An overview of Explicit Number Theory

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L-functions in Analytic Number Theory

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A 4-step guide to solve a conjecture

Conjecture (Goldbach's Ternary Conjecture)

Every odd integer $n \geq 3$ is the sum of at most 3 primes.

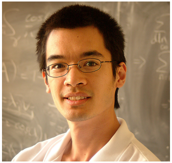
1. A "Neighbour" Result:

Theorem (Shnirel'man -1930)

There exists a constant C, s.t. every integer $n \geq 2$ is the sum of at most C prime numbers.

Authors	C
Shnirel'man (1930)	800 000
Ramaré (1995)	7
Tao (2013)	5

O. Ramaré



T. Tao

2. Partial Answer:

Theorem

Every odd integer $n \leq N$ *can be written as the sum of at most* 3 *primes.*

Authors	N
Ramaré and Saouter (2002)	$1.13 \cdot 10^{22}$
Oliveira e Silva et. al. (2013)	$8.37 \cdot 10^{26}$
Helfgott (2013)	$1.23 \cdot 10^{27}$
K., Lumley (2013)	$7.86 \cdot 10^{27}$
Helfgott, Platt (2013)	$8.87 \cdot 10^{30}$

Oliveira e Silva (Aveiro)

Consequence of explicit short intervals containing primes:

Theorem (K., Lumley - 2013)

Let $x \geq e^{60}$*. Then there exists at least one prime* p *between* $(1 - \frac{1}{\Delta})x$ *and* x*, with* $\Delta = 966\,090\,061$*.*

3. Asymptotic Answer:

Theorem (Vinogradov - 1937)

Every sufficiently large odd integer n can be written as the sum of at most 3 primes.

4. Explicit Answer:

Theorem

The conjecture is proven for every odd $n \geq N'$:

Authors	N'
Liu and Wang (2002)	$10^{1\,350}$
K. (2008, unpub.)	10^{850}
Helfgott (Arxiv 2013)	$8.87 \cdot 10^{30}$

The Ternary Goldbach conjecture is proven:
partial result N = explicit $N' = 8.87 \cdot 10^{30}$.

Observation leading to conjecture:

C. F. Gauss

Counting by intervals of 1000:

I soon recognized that behind all of its fluctuations, this frequency is on the average inversely proportional to the logarithm, so that the number of primes below a given bound n is approximately equal to $\int \frac{dn}{\log(n)}$.

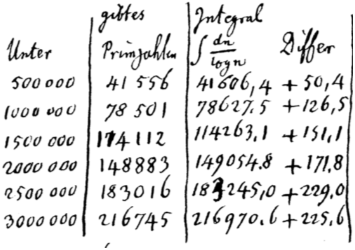
Letter to J. Encke (Dec. 24, 1849)

Gauss Conjecture[1]

$$\pi(x) \sim \int_2^x \frac{dt}{\log t} := \mathrm{Li}(x).$$

Prediction of the error term:

$$\pi(x) - \mathrm{Li}(x) \ll x^{1/2+\epsilon}.$$



Unter	gezählte Primzahlen	Integral $\int \frac{dn}{\log n}$	Differ.
500 000	41 556	41 606,4	+ 50,4
1000 000	78 501	78627,5	+ 126,5
1500 000	114 112	114263,1	+ 151,1
2000 000	148883	149054,8	+ 171,8
2500 000	183016	183245,0	+ 229,0
3000 000	216745	216970,6	+ 225,6

[1] $\pi(x)$ is the number of primes no larger than x, and $\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t}$.

Bridges between number theory and analysis

Dirichlet (1805-1859)

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s},$$

Multiplicative characters χ mod q, where a and q are coprime.

$L(1,\chi)\neq 0 \implies \infty$ many primes of the form $a+nq$.

Riemann (1826-1866)

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_{p\ \text{prime}}\left(1-p^{-s}\right)^{-1}.$$

for $\Re s>1$.

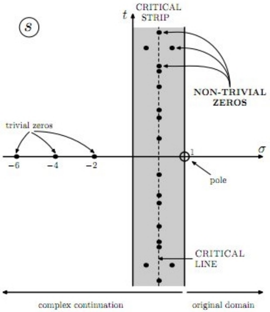
Explicit Formula:

$$\pi(x)=R(x)-\sum_{\varrho}R(x^{\varrho}),$$

with $R(x)=1+\sum_{k\geq 1}\frac{(\log x)^k}{k!k\zeta(k+1)}$.

Bridges between conjectures on the zeros of Zeta and the primes

- Dirichlet series:
$\zeta(s) = \sum_{n\geq 1} n^{-s}$, for $\Re s > 1$.
- Euler product:
$\zeta(s) = \prod_p (1-p^{-s})^{-1}$.
- Functional equation:
$\zeta(s) = \Lambda(1-s)\zeta(1-s)$,
with $\Lambda(s) = \pi^{\frac{1}{2}-s}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1}{2}(1-s))}$.



Conjecture (Riemann Hypothesis (RH) - 1859)

If $\beta + i\gamma$ is a zero of the zeta function with $0 \leq \beta \leq 1$, then $\beta = \frac{1}{2}$.

The Prime Number Theorem (PNT)

Theorem (Hadamard, de la Vallée Poussin - 1896)

$\psi(x) \sim x$ as $x \to \infty$.

Equivalently $\theta(x) \sim x$

and $\pi(x) \sim Li(x) = \int_2^x \frac{dt}{\log t}$.

Hadamard (1865-1963)

de la Vallée Poussin (1866-1962)

Le plus court chemin entre deux vérités dans le domaine réel passe par le domaine complexe. (Hadamard)

The true error term in the Prime Number Theorem

Under Riemann Hypothesis (RH)

Theorem (Von Koch -1901)

RH $\iff \pi(x) - \mathrm{Li}(x) \ll x^{\frac{1}{2}}(\log x)$.

Theorem (Rosser, Schoenfeld - 1976)

RH $\implies |\pi(x) - \mathrm{Li}(x)| \leq \frac{1}{\sqrt{8\pi}}\sqrt{x}(\log x)$ *for all* $x > 2657$.

Numerical Investigation of PNT

Theorem (Büthe - 2018)

For all $x \leq 10^{19}$,

- $|\psi(x) - x| \leq 0.94\sqrt{x}$.
- $0 < \mathrm{Li}(x) - \pi(x) \leq \frac{\sqrt{x}}{\log x}\Big(1.95 + \frac{3.9}{\log x} + \frac{19.5}{(\log x)^2}\Big)$.

(Skewes's number $\leq 1.4 \cdot 10^{316}$*.)*

J. Büthe

Various shapes of explicit bounds for $E(x)$

$E(x) \leq$	$x >$	Authors
$0.026(\log x)^{1.801} e^{-0.185 \frac{(\log x)^{3/5}}{(\log\log x)^{1/5}}}$	23	Johnston-Yang (2022^+)
$23.13(\log x)^{1.503} e^{-0.865\sqrt{\log x}}$	$e^{100\,000}$	Johnston-Yang (2022^+)
$9(\log x)^{1.5} e^{-0.847\sqrt{\log x}}$	2	Fiori-K-Swidinsky (2022^+)
$1.93 \cdot 10^{-12}$	$e^{2\,000}$	Büthe (2018)
$1.34 \cdot 10^{-30}$	$e^{10\,000}$	Fiori-K-Swidinsky (2022^+)
$\frac{1.08 \cdot 10^{-16}}{\log x}$	e^{35}	BKLNW (2021)

A. Yang (UNSW)

S. Broadbent, K. Wilk, J. Swindisky (UofL)

Ingredients of the proof for the PNT

PNT: $\psi(x) := \sum_{p^k \le x} \log p \sim x$, with error term $E(x) := \left|\frac{\psi(x)-x}{x}\right| = o(1)$.

1. An Explicit Formula for ψ, depending on some Mellin transform F:

$$E(x) \le \delta_f + \mathcal{S} + \mathcal{O}(x^{-1}), \text{ with } \mathcal{S} = \sum_{\varrho} |F(\varrho)|\, x^{\beta-1}.$$

   Control the smooth weight f so as to have δ_f and $|F(\varrho)|$ as small as possible.

2. Have a bound as small as possible for β.
   The largest value for β ("zero-free region") is $\beta = 1/2$ under RH, but the key to prove PNT is just $\beta < 1$.

3. Control the sum over zeros in regions of the critical strip, e.g. in rectangles $\sigma_0 < \Re s < \sigma_1$ and $T_0 < |\Im s| < T_1$ ("zero density").
   Classically: bound for $N(T)$ $(\sigma_0 = 0, \sigma_1 = 1, T_0 = 0, T_1 = T)$.

An Explicit Formula for ψ: Comparing it to its average

Rosser, Schoenfeld (1941-1976)[2]
They generalize and make explicit de la Vallée Poussin's proof by comparing $\psi(x)$ to its m^{th}-average in $(x, x+\delta)$
So, F is fixed, depending on m, δ: $\delta_f = \frac{\delta}{2}, |F(\varrho)| \leq \frac{c_m}{\delta^m |\gamma|^{m+1}}$, with $c_m \leq (2m)^m$.


Rosser (1907-1989)


Schoenfeld (1920-2002)

Idea is used for

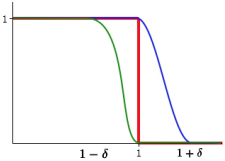
- primes (Dusart, 1998-2016),
- primes in arithmetic progressions (McCurley - 1986, Bennett et.al - 2018).

[2] 1962 article has 33 theorems and over 2000 citations

An Explicit Formula for a "smoothed" ψ_ϕ

- **Faber & K (2014)**
  Choice of family of smoothing weights $f_{\delta,m}$ s.t. $|F(\varrho)| \le \frac{c_m}{\delta^m |\gamma|^{m+1}}$, c_m is close to minimum: $c_m = \frac{\sqrt{(2m)!(2m+1)!}}{m!}$.

  $\psi_f \le \psi \le \psi_f$

  L. Faber (UofL)

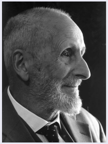
  - Method works better than Rosser and Schoenfeld for smaller x ($\le e^{1000}$) or smaller verification of RH.
  - Method also used by Lumley (MSc. 2015): primes in arithmetic progressions,
    and Das (MSc. 2021): primes in Chebotarev density theorems.
- **Büthe (2016)** uses the Logan function for F, it has a sharp cutoff filter kernel, $\mathcal{S} \simeq \tilde{c}_{F,H_0} \sum_{|\Im\varrho|<H_0} \frac{x^{-1/2}}{|\gamma|}$.
  Method gets better than FK for x smaller ($x \le e^{2500}$).

Ingredient 1: Relating primes to the zeros of zeta

Theorem (Truncated Perron formula (1908))

For $T \geq (\log x)^2$,

$$E(x) = \sum_{|\gamma| \leq T} \frac{x^{\varrho-1}}{\varrho} + \mathcal{O}\left(\frac{(\log x)(\log T)}{T}\right).$$



O. Perron (1880-1975)

Theorem (Explicit Truncated Perron formula)

- *Wolke* (1893)*, Ramaré* $(2016, 2022^{+})$
- *Goldston* (201)*,*
- *Dudek* (2016)*,*

Cully-Hugill and Johnston (2022^{+})*:*
for $x > e^{40}$ *and* $\max(51, \log x) < T < \frac{\sqrt{x}-2}{5}$,

$$E(x) = \sum_{|\gamma| \leq T^{\star}} \frac{x^{\varrho-1}}{\varrho} + \mathcal{O}^{\star}\left(\frac{2.091(\log x)}{T}\right).$$

D. Johnston (UNSW)

Methods used by Platt and Trudgian (2021), Johnston and Yang (2022^{+})

Explicit formula via Perron

Controlling the largest value for β

- Let $H_0 > 0$ s.t. RH is verified for $0 < \Re s < 1$ and $|\Im s| < H_0$.
- $\zeta(s)$ does not vanish in $\Re s > 1 - \frac{1}{f(\log|\Im s|)}$.
- Fix $1/2 < \sigma_0 < 1$ and split the sum at σ_0.

$$\mathcal{S} = \sum_{|\Im \varrho| \le H_0} |F(\varrho)|\, x^{-1/2} + \sum_{\substack{|\Im \varrho| > H_0 \\ \Re \varrho < \sigma_0}} |F(\varrho)|\, x^{\sigma_0 - 1} + \sum_{\substack{|\Im \varrho| > H_0 \\ \Re \varrho > \sigma_0}} |F(\varrho)|\, x^{-\frac{1}{f(\log|\gamma|)}}$$

This was used in K-Faber, Platt-Trudgian, Johnston-Yang. Additional "slicing" (horizontal and vertical) were introduced in Fiori-K.-Swindisky.

Controlling the number of zeros in rectangles $\sigma_0 < \Re s < \sigma_1$ and $T_0 < |\Im s| < T_1$ ("zero density"):

- bound for $N(T)$ $(\sigma_0 = 0, \sigma_1 = 1, T_0 = 0, T_1 = T)$.
- bound for $N(\sigma, T), N(\sigma_0, \sigma_1, T)$.

Partial verification of the Riemann Hypothesis

Siegel (1932): Riemann calculated the first zeros, using the "Riemann-Siegel formula":
$1/2 + i\,14.134725\ldots$, $1/2 + i\,21.022040\ldots$, $1/2 + i\,25.010858\ldots$

Theorem

RH is verified for the zeros satisfying $|\gamma| < H_0$:

Authors	H_0
Wedeniwski (2003)[a]	$2.41 \cdot 10^{11}$
Gourdon (2004)[b]	$2.44 \cdot 10^{12}$
Platt (2017)[c]	$3.06 \cdot 10^{10}$
Platt and Trudgian (2021)	$\mathbf{3 \cdot 10^{12}}$

D. Platt (Bristol)

[a] "ZetaGrid-computational verification of the Riemann hypothesis", Conference in Number Theory in Honour of Prof. Hugh Williams, Banff

[b] unpublished

[c] Rigorous calculations, uses interval arithmetics.

Zero-free regions for $\zeta(s)$

Theorem (Zero-free region)

$\zeta(\sigma + it) \neq 0$ *if* $\sigma > 1 - \frac{1}{f(t)}$ *for all* $|t| \geq 2$.

Author	$f(t)$
de la Vallée Poussin (1899)	$R\,(\log t)$
Stechkin (1970)	9.65
Ford (2000)	8.43
K. (2005)	5.71
Mossinghoff, Trudgian (2015)	5.58 *(for* $t \leq e^{10\,000}$*)*
Korobov-Vinogradov (1958)	$r\,(\log t)^{2/3}(\log\log t)^{1/3}$
Ford (2000)	57.54 *(for* $t \geq e^{10\,000}$*)*

K. Ford (UIUC)

Counting zeros

$N(T)$ = number of zeros $\rho = \beta + i\gamma$ with $0 < \beta < 1, |\gamma| < T$.

Theorem (Explicit bounds for $N(T)$)

$$\left|N(T) - \frac{T}{2\pi}\log\frac{T}{2\pi e} - \frac{7}{8}\right| \le a\log T + b\log\log T + c, \quad \textit{for all } T \ge e.$$

Authors	a	b	c
Rosser (1941)	0.137	0.443	2.463
Trudgian (2014)	0.112	0.278	3.385
Hasanalizade, Shen, Wong (2021)	0.104	0.258	9.368

T. Trudgian (UNSW)

Q. Shen (Shandong), E. Hasanalizade (UofL)

Counting the zeros close to the 1-line

Zero-density of the zeros of zeta:
Let $1/2 \leq \sigma < 1$ be fixed. $N(\sigma, T)$ counts the number of zeros $\rho = \beta + i\gamma$ in the rectangle $\sigma < \beta < 1$ and $|\gamma| \leq T$.

Theorem (Bohr-Landau's zero density - 1914)

For every $\sigma > \frac{1}{2}$,
$N(\sigma, T) \ll \frac{T}{\sigma - 1/2}$.


Bohr (1887-1951)


Landau (1877-1938)

Explicit version: $N(\sigma, T) \leq a_\sigma T + b_\sigma \log T + c_\sigma$.

Theorem (K - 2012)

$$N(0.90, T) \leq 0.45T + 0.65 \log T - 3.63 \cdot 10^5.$$

Used for Faber-K bounds for error term in PNT.
Proof uses bound for the **second moment of zeta on** $1/2$*-line.*

Zeta on the $1/2$-line, zero-density, and prime gap

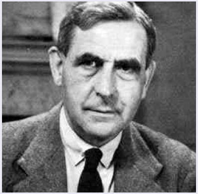
Let $\sigma \geq 1/2$.

Theorem (Ingham - 1937)

If $\exists c > 0$ *s.t.* $|\zeta(1/2+it)| \ll t^{c}$,

then $N(\sigma, T) \ll T^{(2+4c)(1-\sigma)+\epsilon}$
uniformly for all $1/2 \leq \sigma \leq \frac{1}{2}$,

and then $p_{n+1} - p_n \ll p_n^{\frac{1+4c}{2+4c}+\epsilon}$.



A. Ingham (1900-1967)

RH

$\Longrightarrow$ **Lindelöf Hypothesis** $\zeta(1/2+it) \ll t^{\epsilon}$
$\Longrightarrow$ **Density Hypothesis** $N(\sigma, T) \ll T^{2(1-\sigma)+\epsilon}$
$\Longrightarrow$ **Prime gap** $p_{n+1} - p_n \ll p_n^{\frac{1}{2}+\epsilon}$.

Bounds for zeta on the $1/2$-line

- Lindelöf Hypothesis: $\zeta(1/2+it) \ll t^{\epsilon}$
- Lindelöf (1908): $\zeta(1/2+it) \ll t^{\frac{1}{4}+\epsilon}$
- Hardy-Littlewood (1921):

$$|\zeta(1/2+it)| \ll t^{\frac{1}{6}+\epsilon}.$$

- Bourgain (2014):

$$|\zeta(1/2+it)| \ll t^{\frac{53}{342}+\epsilon}.$$

- Hiary (2016): for all $3 \le t \le 200$,

$$|\zeta(1/2+it)| \le 0.595 t^{\frac{1}{6}} (\log t).$$

Theorem (Hiary, Patel and Wang -2022^{+})

For all $t \ge 3$,
$|\zeta(1/2+it)| \le 0.618 t^{\frac{1}{6}} (\log t)$.

G. Hiary (Ohio SU)

Zero density for zeta close to 1-line

- Density Hypothesis: $N(\sigma, T) \ll T^{2(1-\sigma)+\epsilon}$
- Ingham zero-density theorem (1921):
  $N(\sigma, T) \ll T^{\frac{8}{3}(1-\sigma)}(\log T)^5$ with $8/3 = 2.666\ldots$.
- Bourgain (2017): $2.619\ldots$.

Theorem

For all $T \geq 2$ and $\sigma \geq 0.52$, $N(\sigma, T) \leq \mathcal{C}_1(\sigma)\, T^{\frac{8}{3}(1-\sigma)}(\log T)^{5-2\sigma} + \mathcal{C}_2(\sigma)\, (\log T)^2$.

$\mathcal{C}_1(0.90)$	$\mathcal{C}_2(0.90)$	Author
$1\,294$	52	Ramaré (2016)
11.5	3.2	K., Lumley, Ng (2018)

A. Lumley (York)

N. Ng (UofL)

Used in Platt-Trudgian, Johnston-Yang, Fiori-K.-Swindisky. Note: Choice of weight improved Ingham and Ramaré.

Second moment of zeta inside the critical strip

Let $1/2 < \sigma < 1$.

$$I(\sigma, T) := \int_1^T |\zeta(\sigma + it)|^2 dt$$

Theorem (Landau and Schnee - 1909)

$I(\sigma, T) \sim \zeta(2\sigma)T$

Theorem (K -2012)

$$I(\sigma, T) \leq \zeta(2\sigma)T + \tfrac{2}{1-\sigma}T^{2-2\sigma}(\log T + \tfrac{2}{1-\sigma})$$

Theorem (Dona, Helfgott and Zuniga Alterman -2022)

$$I(\sigma, T) \leq \zeta(2\sigma)T + \frac{5.22}{(\sigma-\frac{1}{2})(1-\sigma)^2}\max(T^{2-2\sigma}(\log T), \sqrt{T}).$$

Second moment of zeta on the critical line

$I(1/2, T) \leq T \log T - (1 + \log(2\pi) - 2\gamma)T + E(T)$, with $E(T) \ll T^{\theta+\epsilon}$.
$\theta = \frac{1}{4}$ (Conjecture)
$\theta \leq \frac{1}{2}$ (Ingham, Titchmarsh, Balasubramanian),
$\theta < \frac{1}{3}$ (Jutila, Motohashi, Watt - 2010: $\theta = \frac{131}{416}$)

Explicit bounds for $E(T)$:

Theorem (Simonič - 2019)

$E(T) \leq 70.26 T^{3/4} \sqrt{\log \frac{T}{2\pi}}$
with application to zero density $N(\sigma, T)$ for $0.5 \leq \sigma \leq 0.569$.

A. Simonič (UNSW)

Theorem (Dona, Zuniga Alterman - 2022^{+})

$E(T) \leq 18.169\sqrt{T}(\log T)^2$.

A parenthesis about Dirichlet Divisor Problem [3]

$$\Delta(x) = \sum_{d \leq x} d(n) - (x(\log x) + (2\gamma - 1)x) \, .$$

$\Delta(x) \ll \sqrt{x}$ (Dirichlet - 1849)
$\Delta(x) \ll x^{\frac{1}{3}}(\log x)$ (Voronoï - 1903)
$\Delta(x) \ll x^{\frac{1}{4}+\epsilon}$ (Divisor Conjecture)
$\Delta(x) \ll x^{\frac{131}{416}+\epsilon}$ (Huxley - 2003)

Theorem

$\Delta(x) \leq 0.764x^{1/3}(\log x)$ *for all* $x \geq 5$
by Berkane, Bordellès, and Ramaré (2012)

O. Bordellès

[3] Subconvexity for the Riemann zeta-function and the divisor problem, by Huxley and Ivič

The Prime Number Theorem in Arithmetic Progressions

Theorem (Ramaré and Rumely - 1996)

For a large collection of moduli $q \leq q_0 = 432$, for $x_0 > 0$, there exists $\epsilon_{q,x_0} > 0$ s.t, for all $x \geq x_0$,

$$\left| \frac{\psi(x;q,a) - x/\varphi(q)}{x/\varphi(q)} \right| \leq \epsilon_{q,x_0}.$$

Sample values for ϵ_{q,x_0}:

(q, x_0)	Ramaré & Rumely (1996)	K. & Lumley (2014, unpub)
$(100, 10^{30})$	$9.64 \cdot 10^{-3}$	$2.23 \cdot 10^{-5}$
$(400\,000, 10^{30})$	-	$6.56 \cdot 10^{-2}$

Theorem (Bennett, Martin, O'Bryant, Rechnitzer - 2018)

Let $q \geq 10^5$. Then for all $x \geq \exp(4R(\log q)^2)$

$$\left| \frac{\psi(x;q,a) - x/\varphi(q)}{x/\varphi(q)} \right| \leq \frac{1.02}{\phi(q)} x^{\beta_0} + 1.457x\sqrt{\frac{\log x}{R}} \exp\left(-\sqrt{\frac{\log x}{R}}\right),$$

where β_0 term is present only if some Dirichlet L-function $(\bmod q)$ has an exceptional zero β_0, R constant from zero-free region for Dirichlet L-functions.

Chebotarev's Density Theorem

- K be a number field, $K \neq \mathbb{Q}$, and L/K be a finite Galois extension,
  $G = Gal(L/K)$,
  $n_L = [L : \mathbb{Q}]$, $d_L = |disc(L/\mathbb{Q})|$.
- Let $\mathfrak{p}$ be an unramified prime of K, $\left[\frac{L/K}{\mathfrak{p}}\right]$ its associated Artin symbol, and $N_{K/\mathbb{Q}}\mathfrak{p}$ the absolute norm of K.
- For C a fixed conjugacy class in G,

$$\mathscr{P}_C = \left\{ \mathfrak{p} \mid \mathfrak{p} \text{ prime in } K, \text{ unramified in } L, \text{ of degree one}, \left[\frac{L/K}{\mathfrak{p}}\right] = C \right\}.$$

$$\pi_C(x) = \#\left\{ \mathfrak{p} \in \mathscr{P}_C : N_{K/\mathbb{Q}}\mathfrak{p} \le x \right\}.$$

Theorem (Chebotarev's Density Theorem - 1922)

$$\pi_C(x) \sim \frac{|C|}{|G|} Li(x) \text{ as } x \to \infty.$$

Lagarias and Odlyzko's effective versions of Chebotarev density theorem

CDT: $\psi_C(x) \sim \frac{|C|}{|G|} x$ as $x \to \infty$.

Theorem (Lagarias and Odlyzko - 1977)

There exists $c_1, c_2 > 0$ constants s.t. if $x \geq \exp(10 n_L (\log d_L)^2)$, then

$$\left| \psi_C(x) - \frac{|C|}{|G|} x \right| \leq \frac{|C|}{|G|} x^{\beta_0} + c_1 x \exp\left(-c_2 \sqrt{\frac{\log x}{n_L}} \right),$$

where β_0 term is present only if Dedekind zeta function ζ_L has an exceptional zero β_0.

	$\log x \geq$	error term
Winckler (PhD 2018)	$1545 n_L (\log d_L)^2$	$1.5 \cdot 10^{12} \exp\left(-0.01435 \sqrt{\frac{\log x}{n_L}}\right)$
Das (MSc 2021)	$\frac{73925}{n_L} (\log d_L)^2$	$0.22 (\log x)^{3/2} \exp\left(-0.25 \sqrt{\frac{\log x}{n_L}}\right)$

Conditional results

Theorem (Grenié, Molteni - 2019)

Let $x \geq 1$. Assuming GRH, we have

$$\left|\psi_C(x) - \frac{|C|}{|G|}x\right| \leq \frac{|C|}{|G|}\sqrt{x}\left[\left(\frac{\log x}{2\pi} + 2\right)\log d_L + \left(\frac{(\log x)^2}{8\pi} + 2\right)n_L\right],$$

Theorem (Ernvall-Hytönen, Palojärvi - 2021)

Let $q \geq 3$ and $x \geq q$. Assuming GRH, we have

$$\left|\pi(x;q,a) - \frac{li(x)}{\phi(q)}\right| \leq \left(\frac{1}{8\pi\phi(q)} + \frac{1}{6\pi}\right)\sqrt{x}\log x + (0.184\log q + 12969.946)\sqrt{x} - 237.936.$$

A-M. Ernvall-Hytönen (Helsinky)

N. Palojärvi (Helsinky)

The least prime $P(a, q)$ in the arithmetic progression $a \bmod q$

- Theorem (Chowla - 1934): **GRH** $\implies P(a, q) \ll_\epsilon q^{2+\epsilon}$.
- Conjecture (Heath-Brown - 1978): $P(a, q) \ll q(\log q)^2$.
- Numerical Verification (Wagsatff - 1979): for all moduli up to $5 \cdot 10^4$. Supports the heuristic $P(a, q) \sim \phi(q)(\log q)(\log \phi(q))$.

Assuming that q is sufficiently large:

Theorem (Linnik - 1944)

There exists an absolute constant $A > 0$, s.t. $P(a, q) \ll q^A$.

Author	A
Pan - 1957	10 000
Jutila - 1977	80
Heath-Brown - 1992	5.5
Xylouris - 2009	5.2


R. Heath-Brown (Oxford)

For all moduli q:
(K - 2008) $P(a, q) \le eq^{7(\log q)}$.

How big do we expect the least prime ideal ideal in CDT to be?

Theorem (Lagarias, Odlyzko - 1977)

Assuming GRH, there exists $\mathfrak{p} \in \mathscr{P}_C$ *s.t.*
$N\mathfrak{p} \ll (\log d_L)^2 (\log\log d_L)^4$ *for* d_L *sufficiently large.*

Theorem (Bach, Sorenson - 1996)

Assuming GRH, there exists $\mathfrak{p} \in \mathscr{P}_C$ *s.t.*
$N\mathfrak{p} \le (1 + o(1)) \left(\log d_L + 2n_L\right)^2.$

Theorem (Fiori - 2019)

There exists an infinite family of number fields L*, Galois over* $\mathbb{Q}$*, for which the smallest prime* p *of* $\mathbb{Q}$ *which splits completely in* L *has size at least* $\left(\frac{3e^\gamma}{2\pi}\right)\left(\log d_L \frac{\log\log\log d_L}{\log\log d_L}\right)^2$*, as* $d_L \to \infty$.

A. Fiori (UofL)

The least prime in the Chebotarev Density Theorem

Theorem (Lagarias, Montgomery, Odlyzko - 1979)

There exists a positive constant B *and a prime ideal* $\mathfrak{p} \in \mathscr{P}_C$ *s.t.* $N\mathfrak{p} \leq d_L^B$ *for* d_L *sufficiently large.*

Assuming d_L is sufficiently large:

Author	B
Zaman (2017)	40
K - Ng - Wong (2019)	16

For all d_L:

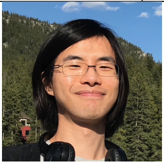
Author	B
Ahn - Kwon (2019)	12 577
K - Wong (2022)	310

In particular if there are **no-exceptional zeros**, we have the refinement

Author	B
Zaman (2017)	7.5

Author	B
K - Wong (2022)	10.5

A. Zaman (UofT)



P. J. Wong (NSYS, Taiwan)

Ingredients to prove a bound for the least prime

1. A numerical verification for the first fields.
   Walgstaff (1979), Fiori (2022)
2. An explicit inequality between prime ideals and zeros of Dedekind zeta functions.
3. Weight to locate the least prime ideal.
   Heath-Brown (1992), Zaman (2017), K-Wong (2022).
4. Counts of zeros of Dirichlet or Dedekind zeta functions.
   Hasanalizade-Shen-Wong (2022)
5. **Zero-free regions for Dedekind zeta functions**,
6. **Deuring-Heilbronn phenomenon:**
   the potential exceptional zero β_0 has the effect of pushing other zeros further away from the line $\Re s = 1$.

Explicit results about the zeros of Dirichlet L-functions

1. **Partial verification of GRH**
   Ramaré, Rumely (1996), Bennett (2003)
   Platt (2013): for all $3 \le q \le 400\,000$, up to height $H_q = \frac{10^8}{q}$.
2. **Zero-free regions for** $L_q(s) := \prod_{\chi^* \bmod q} L(s,\chi)$
   $L_q(s) \neq 0$ for $\sigma \ge 1 - \frac{1}{f(t,q)}$ with
   K (2017):
   $f(t,q) = 5.60 \log(q \max(|t|,1))$, for all $3 \le q \le 400\,000$.
   Khale (2022^+):
   $f(t,q) = 10.5 \log q + 61.5 (\log|t|)^{2/3} (\log\log|t|)^{1/3}, \ |t| \ge 10$, for all $q \ge 3$.
3. **Log-free density estimates:**

Theorem (Thorner, Zaman - 2022^+)

The number $N^*(\sigma,Q)$ of zeros $\beta + i\gamma \neq \varrho_0$ of $\prod_{q\le Q} L_q$ s.t. $\beta > \sigma, |\gamma| \le Q$:

$$N^*(\sigma,Q) \le 6 \cdot 10^{92} (1-\beta_1)(\log q)(10^{466} q^{170})^{1-\sigma}.$$

Zero-free region for Dedekind zeta functions

Theorem (Low-lying zeros)

$\zeta_L(s)$ vanishes at most at the "exceptional zero" ϱ_0 in the region

- $\Re s > 1 - \dfrac{1}{c \log d_L}$ *and* $|\Im s| < \dfrac{1}{c \log d_L}$,
  For d_L is sufficiently large: Stark (1974) $c = 4$.
  For all $L \neq \mathbb{Q}$: K - Wong (2021) $c = 1.7$.
- $\Re s > 1 - \dfrac{1}{r \log d_L}$ *and* $|\Im s| \leq 1$.
  For d_L is sufficiently large: Lee (2021) $r = 12.5$.
  For all $L \neq \mathbb{Q}$: Ahn - Kwon (2019) $r = 29.6$.

Theorem (Higher zeros: when $|\Im s| \geq 1$:)

$\zeta_L(s)$ does not vanish in

$$\Re s > 1 - \left(c_1 \log d_L + c_2 n_L \log |\Im s| + c_3 n_L + c_4\right)^{-1}.$$

For d_L is sufficiently large:
Lee (2021) $c_1 = 12.3, c_2 = 9.6, c_3 = 0.1, c_4 = 2.3$.

E. Lee (UNSW)

Deuring-Heilbronn phenomenon: repulsion of low-lying zeros

Assume $\zeta_L(s)$ admits an exceptional real zero β_0.

Theorem (Lagarias, Montgomery, Odlyzko - 1979)

Let $\beta' + i\gamma'$ be another zero of $\zeta_L(s)$ satisfying $1/2 \leq \beta' < 1$ and $|\gamma'| \leq 1$.

Then there exists $\kappa, C > 0$ s.t. $\beta' \leq 1 - \dfrac{\log\left(\frac{\kappa}{(1-\beta_0)\log d_L}\right)}{C \log d_L}$.

- For Dedekind zeta functions:
  - For d_L is sufficiently large: $C = 35.80$ (Zaman -2017) .
  - For all d_L: $C = 20$ (K. and Wong -2022, Turán's power sum method improves LMO).
- For Dirichlet L-functions, with $q \geq 400\,000$: Thorner, Zaman (2022^+)

$$\beta' \leq 1 - \frac{\log\left(\frac{\kappa_q}{(1-\beta_1)}\right)}{54.2(\log q) + 104.7}.$$

Examples of results used in

- Eric Bach *Explicit bounds for primality testing and related problems* (Math. Comp. 1990)
- M. Bennett's *Rational approximation to algebraic numbers of small height: the Diophantine equation* $|ax^n - by^n| = 1$ (Crelle 2003)
- B. Hough's *Solution of the minimum modulus problem for covering systems* (Annals, 2015)
- T. Tao's *Every odd number greater than 1 is the sum of at most five primes* (Math. Comp. 2013)
- H. Helfgott's proof of Goldbach's Ternary conjecture (2013):
- Samir Siksek's *Every integer greater than 454 is the sum of at most seven positive cubes* (Algebra Number Theory, 2016).
- Ben Green *On Sárközy's theorem for shifted primes* (Arxiv June 2022)

Primes between consecutive cubes

Conjecture (Legendre's conjecture:)

For each positive integer n, there is at least one prime between n^2 and $(n+1)^2$.

Undecided even under RH.

Theorem

There exists a prime between n^3 and $(n+1)^3$,

- *when n is large enough: $n \geq N$ (Ingham).*
- *for all n, conditionally under RH (Caldwell and Cheng - 2005).*
- *for all n up to $4 \cdot 10^{18}$ (Oliveira e Silva, Herzog, and Pardi - 2014).*
- *for all $n \geq N$, unconditionally,*
  with $N = e^{e^{33.217}}$ (Dudek - 2014).
  with $N = e^{e^{32.892}}$ (Cully-Hugill - 2021).

Ingham's theorem and Hardy-Littlewood's sub-convexity bound for $\zeta(1/2+it)$ (1921): implies a prime gap $p_{n+1} - p_n \ll p_n^{\frac{5}{8}+\epsilon}$ for n large enough. Conclude with $(n^3, n^3 + (n^3)^{5/8}) \subset (n^3, (n+1)^3)$.

Primes between consecutive powers

Explicit tools:
inversion formula (truncated Perron formula),
short interval containing primes,
zero-free regions (Korobov-Vinogradov),
zero-density theorems (Ingham).

Theorem

For any $n \geq 2$*, there is a prime between* n^k *and* $(n+1)^k$ *for any* $k \geq K$*, with*

- $K = 5 \cdot 10^9$ *(Dudek - 2014).*
- $K = 180$ *(Cully-Hugill - 2021).*

M. Cully-Hugill (UNSW)

Lots going on, lots to do!

The TME-EMT project:
https://ramare-olivier.github.io/TME-EMT/accueil.html

Théorie Multiplicative Explicite des nombres / Explicit Multiplicative number Theory

Fully explicit results in multiplicative number theory are often scattered through the litterature. The aim of this site is to present annoted bibliographies in order to keep track of the current knowledge.

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