Quantum channels with (quantum) group symmetry

Joint work with Sang-gyun Youn

Hun Hee Lee
Seoul National University

CAHAS 2022

Quantum states, channels and symmetry

- $\mathcal{H}$, $\mathcal{H}_A$, $\mathcal{H}_B$: finite dimensional Hilbert spaces
  - A **(quantum) state** ρ on $\mathcal{H}$ is a positive element in $B(\mathcal{H})$ with $\mathrm{Tr}(\rho) = 1$. We denote by $\mathcal{D}(\mathcal{H})$ the set of all states on $\mathcal{H}$.
  - A **(quantum) channel** $\Phi : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ is a **CPTP** (**completely positive, trace-preserving**) map.
- Conservation of symmetry is a central theme in quantum theory, where **group action** (or **representation**) play an important role.
- For a fixed state ρ on $\mathcal{H}$ the set $\{U \in \mathcal{U}(\mathcal{H}) : U\rho U^* = \rho\}$ form a subgroup G of $\mathcal{U}(\mathcal{H})$, i.e. we get a **unitary representation** $\pi : G \to \mathcal{U}(\mathcal{H})$.
- G: a **compact group** $\Rightarrow$ rich finite dimensional representation theory!

Bipartite states and channels under symmetry

- For bipartite states acting on the **composite** system $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ it is natural to consider "**local**" unitaries of the form $U \otimes V$.
- $\begin{cases} \pi_A : G \to \mathcal{U}(\mathcal{H}_A) \\ \pi_B : G \to \mathcal{U}(\mathcal{H}_B) \end{cases}$ Unitary representations on a compact group G.
- (**Def**) A bipartite state ρ on $\mathcal{H}_{AB}$ is called (π_A, π_B)-**invariant** if

$$[\pi_A(x) \otimes \pi_B(x)]\, \rho\, [\pi_A(x)^* \otimes \pi_B(x)^*] = \rho, \ \ \forall x \in G.$$

  In other words, ρ is invariant under the representation $\pi_A \otimes \pi_B$.
- (**Def**) A channel $\Phi : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ is called (π_A, π_B)-**covariant** if

$$\Phi(\pi_A(x)\, \rho\, \pi_A(x)^*) = \pi_B(x)\Phi(\rho)\pi_B(x)^*, \ \ \forall x \in G, \ \ \forall \rho \in \mathcal{D}(\mathcal{H}_A).$$

- The above concepts can obviously be extended to any operators on $\mathcal{H}_{AB}$ and any linear maps from $B(\mathcal{H}_A)$ into $B(\mathcal{H}_B)$

The CJ-map and the state-channel duality

- (Notations)
  - $\mathcal{L}(A, B)$: linear maps from $B(\mathcal{H}_A)$ into $B(\mathcal{H}_B)$.
  - $CP(A, B)$, $CPTP(A, B)$: CP, CPTP maps from $B(\mathcal{H}_A)$ into $B(\mathcal{H}_B)$.

- (The **CJ-map**) $\mathcal{L}(A, B) \to B(\mathcal{H}_{AB}),\ \Phi \mapsto C_\Phi := \dfrac{1}{d_A} \displaystyle\sum_{i,j=1}^{d_A} e_{ij} \otimes \Phi(e_{ij})$.

- (**Choi-Jamiołkowski**) For $\Phi \in \mathcal{L}(A, B)$

$$\Phi \in CP(A, B) \Leftrightarrow C_\Phi \in B(\mathcal{H}_{AB})_+.$$

- (**State-channel duality**) For $\Phi \in \mathcal{L}(A, B)$

$$\Phi \in CPTP(A, B) \Leftrightarrow C_\Phi \in \mathcal{D}(\mathcal{H}_{AB}) \text{ with } \mathrm{Tr}_B(C_\Phi) = id_A/d_A.$$

- (**Prop**) $\Phi \in \mathcal{L}(A, B)$ is (π_A, π_B)-covariant $\Leftrightarrow$ C_Φ is $(\overline{\pi_A}, \pi_B)$-invariant.
- Recall $\overline{\pi}$ is the **conjugate representation** given by $\overline{\pi}(x) = \pi(x^{-1})^T$, where X^T is the transpose of X.

Questions and previous works

- (**Q**) (1) Can we **characterize** invariant bipartite states/covariant channels? (2) Can we **describe QIT properties** of those states/channels **more effectively**?
- (**Werner, Holevo, Vollbrecht, Keyl,** $\cdots$)
  - U: the fundamental representation (identity map) of $U(n)$
    O: the fundamental representation of $O(n)$
  - (**Werner states/Werner-Holevo channels**)
    - ★ (U, U)-invariant states $\rho_\lambda = \lambda p_0 + (1-\lambda)p_1 \in \mathcal{D}(\mathbb{C}^n \otimes \mathbb{C}^n)$, $0 \le \lambda \le 1$ for some projections p_0, p_1 on $\mathbb{C}^{2n}$. $\Rightarrow$ a **line-segment** (or a **1-simplex**)
    - ★ $(\overline{U}, U)$-covariant channels $\Phi_\lambda : B(\mathbb{C}^n) \to B(\mathbb{C}^n)$ with $C_{\Phi_\lambda} = \rho_\lambda$.
  - (**Isotropic states/depolarizing channels**)
    - ★ $(\overline{U}, U)$-invariant states $\rho_t = tq_0 + (1-t)q_1 \in \mathcal{D}(\mathbb{C}^n \otimes \mathbb{C}^n)$, $0 \le t \le 1$ for some projections q_0, q_1 on $\mathbb{C}^{2n}$. $\Rightarrow$ a **line-segment** (or a **1-simplex**)
    - ★ (U, U)-covariant channels $\Psi_t : B(\mathbb{C}^n) \to B(\mathbb{C}^n)$ with $C_{\Psi_t} = \rho_t$ depolarizing channels.
  - (O, O)-inv. states (resp. (O, O)-cov. channels) form a **2-simplex**.

Previous works: continued

- (**Al Nuwairan '14**, $SU(2)$ **case**)
  $\widehat{SU(2)} = \{\pi_n : n \geq 1, \ \dim \pi_n = n+1\}$.
    - A complete description of extreme points of the convex set of (π_n, π_m)-covariant channels.
    - Made connection to invariant states by Vollbrecht-Werner.
    - Serious representation theory beyond the fundamental ones is used.
- (**Datta et al., '17, finite group case**)
    - Finite group cases (e.g. S_n, $n = 3, 4$) have been examined.
    - **Multiplicity free condition** of tensor decomposition is used.
- (**Def**) We say that the irreducible decomposition $\pi_A \otimes \pi_B \cong \pi_1 \oplus \cdots \oplus \pi_n$ for two finite dimensional representations π_A, π_B of G is called **multiplicity free** if π_j and π_k are not equivalent for any $j \neq k$.

Main results and the Clebsch-Gordan channels

- (**Thm**) Let π_A, π_B be **irreducible** unitary representations of a cpt group G s.t. $\overline{\pi_A} \otimes \pi_B$ is **multiplicity-free**. Then, the set of (π_A, π_B)-covariant channels is a **simplex** whose **extreme points** are exactly the **Clebsch-Gordan channels**.
- For $\overline{\pi_A} \otimes \pi_B \cong \pi_1 \oplus \cdots \oplus \pi_n$
  $\Rightarrow \pi_A$ is a subrepresentation of $\pi_B \otimes \overline{\pi_j}$ for each $1 \leq j \leq n$
  $\Rightarrow$ there is an isometry $V_j : \mathcal{H}_A \to \mathcal{H}_B \otimes \overline{\mathcal{H}_j}$ such that

$$V_j^* [\pi_B(x) \otimes \overline{\pi_j}(x)] V_j = \pi_A(x),\ x \in G.$$

- (**Def**) The channel $\Phi_j \in CPTP(A, B)$ with V_j as the **Stinespring isometry**, i.e.

$$\Phi_j(X) := (Id \otimes \mathrm{Tr})(V_j X V_j^*),\ X \in B(\mathcal{H}_A)$$

  is called the **Clebsch-Gordan channels** (shortly, **CG-channels**).
- (**Brannan/Collins/L./Youn, CMP ’20**) $SU(2)$-CG-channels are not “coming from” known examples including quantum erasure, amplitude damping, dephasing and depolarizing channels.

Remarks on the proof

- (Step 1) We first focus on the $(\overline{\pi_A}, \pi_B)$-invariant states acting on $\mathcal{H}_{AB}$ and observe that they form a simplex with extreme points being orthogonal projections onto the subspaces appearing in the decomposition of $\overline{\pi_A} \otimes \pi_B$.
- (Step 2) We check that the mentioned projections are exactly the images of CG-channels through the CJ-map.
- (Our contributions) We highlighted the role of the CJ-map and the CG-channels.

When do we get the multiplicity-free condition?

- ($SU(2)$ case) Recall $\widehat{SU(2)} = \{\pi_n : n \geq 1,\ \dim \pi_n = n+1\}$.

$$\pi_n \otimes \pi_m \cong \pi_{n+m} \oplus \cdots \oplus \pi_{|n-m|}$$

  is multiplicity-free! Note $\overline{\pi_n} \cong \pi_n$.

- ($U(n)$, $O(n)$ cases) $\overline{U} \otimes U$, $U \otimes U$, $O \otimes O$ are multiplicity free. Note $\overline{O} = O$.
- (S_n case) Recall that the the fundamental representation of S_n, i.e. permutation matrices decomposes into $(n) \oplus (n-1,1)$. The $(n-1)$-dimensional component $(n-1,1)$ will be denoted by V.

$$V \otimes V \cong (n) \oplus (n-1,1) \oplus (n-2,2) \oplus (n-2,1,1)$$

  is multiplicity free. Note $\overline{V} = V$.
  - Thus, (V, V)-covariant channels form a 3-simplex.
  - When we can specify the intertwining isometries, we can write down the corresponding CG-channels. For example, $n = 4$ case has been examined in the paper [LY].

The case of projective representations

- All the above results can be extended to projective representations.
- Recall that $\sigma : G \to \mathbb{T}$ is called a **2-cocycle** if

$$\sigma(s,t)\sigma(st,u) = \sigma(s,tu)\sigma(t,u),\ \sigma(s,e) = \sigma(e,t) = 1,\ s,t,u \in G.$$

We say a strongly continuous map $\pi : G \to \mathcal{U}(\mathcal{H})$ is a **projective representation w.r.t. σ (shortly, σ-rep.)** if

$$\pi(st) = \pi(s)\pi(t)\sigma(s,t),\ s,t \in G.$$

- F: finite abelian group, $G := F \times \widehat{F}$
  - $\sigma((x,\gamma),(y,\delta)) := \gamma(y)$.
  - The unique σ-rep. $W : G \to \mathcal{U}(\ell^2(F))$ is given by $W(x,\gamma) := M_\gamma T_x$, where $T_x f(y) = f(y-x)$, $M_\gamma f(y) = \gamma(y) f(y)$, $f \in \ell^2(F)$.
- $\overline{W} \otimes W$: multiplicity-free with the associated CG-channels $\mathrm{Ad}_{W(s)}$, $s \in G$.
- (W, W)-covariant channels for $G = \mathbb{Z}_d$, $d \geq 1$ are called the **Weyl covariant channels** in the literature.

Applications (in progress): Degradability

- A channel Φ with its **Stinespring representation** $\Phi(X) = id \otimes \mathrm{Tr}(VXV^*)$ has a **complementary channel** Φ^c given by $\Phi^c(X) = \mathrm{Tr} \otimes id(VXV^*)$. We say that Φ is **degradable** if there is another channel Ψ such that $\Psi \circ \Phi = \Phi^c$.
- (**Remarks**)
  - When a (π_A, π_B)-cov. channel Φ is degradable, we can take Ψ with a suitable covariance (with the multiplicity free assumption)!
  - Covariance is well-preserved by composition.
- (**Thm**) π_2: irreducible $SU(2)$-representation with dim$= 3$. There are only two degradable channels among (π_2, π_2)-covariant channels.
- We use composition rules of CG-channels and the above remarks for a systematic approach.

Applications (in progress): EBT and PPT

- Recall that a channel Φ is called **EBT** (entanglement breaking) and **PPT** if C_Φ is separable and PPT, respectively.
- (**Recall**)
  - Φ is EBT $\Leftrightarrow$ $\Psi \circ \Phi$ is CP for any positive Ψ
  - Φ is PPT $\Leftrightarrow$ $T \circ \Phi$ is CP, where T is the transpose map.
- (**Rem**)
  - Covariance is actually a property of a linear map and positive covariance maps have similar structure (with the multiplicity free assumption)
  - Sometimes taking a unitary conjugate after T can be covariant.
- (**Thm**) EBT $=$ PPT among (π_2, π_2)-covariant channels.
- We use, again, composition rules of CG-channels and the above remarks for a systematic approach.

Quantum groups

- All the results in the above are true (except the projective representation case) for compact quantum groups of Kac-type.
- For non-Kac type quantum groups (e.g. $SU_q(2)$) we can adapt the **Heisenberg picture**, i.e. **UCP maps** for quantum channels. The Clebsch-Gordan maps still play an important role, but we need to use quantum trace instead of the usual trace.
- The usual Schrödinger picture is not suitable for non-Kac case since we can show that $SU_q(2)$-covariant channels are rarely TP.

Thank you for your attention.