Cop Numbers of Generalised Petersen Graphs

Joy Morris

University of Lethbridge

June 4, 2022

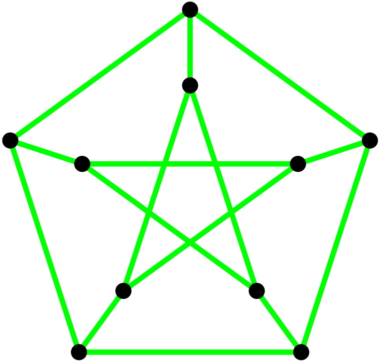
Alberta–Montana Combinatorics and Algorithms Day

Overview

The game

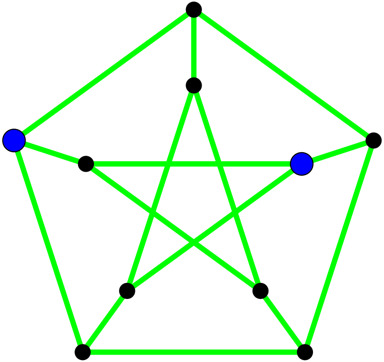
Rules

Start with a graph.



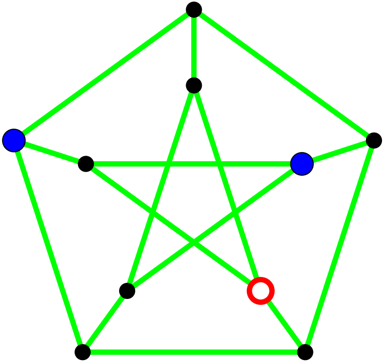
Rules

Start with a graph. The pursuer [cop] places their pieces.



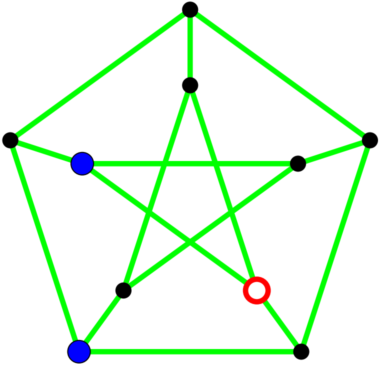
Rules

Start with a graph. The pursuer [cop] places their pieces. The evader [robber] places their piece.



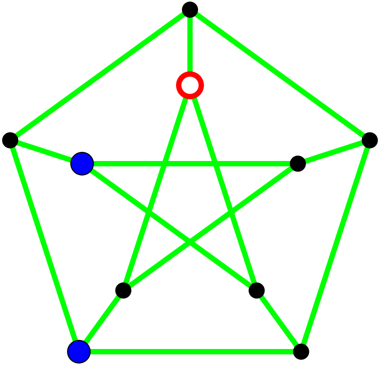
Rules

Start with a graph. The pursuer [cop] places their pieces. The evader [robber] places their piece. The pursuer and evader take turns moving their piece(s).



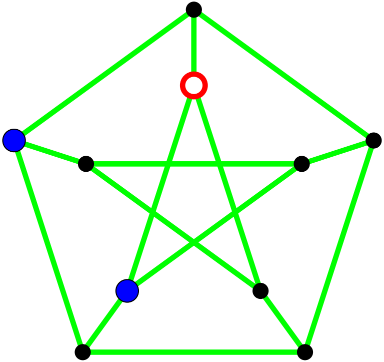
Rules

Start with a graph. The pursuer [cop] places their pieces. The evader [robber] places their piece. The pursuer and evader take turns moving their piece(s).



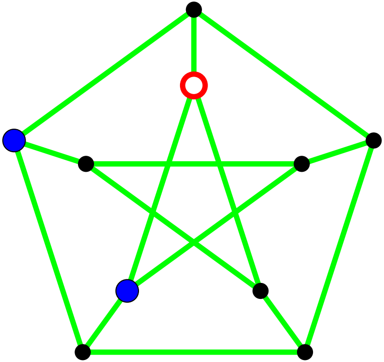
Rules

Start with a graph. The pursuer [cop] places their pieces. The evader [robber] places their piece. The pursuer and evader take turns moving their piece(s).



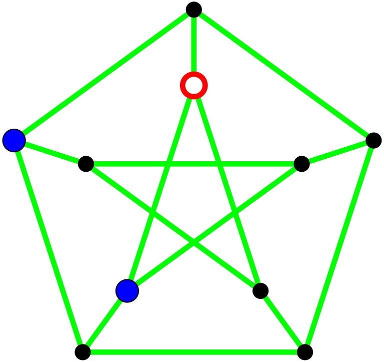
Rules

Start with a graph. The pursuer [cop] places their pieces. The evader [robber] places their piece. The pursuer and evader take turns moving their piece(s). The pursuer wins if any of their pieces is ever on top of the evader.



Rules

Start with a graph. The pursuer [cop] places their pieces. The evader [robber] places their piece. The pursuer and evader take turns moving their piece(s). The pursuer wins if any of their pieces is ever on top of the evader. The evader wins if they can evade capture forever.



Definition

The cop number of a graph is the minimum number of cops required to ensure that the cops have a winning strategy.

Definition

The cop number of a graph is the minimum number of cops required to ensure that the cops have a winning strategy.

Graphs with cop number 1 have been completely characterised (they must contain a pitfall).

Generalised Petersen graphs

Definition

Generalised Petersen graphs were introduced by Coxeter in 1950, and named by Watkins in 1969.

Definition

Generalised Petersen graphs were introduced by Coxeter in 1950, and named by Watkins in 1969.

Definition

The Generalised Petersen graph $GP(n, k)$ has $2n$ vertices:

Definition

Generalised Petersen graphs were introduced by Coxeter in 1950, and named by Watkins in 1969.

Definition

The Generalised Petersen graph $GP(n,k)$ has $2n$ vertices:

$$u_0, \ldots, u_{n-1} \text{ and } v_0, \ldots, v_{n-1}.$$

Definition

Generalised Petersen graphs were introduced by Coxeter in 1950, and named by Watkins in 1969.

Definition

The Generalised Petersen graph $GP(n,k)$ has $2n$ vertices:

$$u_0, \ldots, u_{n-1} \textit{ and } v_0, \ldots, v_{n-1}.$$

For $0 \leq i \leq n-1$ it has the edges:

$$u_i v_i,$$

Definition

Generalised Petersen graphs were introduced by Coxeter in 1950, and named by Watkins in 1969.

Definition

The *Generalised Petersen graph* $GP(n,k)$ *has* $2n$ *vertices:*

$$u_0, \ldots, u_{n-1} \text{ and } v_0, \ldots, v_{n-1}.$$

For $0 \le i \le n-1$ *it has the edges:*

$$u_i v_i, u_i u_{i+1},$$

Definition

Generalised Petersen graphs were introduced by Coxeter in 1950, and named by Watkins in 1969.

Definition

The *Generalised Petersen graph* $GP(n,k)$ *has* $2n$ *vertices:*

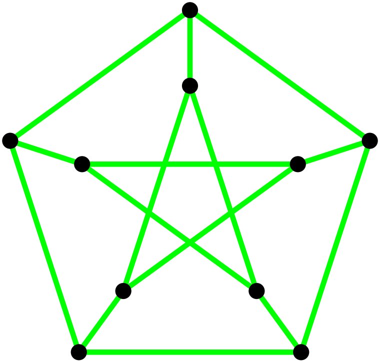
$$u_0, \dots, u_{n-1} \text{ and } v_0, \dots, v_{n-1}.$$

For $0 \le i \le n-1$ *it has the edges:*

$$u_i v_i, u_i u_{i+1}, \text{ and } v_i v_{i+k} \pmod{n}.$$

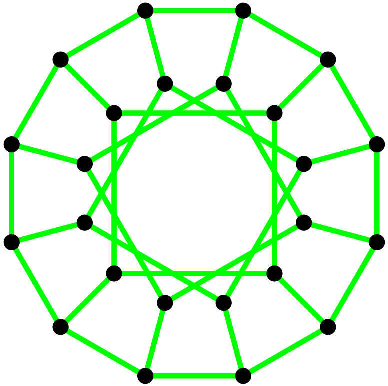
Example

So the Petersen graph is $GP(5, 2)$:



Example

Here is $GP(12,3)$:



Isomorphism

Theorem (Steimle and Stanton, 2009)

$GP(n,k)$ and $GP(n,\ell)$ are isomorphic if and only if $k = \ell$ or $k\ell \equiv \pm 1 \pmod{n}$.

Previous results

Cop number of $GP(n, k)$

These results were shown by Ball, Bell, Guzman, Hanson-Colvin, and Schonscheck, 2017.

Cop number of $GP(n, k)$

These results were shown by Ball, Bell, Guzman, Hanson-Colvin, and Schonscheck, 2017.

- $GP(n, k)$ never has cop number 1 (Generalised Petersen graphs do not have pitfalls).

Cop number of $GP(n, k)$

These results were shown by Ball, Bell, Guzman, Hanson-Colvin, and Schonscheck, 2017.

- $GP(n, k)$ never has cop number 1 (Generalised Petersen graphs do not have pitfalls).
- $GP(n, k)$ has cop number 2 when $k = 1$, or when $(n, k) \in \{(6, 2), (8, 2), (9, 3), (12, 3)\}$.

Cop number of $GP(n, k)$

These results were shown by Ball, Bell, Guzman, Hanson-Colvin, and Schonscheck, 2017.

- $GP(n, k)$ never has cop number 1 (Generalised Petersen graphs do not have pitfalls).
- $GP(n, k)$ has cop number 2 when $k = 1$, or when $(n, k) \in \{(6, 2), (8, 2), (9, 3), (12, 3)\}$.
- if $GP(n, k)$ has cop number 2 then either $k = 1$, $n = 3k$, or $n = 4k$.

Cop number of $GP(n, k)$

These results were shown by Ball, Bell, Guzman, Hanson-Colvin, and Schonscheck, 2017.

- $GP(n, k)$ never has cop number 1 (Generalised Petersen graphs do not have pitfalls).
- $GP(n, k)$ has cop number 2 when $k = 1$, or when $(n, k) \in \{(6, 2), (8, 2), (9, 3), (12, 3)\}$.
- if $GP(n, k)$ has cop number 2 then either $k = 1$, $n = 3k$, or $n = 4k$.
- $GP(n, 3)$ has cop number 3 except for $GP(9, 3)$ and $GP(12, 3)$.

Cop number of $GP(n, k)$

These results were shown by Ball, Bell, Guzman, Hanson-Colvin, and Schonscheck, 2017.

- $GP(n, k)$ never has cop number 1 (Generalised Petersen graphs do not have pitfalls).
- $GP(n, k)$ has cop number 2 when $k = 1$, or when $(n, k) \in \{(6, 2), (8, 2), (9, 3), (12, 3)\}$.
- if $GP(n, k)$ has cop number 2 then either $k = 1$, $n = 3k$, or $n = 4k$.
- $GP(n, 3)$ has cop number 3 except for $GP(9, 3)$ and $GP(12, 3)$.
- $GP(n, k)$ has cop number at most 4.

Cop number of $GP(n,k)$

These results were shown by Ball, Bell, Guzman, Hanson-Colvin, and Schonscheck, 2017.

- $GP(n,k)$ never has cop number 1 (Generalised Petersen graphs do not have pitfalls).
- $GP(n,k)$ has cop number 2 when $k = 1$, or when $(n,k) \in \{(6,2),(8,2),(9,3),(12,3)\}$.
- if $GP(n,k)$ has cop number 2 then either $k = 1$, $n = 3k$, or $n = 4k$.
- $GP(n,3)$ has cop number 3 except for $GP(9,3)$ and $GP(12,3)$.
- $GP(n,k)$ has cop number at most 4.
- the cop number of $GP(n,k)$ is determined for every $n \leq 40$.

Cop number of $GP(n, k)$

These results were shown by Ball, Bell, Guzman, Hanson-Colvin, and Schonscheck, 2017.

- $GP(n, k)$ never has cop number 1 (Generalised Petersen graphs do not have pitfalls).
- $GP(n, k)$ has cop number 2 when $k = 1$, or when $(n, k) \in \{(6, 2), (8, 2), (9, 3), (12, 3)\}$.
- if $GP(n, k)$ has cop number 2 then either $k = 1$, $n = 3k$, or $n = 4k$.
- $GP(n, 3)$ has cop number 3 except for $GP(9, 3)$ and $GP(12, 3)$.
- $GP(n, k)$ has cop number at most 4.
- the cop number of $GP(n, k)$ is determined for every $n \leq 40$.

So the cop number is 2, 3, or 4.

Cop number of $GP(n, k)$

These results were shown by Ball, Bell, Guzman, Hanson-Colvin, and Schonscheck, 2017.

- $GP(n, k)$ never has cop number 1 (Generalised Petersen graphs do not have pitfalls).
- $GP(n, k)$ has cop number 2 when $k = 1$, or when $(n, k) \in \{(6, 2), (8, 2), (9, 3), (12, 3)\}$.
- if $GP(n, k)$ has cop number 2 then either $k = 1$, $n = 3k$, or $n = 4k$.
- $GP(n, 3)$ has cop number 3 except for $GP(9, 3)$ and $GP(12, 3)$.
- $GP(n, k)$ has cop number at most 4.
- the cop number of $GP(n, k)$ is determined for every $n \leq 40$.

So the cop number is 2, 3, or 4. From the data it appears that the cop number is 2 only in the cases mentioned above. But when is it 3 and when is it 4?

Girth and our results

Girth and Generalised Petersen graphs

Definition

The girth of a graph is the length of the shortest cycle.

Girth and Generalised Petersen graphs

Definition

The girth of a graph is the length of the shortest cycle.

Generalised Petersen graphs have girth at most 8.

Girth and Generalised Petersen graphs

Definition

The girth of a graph is the length of the shortest cycle.

Generalised Petersen graphs have girth at most 8.

$(u_0, u_1, v_1, v_{k+1}, u_{k+1}, u_k, v_k, v_0, u_0)$

Girth of $GP(n,k)$

(Boben, Pisanski, and Žitnik, 2005. Showing smallest k up to isomorphism.)

Girth 3	Girth 4	Girth 5	Girth 6	Girth 7	Girth 8
$n = 3k$	$n = 4k$	$n = 5k$	$n = 6k$	$n = 7k$	otherwise
	$k = 1$	$k = 2$	$k = 3$	$k = 4$	
		$n = 5k/2$	$n = 2k + 2$	$n = 7k/2$	
				$n = 7k/3$	
				$n = 2k + 3$	
				$n = 3k \pm 2$	

Girth and cop number

Generalised Petersen graphs with cop number 4, up to $n = 40$:

n	k	girth	n	k	girth
25	7	8	34	6, 10, 13, 14	8, 8, 8, 8
26	10	8	35	6, 8, 10, 13, 15	8, 8, 7, 8, 7
27	6	8	36	8, 10, 14, 15	8, 8, 8, 8
28	6, 8	8, 7	37	6, 7, 8, 10, 11, 14, 16	all 8
29	8, 11, 12	8, 8, 8	38	6, 7, 8, 11, 14, 16	all 8
31	7, 9, 12, 13	8, 8, 8, 8	39	6, 7, 9, 11, 15, 16, 17	all 8
32	6, 7, 9, 12	8, 8, 8, 8	40	6, 7, 9, 11, 12, 15, 17	all 8
33	6, 7, 9, 14	8, 8, 8, 8			

Main results

Theorem (M., Runte, Skelton, 2022)

Let G be a cubic graph of girth at least 8. Unless G contains two cycles of length 8 whose intersection is a path of length 2, the cop number of G is at least 4.

Main results

Theorem (M., Runte, Skelton, 2022)

Let G be a cubic graph of girth at least 8. Unless G contains two cycles of length 8 whose intersection is a path of length 2, the cop number of G is at least 4. If G is a Generalised Petersen graph, then its cop number is 4.

Main results

Theorem (M., Runte, Skelton, 2022)

Let G be a cubic graph of girth at least 8. Unless G contains two cycles of length 8 whose intersection is a path of length 2, the cop number of G is at least 4. If G is a Generalised Petersen graph, then its cop number is 4.

Theorem (M., Runte, Skelton, 2022)

If G is a connected graph of minimum valency $\delta \geq 3$ and girth at least 9, then its cop number is greater than δ.

Main results

Theorem (M., Runte, Skelton, 2022)

Let G be a cubic graph of girth at least 8. Unless G contains two cycles of length 8 whose intersection is a path of length 2, the cop number of G is at least 4. If G is a Generalised Petersen graph, then its cop number is 4.

Theorem (M., Runte, Skelton, 2022)

If G is a connected graph of minimum valency $\delta \geq 3$ and girth at least 9, then its cop number is greater than δ. (Better than Frankl bound.)

Main results

Theorem (M., Runte, Skelton, 2022)

Let G be a cubic graph of girth at least 8. Unless G contains two cycles of length 8 whose intersection is a path of length 2, the cop number of G is at least 4. If G is a Generalised Petersen graph, then its cop number is 4.

Theorem (M., Runte, Skelton, 2022)

If G is a connected graph of minimum valency $\delta \geq 3$ and girth at least 9, then its cop number is greater than δ. (Better than Frankl bound.)

Theorem (M., Runte, Skelton, 2022)

$GP(n, 2)$ has cop number 3, except for $GP(6, 2)$ and $GP(8, 2)$.

Main results

Theorem (M., Runte, Skelton, 2022)

Let G be a cubic graph of girth at least 8. Unless G contains two cycles of length 8 whose intersection is a path of length 2, the cop number of G is at least 4. If G is a Generalised Petersen graph, then its cop number is 4.

Theorem (M., Runte, Skelton, 2022)

If G is a connected graph of minimum valency $\delta \geq 3$ and girth at least 9, then its cop number is greater than δ. (Better than Frankl bound.)

Theorem (M., Runte, Skelton, 2022)

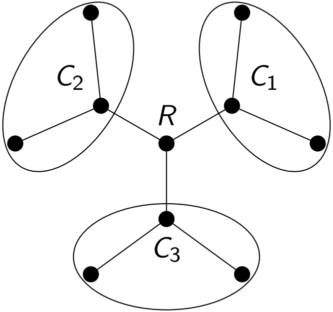
$GP(n, 2)$ has cop number 3, except for $GP(6, 2)$ and $GP(8, 2)$.

Theorem (H. Morris, M., 2022)

Suppose that $n = 7k/i$ where $i \in \{1, 2, 3\}$, and $n \geq 42$ or $(n, k) \in \{(28, 8), (35, 10), (35, 15)\}$. Then the cop number of the graph $GP(n, k)$ is 4.

Key Ideas

Trapped!



This evader is trapped.

Main idea

We consider various cases for possible configurations for three pursuers relative to the the evader (other than trapped).

Main idea

We consider various cases for possible configurations for three pursuers relative to the the evader (other than trapped). In each case, we show that the evader has a move that does not leave them trapped.

Main idea

We consider various cases for possible configurations for three pursuers relative to the the evader (other than trapped). In each case, we show that the evader has a move that does not leave them trapped. Showing that the evader always has a non-trapped starting position proves inductively that the evader can win.

Main idea

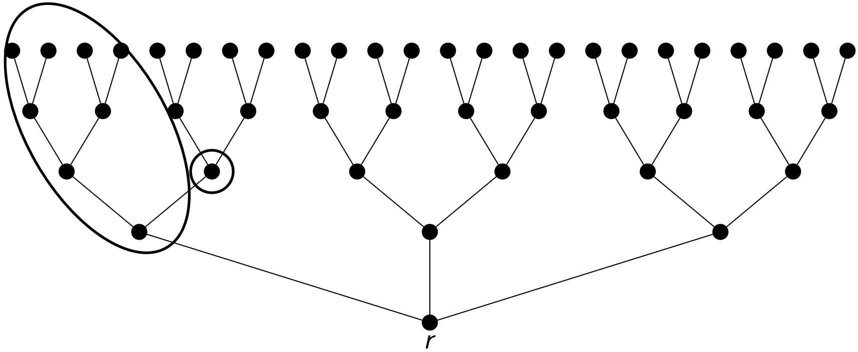
We consider various cases for possible configurations for three pursuers relative to the the evader (other than trapped). In each case, we show that the evader has a move that does not leave them trapped. Showing that the evader always has a non-trapped starting position proves inductively that the evader can win. Thus, the cop number is at least 4.

Case 1

There is a neighbour of the evader's vertex that has no pursuer on one of its branches, and no pursuer within distance 2 of the evader on the other branch.

Case 1

There is a neighbour of the evader's vertex that has no pursuer on one of its branches, and no pursuer within distance 2 of the evader on the other branch. (No pursuer on any of the circled vertices.)

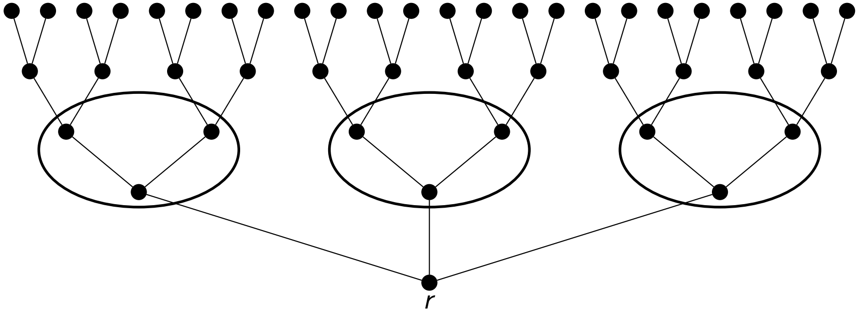


Case 2

No pursuer is within distance 2 of the evader.

Case 2

No pursuer is within distance 2 of the evader. (No pursuer on any of the circled vertices.)

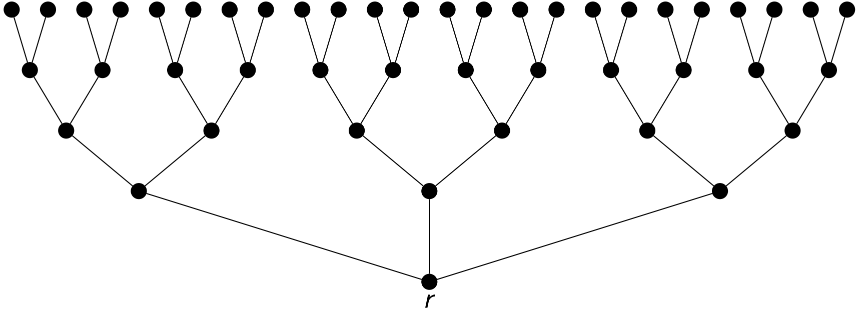


Case 3

For each of the evader's neighbouring vertices, there is either a pursuer within distance 2 of the evader (this happens somewhere), or there is at least one pursuer on each branch.

Case 3

For each of the evader's neighbouring vertices, there is either a pursuer within distance 2 of the evader (this happens somewhere), or there is at least one pursuer on each branch.



Summary of Results for $n > 40$

$c = 2$	$2 \leq c \leq 4$	$c = 3$	$3 \leq c \leq 4$	$c = 4$
$k = 1$	$n = 3k$	$k = 2$	$4 \leq k \leq 5$	otherwise
	$n = 4k$	$k = 3$	$n = 2k + i, i \in \{2, 3, 4\}$	
			$n = 3k + i, i \in \{\pm 2, \pm 3\}$	
			$n = 4k + i, i \in \{\pm 2\}$	
			$n = 5k/i, i \in \{1, 2\}$	
			$n = 6k$	

Also worth noting

Also worth noting

Theorem

Any connected graph of minimum valency $\delta \geq 3$ and girth at least 9 has cop number greater than δ.

Open Problems

Open problems

Are there other Generalised Petersen graphs with cop number 2?

Open problems

Are there other Generalised Petersen graphs with cop number 2?
Are there other Generalised Petersen graphs with cop number 4?

Open problems

Are there other Generalised Petersen graphs with cop number 2?
Are there other Generalised Petersen graphs with cop number 4?
The cop number of an I-graph (even more general family) is at most 5.
Which I-graphs have cop number 5?

Open problems

Are there other Generalised Petersen graphs with cop number 2?
Are there other Generalised Petersen graphs with cop number 4?
The cop number of an I-graph (even more general family) is at most 5.
Which I-graphs have cop number 5?
What does the "lazy" cop number look like on Generalised Petersen graphs? [Only 1 cop can move in a turn.]

Thank you!

NSERC
CRSNG