At most 3.55^n stable matchings

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University of Montana

Alberta-Montana Combinatorics & Algorithms Days at BIRS

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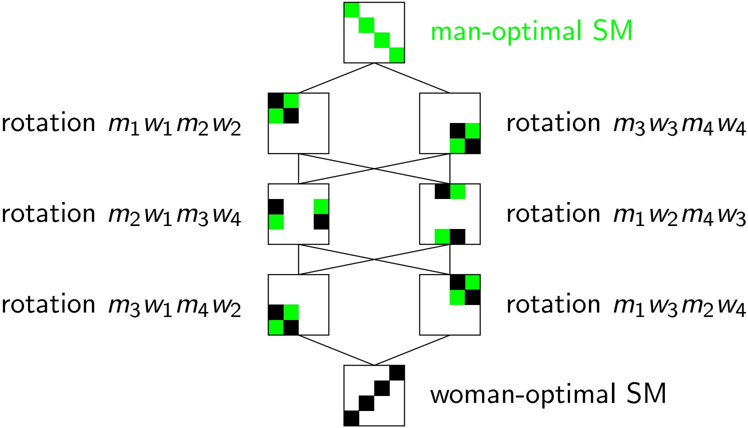
- Palmer-Pálvölgyi '22+: $SM(n) = O(3.55^n)$.

Rotation poset



Irving-Leather '86: Rotations form a poset

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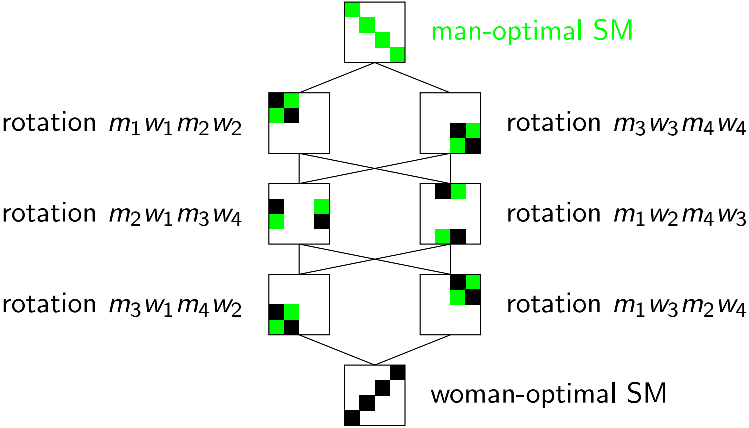

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Key fact: Poset downsets 1–1 stable matchings

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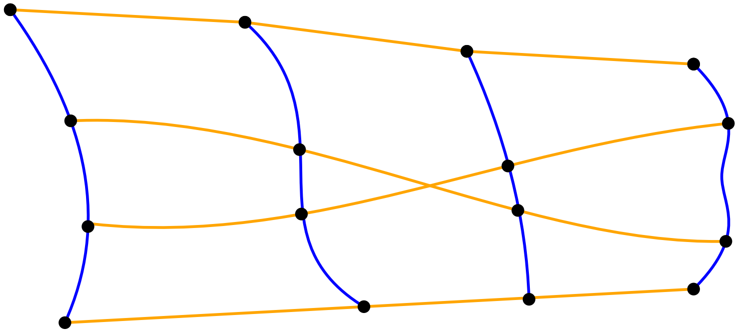

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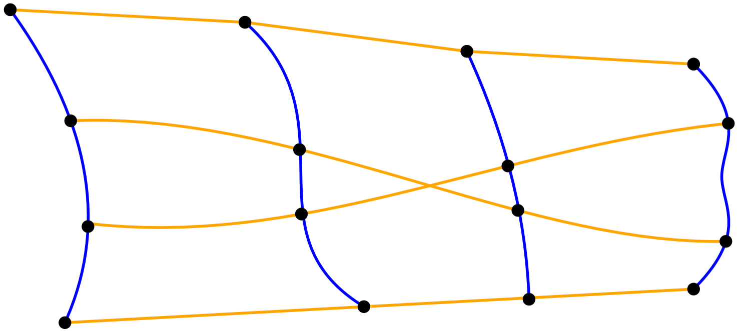

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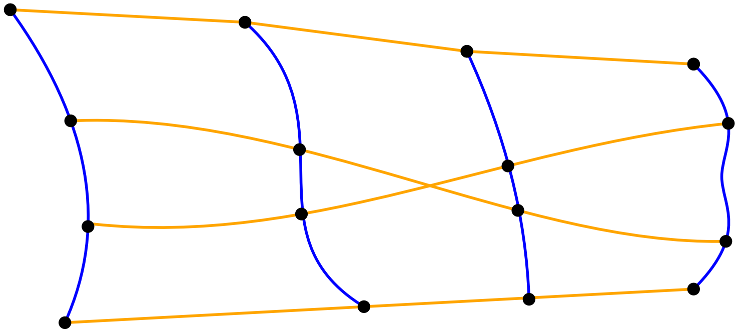
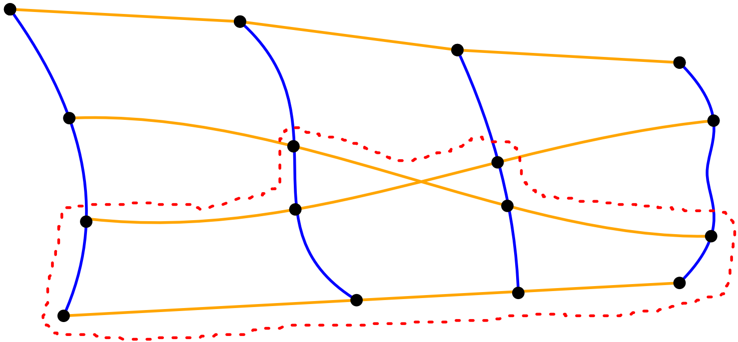
- Instead, we investigate the simpler tangled grid poset.
- Lemma: Tangled grid contains the rotation poset.
- The tangled grid is composed of two *n*-member chain decompositions – m-chains and w-chains – such that every m-chain and w-chain intersect in exactly one poset element.



Downsets in TG



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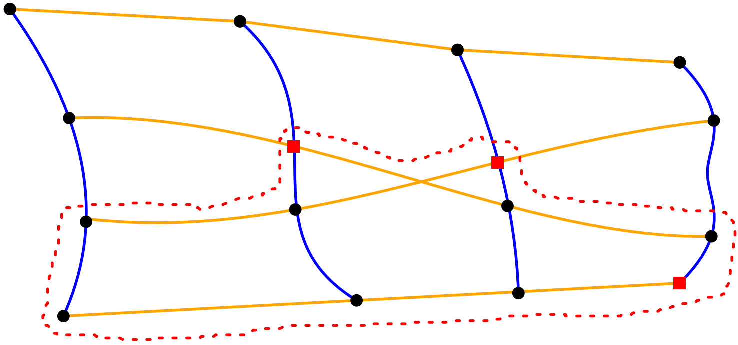
- We can encode a downset D by its maximal intersections with each m-chain or each w-chain.



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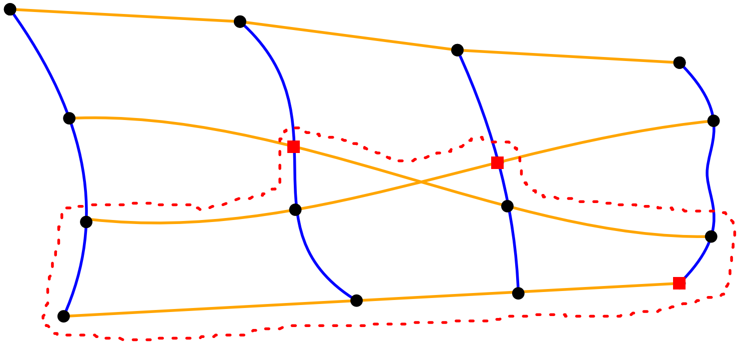

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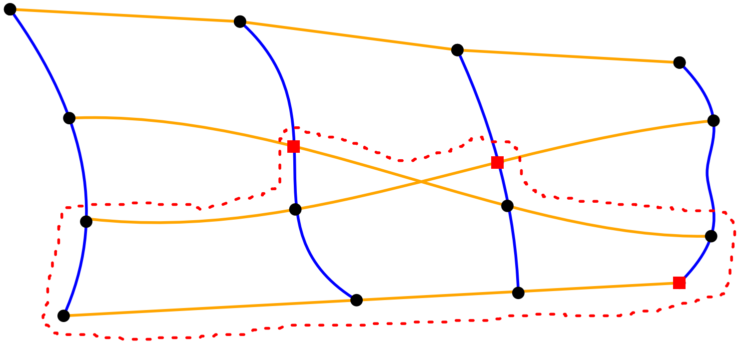
- Trivial bound: $(n+1)^n$. ☹



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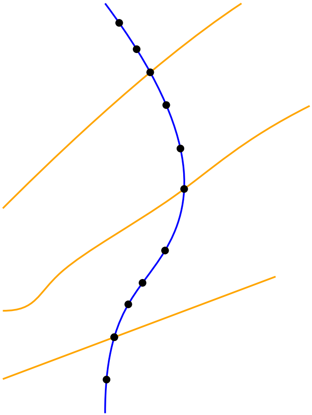
- Randomly order the m-chains and w-chains.



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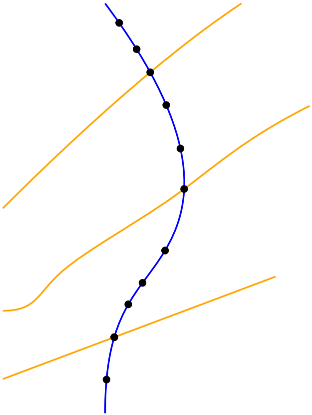

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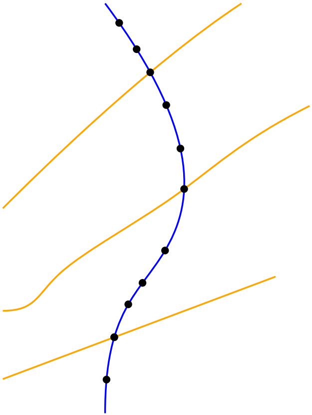

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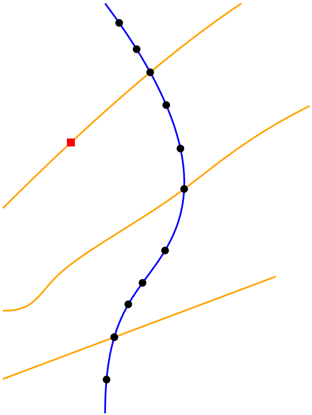

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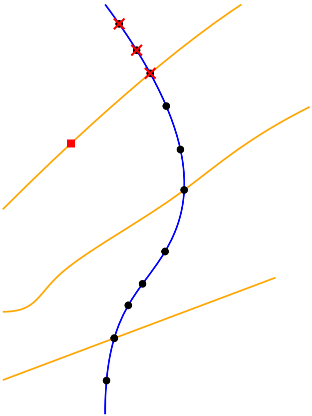

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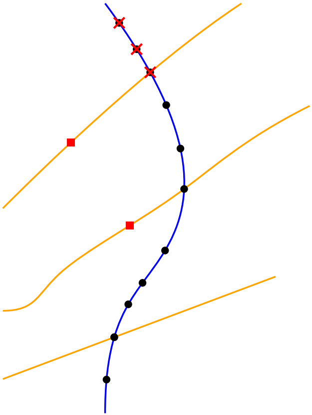

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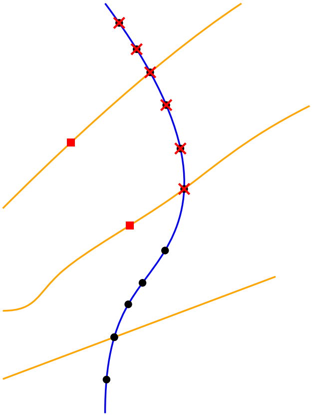

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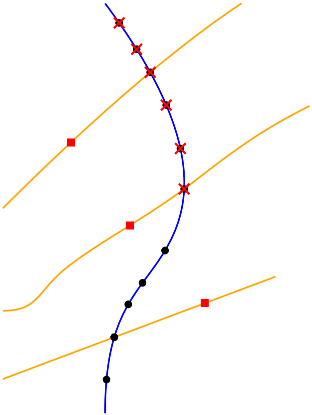

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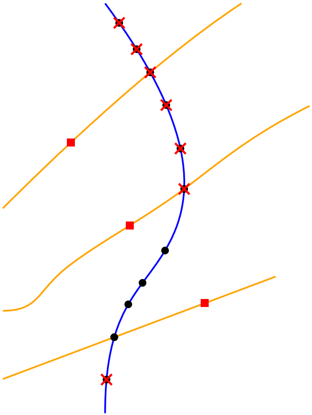

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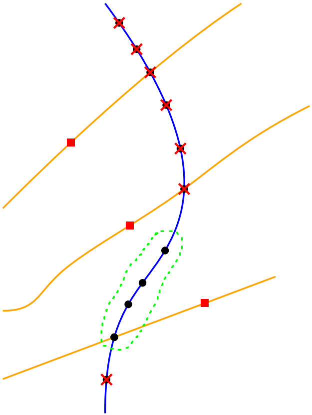
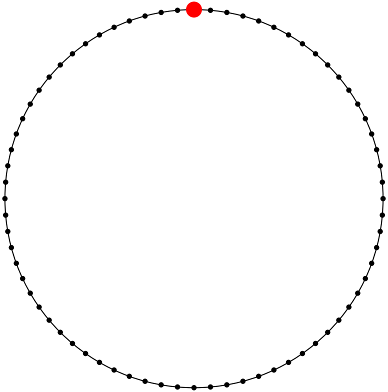


Related Puzzle



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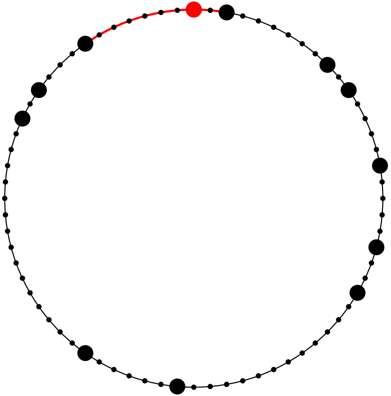

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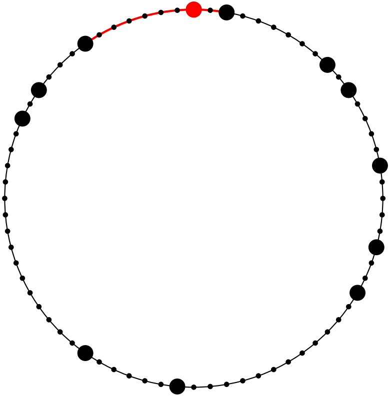

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$$\sum_k k \frac{\binom{n-k}{j-2}}{\binom{n+1}{j}} k$$

First Bound

$$E[\log X] \leq \frac{3}{2n}\log(n+1) + \frac{1}{n}\sum_{j=2}^{n}\sum_{k} k\frac{\binom{n-k}{j-2}}{\binom{n+1}{j}}\log k$$

$$\xrightarrow{n\to\infty} \int_0^1 \sum_k (\log k) k x^2 (1-x)^{k-1}\, dx$$

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$$\#\text{encodings} \le e^{2.4076n} \lesssim 11.11^n.$$

□

Main Lemma

$S \subset \bigtimes_i A_i, \quad X_i(s,\pi) = |\{x_i \mid x \in S, x_j = s_j \text{ } if \text{ } \pi^{-1}(j) < \pi^{-1}(i)\}|$
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Proof 2 [à la Shannon]:

$$\log |S| = H(s) = \sum_{i=1}^{n} H(s_i \mid s_j \ for\ j\ satisfying\ \pi^{-1}(j) < \pi^{-1}(i))$$

$$\le \sum_{i=1}^{n} H(s_i \mid X_i(s,\pi)) = \sum_{i=1}^{n}\sum_{k} Pr_s[X_i(s,\pi) = k] \cdot H(s_i \mid X_i(s,\pi) = k)$$

$$\le \sum_{i=1}^{n}\sum_{k} Pr_s[X_i(s,\pi) = k] \cdot \log X_i(s,\pi) = \sum_{i=1}^{n} E_s[\log X_i(s,\pi)].$$

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