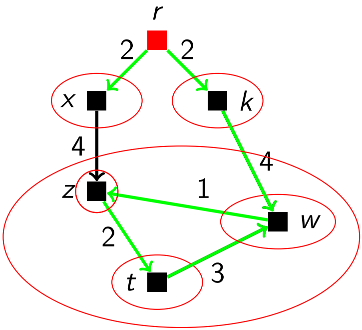

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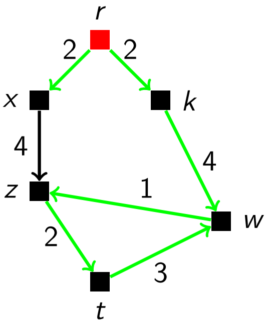

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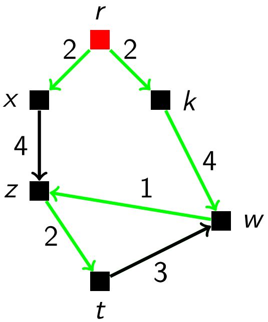

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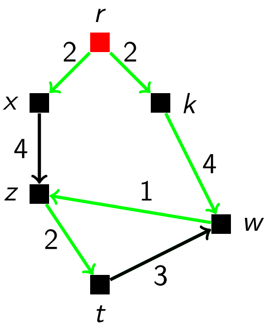

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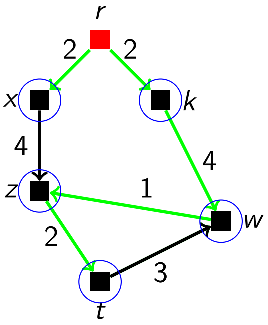

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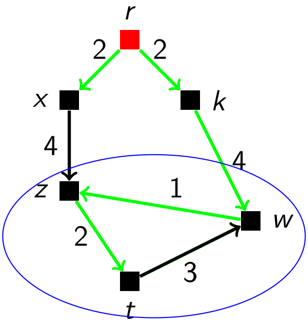

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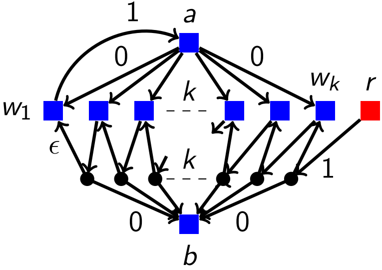
- Natural thing to try is to use the "same" primal-dual algorithm for Arborescence here!

There is always an obstacle!



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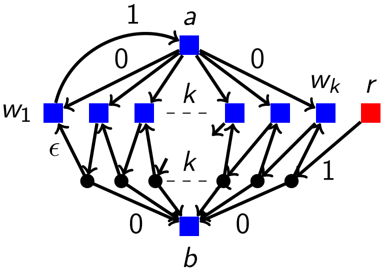

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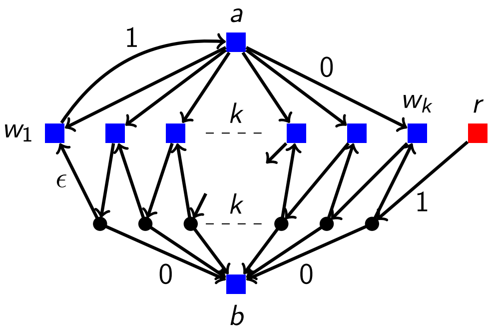

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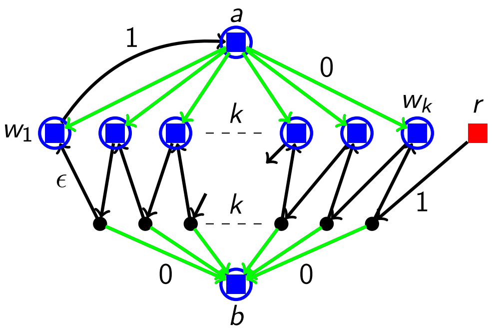

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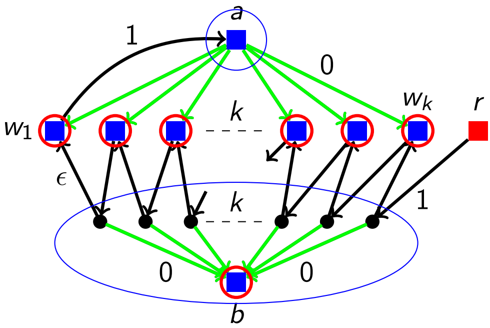

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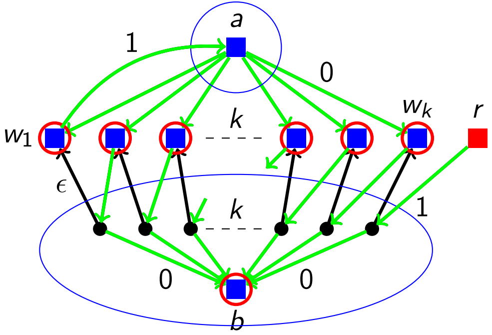

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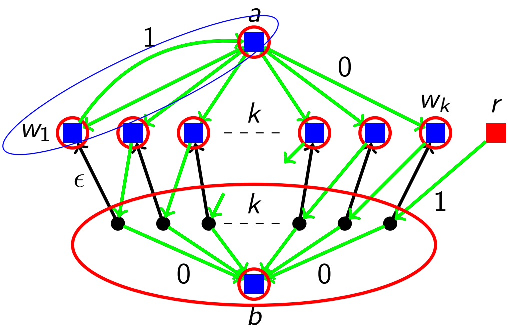

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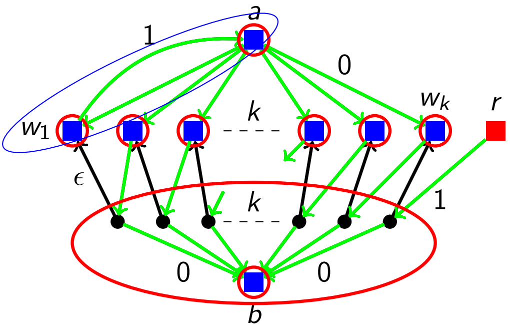
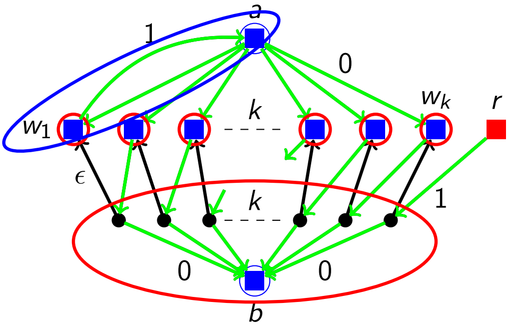


- Bottom set bought too many edges that aren't used for its connectivity

But sometimes there is a bypass!



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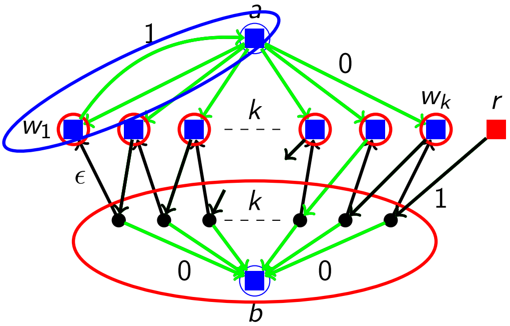

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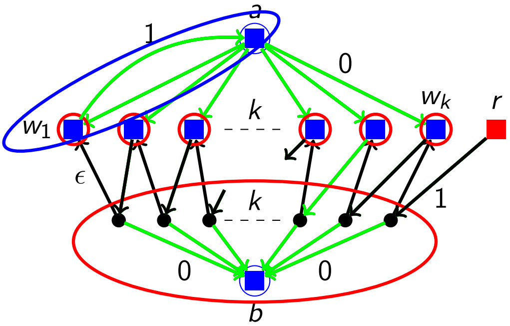

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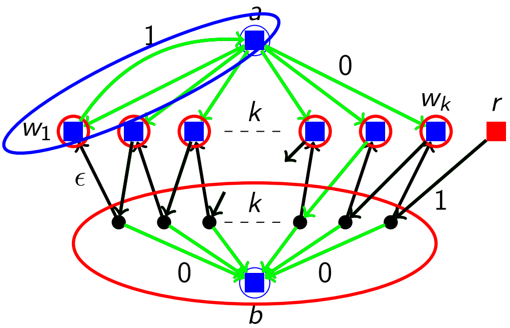

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- How to formalize this?
- Introducing two buckets: expansion and killer!

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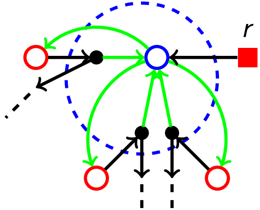

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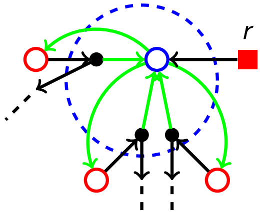

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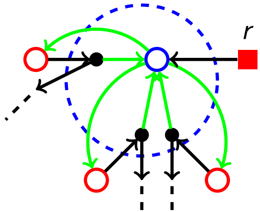

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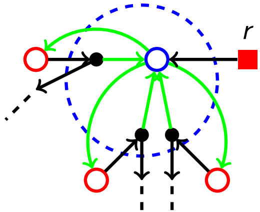

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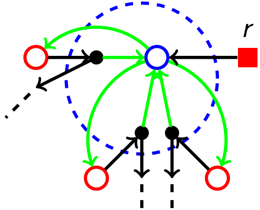

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