Moments of L-functions and Automorphic Forms

Collaborative Research Group (CRG)
L-functions in Analytic Number Theory

CRG Launch Event - November 19, 2022

Banff International Research Station
for Mathematical Innovation and Discovery

PIMS

UBC

University of Lethbridge

FIAT LUX

UNBC

Table of Contents

Moments of $\zeta(\frac{1}{2}+it)$

We define the $2k$-th moments of $|\zeta(\frac{1}{2}+it)|$ as

$$I_k(T) = \int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\, dt$$

Lindelöf Hypothesis

$$\Updownarrow$$

for any $\epsilon > 0,\ I_k(T) = O(T^{1+\epsilon})\ \ \forall\, k \in \mathbb{N}$

A Folklore Conjecture

It is believed that

Conjecture

$$I_k(T) = \int_0^T \left|\zeta(\tfrac{1}{2} + it)\right|^{2k} dt \sim c_k T \left(\log T\right)^{k^2}$$

for some unspecified constant c_k.

What is known?

A Folklore Conjecture

It is believed that

Conjecture

$$I_k(T) = \int_0^T |\zeta(\tfrac{1}{2} + it)|^{2k}\, dt \sim c_k T \left(\log T\right)^{k^2}$$

for some unspecified constant c_k.

What is known?

- *Hardy-Littlewood (1918):* $I_1(T) \sim T \log T$
- *Ingham (1926):* $I_1(T) = T P_1(\log T) + O(T^{\frac{1}{2}+\epsilon})$

A Folklore Conjecture

It is believed that

Conjecture

$$I_k(T) = \int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt \sim c_k T\,(\log T)^{k^2}$$

for some unspecified constant c_k.

What is known?

- *Hardy-Littlewood (1918):* $I_1(T) \sim T\log T$
- *Ingham (1926):* $I_1(T) = TP_1(\log T) + O(T^{\frac{1}{2}+\epsilon})$

- *Ingham (1926):* $I_2(T) \sim \frac{1}{2\pi^2}T\,(\log T)^4$
- *Heath-Brown (1979):* $I_2(T) = TP_2(\log T) + O(T^{\frac{7}{8}+\epsilon})$

Asymptotic Bounds

We have the lower bound

- *Radziwiłł-Soundararajan (2013):*
  For all $k > 0$, *we have* $I_k(T) \gg T\,(\log T)^{k^2}$.

and the upper bounds

- *Soundararajan (2008):*
  Under RH, for any $\epsilon > 0$ *we have* $I_k(T) \ll T(\log T)^{k^2+\epsilon}$.
- *Harper (2013):*
  Under RH, we have $I_k(T) \ll T(\log T)^{k^2}$

Conjectural Asymptotic Formulae

Conjecture (Conrey-Ghosh, 1998)

$$I_3(T) \sim \frac{g_3}{9!} a_3 \cdot T(\log T)^9$$

where $g_3 = 42$ *and* $a_3 = \prod_p \left(1 - p^{-1}\right)^4 \left(1 + 4p^{-1} + p^{-2}\right)$

Conjectural Asymptotic Formulae

Conjecture (Conrey-Ghosh, 1998)

$$I_3(T) \sim \frac{g_3}{9!} a_3 \cdot T(\log T)^9$$

where $g_3 = 42$ *and* $a_3 = \prod_p \left(1 - p^{-1}\right)^4 \left(1 + 4p^{-1} + p^{-2}\right)$

Conjecture (Conrey-Gonek, 2001)

$$I_4(T) \sim \frac{g_4}{16!} a_4 \cdot T(\log T)^{16}$$

where $g_4 = 24024$ *and* $a_4 = \prod_p \left(1 - p^{-1}\right)^9 \left(1 + 9p^{-1} + 9p^{-2} + p^{-3}\right)$

Asymptotic Formulae for Higher Moments?

Conjecture (Keating and Snaith, 2000)

$$I_k(T) = \int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt \sim \frac{g_k}{(k^2)!}\cdot a_k \cdot T(\log T)^{k^2}$$

where

$$g_k = (k^2)!\prod_{j=0}^{k-1}\frac{j!}{(j+k)!}$$

and

$$a_k = \prod_p \left(1-\frac{1}{p}\right)^{(k-1)^2}\sum_{j=0}^{k-1}\binom{k-1}{j}^2 p^{-j}.$$

Conditional Results

Theorem (Ng, 2021)

Under a ternary additive divisor conjecture

$$I_3(T) \sim \frac{g_3}{9!} a_3 \cdot T(\log T)^9$$

as $T \to \infty$

Theorem (Ng-Shen-Wong, 2022+)

Under the Riemann Hypothesis and a quaternary additive divisor conjecture

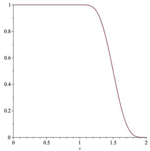
$$I_4(T) \sim \frac{g_4}{16!} a_4 \cdot T(\log T)^{16}$$

as $T \to \infty$.

Tools: Smooth additive divisor sums, smooth AFE

- **Smooth AFE.** Heath-Brown (1979): $N = t^k$.

$$|\zeta(\tfrac{1}{2}+it)|^{2k} = \sum_{m,n=1}^{\infty} \frac{d_k(m)d_k(n)}{m^{\frac{1}{2}}n^{\frac{1}{2}}} \left(\frac{m}{n}\right)^{-it} \varphi\left(\frac{mn}{N}\right) + O(\exp(-t^2/2))$$



smooth weight φ

- **Smooth additive divisor sums** DFI (1994) and Aryan (2017)

$$\tilde{D}_{k,\ell}(f,r) = \sum_{m-n=r} d_k(m)d_\ell(n)f(m,n)$$

where $f : [M,2M] \times [N,2N] \to \mathbb{R}$ is smooth.

Mean values of long Dirichlet polynomials

Theorem (Hamieh-Ng, 2021)

Let $1 < \eta < \frac{4}{3}$, $N = T^\eta$, $L = \log(\frac{t}{2\pi})$, and ω is a smooth weight.

$$\int_{-\infty}^{\infty} \omega(t) \Big| \sum_{n=1}^{N} \frac{d_2(n)}{n^{\frac{1}{2}+it}} \Big|^2 dt \sim \frac{1}{\zeta(2)} \int_{-\infty}^{\infty} \omega(t) \frac{1}{4!} P(\eta) L^4 \, dt$$

where $P(\eta) = -\eta^4 + 8\eta^3 - 24\eta^2 + 32\eta - 14$.

- special case of more general theorem with d_k.
- Conrey-Keating series of papers on these mean values.
- Baluyot-Turnage-Butterbaugh (2022): Dirichlet L-functions
- Conrey-Rodgers (2022): quadratic Dirichlet L-functions
- Conrey-Fazzari (2022): modular L-functions

Moments of $L(\frac{1}{2}, \chi_d)$

Conjecture (Conrey-Farmer-Keating-Rubinstein-Snaith, 2005)

For any $k \in \mathbb{N}^*$,

$$\sum_{0<d\leq X}^{\flat} L(\tfrac{1}{2}, \chi_d)^k = X P_{\frac{k(k+1)}{2}}(\log X) + O(X^{\frac{1}{2}+\varepsilon}) \sim c_k X (\log X)^{\frac{k(k+1)}{2}}$$

where $P_{\frac{k(k+1)}{2}}$ is a polynomial of degree $\frac{k(k+1)}{2}$.

- $k = 1$: $O(X^{\frac{1}{4}+\varepsilon})$, $k = 2$: $O(X^{\frac{1}{4}+\varepsilon})$
- Diaconu-Twiss (2020). $k \geq 3$: conjecture error term is $O(X^{\frac{3}{4}}(\log X)^{C_k})$

History

Theorem (Jutila, 1981)

$$\sum_{0<d\leq X}^{\flat} L(\tfrac{1}{2},\chi_d) = XP_1(\log X) + O(X^{\frac{3}{4}+\varepsilon}),$$

where $P_1(t)$ is a linear polynomial with explicit coefficients.

History

Theorem (Jutila, 1981)

$$\sum_{0<d\leq X}^{\flat} L(\tfrac{1}{2}, \chi_d) = XP_1(\log X) + O(X^{\frac{3}{4}+\varepsilon}),$$

where $P_1(t)$ is a linear polynomial with explicit coefficients.

- Goldfeld-Hoffstein, 1985: $O(X^{\frac{19}{32}+\varepsilon})$.
- Young, 2009: $O(X^{\frac{1}{2}+\varepsilon})$ (smooth version).
- Florea, 2017: $eX^{\frac{1}{3}} + O(X^{\frac{1}{4}+\varepsilon})$ (function field), where e is a constant.

Theorem (Jutila, 1981)

$$\sum_{0<d\le X}^{\flat} L(\tfrac{1}{2}, \chi_d)^2 = c_2 X (\log X)^3 + O(X (\log X)^{\frac{5}{2}+\varepsilon}),$$

where c_2 is a explicit constant.

Theorem (Jutila, 1981)

$$\sum_{0<d\leq X}^{\flat} L(\tfrac{1}{2}, \chi_d)^2 = c_2 X (\log X)^3 + O(X (\log X)^{\frac{5}{2}+\varepsilon}),$$

where c_2 is a explicit constant.

- Soundararajan, 2000: main term + $O(X^{\frac{5}{6}+\varepsilon})$.
- Florea, 2015: $O(X^{\frac{1}{2}+\varepsilon})$ (function field).
- Sono, 2019: $O(X^{\frac{1}{2}+\varepsilon})$ (smooth version).

Theorem (Soundararajan, 2000)

$$\sum_{\substack{0<d\leq X \\ (d,2)=1}}^{*} L(\tfrac{1}{2}, \chi_{8d})^3 = XR_6(\log X) + O(X^{\frac{11}{12}+\varepsilon}),$$

where $R_6(t)$ is a polynomial of degree 6 and $\sum^*$ is the sum over square-free integers.

Theorem (Soundararajan, 2000)

$$\sum_{\substack{0<d\leq X\\(d,2)=1}}^{*} L(\tfrac{1}{2},\chi_{8d})^3 = XR_6(\log X) + O(X^{\frac{11}{12}+\varepsilon}),$$

where $R_6(t)$ is a polynomial of degree 6 and $\sum^*$ is the sum over square-free integers.

- Diaconu-Goldfeld-Hoffstein, 2003: $O(X^{0.853366+\cdot+\varepsilon})$.
- Zhang, 2005: $e_1X^{\frac{3}{4}}+$ Error, under technical assumptions.
- Young, 2013: $O(X^{\frac{3}{4}+\varepsilon})$ (smooth version).
- Florea, 2015: $O(X^{\frac{3}{4}+\varepsilon})$ (function field).
- Diaconu, 2018: $e_2X^{\frac{3}{4}}+O(X^{\frac{2}{3}+\varepsilon})$ (function field).
- Diaconu-Whitehead, 2018: $e_3X^{\frac{3}{4}}+O(X^{\frac{2}{3}+\varepsilon})$ (smooth version).

Theorem (Shen, 2020)

Under GRH,

$$\sideset{}{^*}\sum_{\substack{0<d\leq X\\(d,2)=1}} L(\tfrac{1}{2},\chi_{8d})^4 = c_1 X(\log X)^{10} + O\left(X(\log X)^{9.75+\varepsilon}\right).$$

- Florea, 2017 (function field) obtains the first three terms using recursive method.
- Xiannan Li (2022) removes GRH assumption in evaluation of $\sideset{}{^*}\sum_{\substack{0<8d<X\\(d,2)=1}} L(\frac{1}{2}, f\otimes\chi_{8d})^2$
- Shen-Stucky (in progress) remove GRH and obtain some lower order terms.
- Diaconu-Pasol-Popa (2022) exact formula for weighted 4th moment (function field).

Modular Forms

- Consider the congruence subgroup $\Gamma_0(q) \subset \mathrm{SL}_2(\mathbb{Z})$:

$$\Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \mod q \right\}.$$

- For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathfrak{h}$, let

$$\gamma \cdot z = \frac{az + b}{cz + d}.$$

- Let $f : \mathfrak{h} \to \mathbb{C}$ be holomorphic. We say f is a modular form of weight k with respect to the congruence subgroup $\Gamma_0(q)$ and a character χ modulo q if f is holomorphic at all cusps and

$$f(\gamma \cdot z) = \chi(d)(cz + d)^k f(z) \quad \forall\, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q).$$

Space of Cusp Forms

- We say f is a cusp form if f vanishes at all the cusps.
- The space of cusp forms of weight k for $\Gamma_0(q)$ and $\chi \mod q$ is denoted by $S_k(\Gamma_0(q), \chi)$.
  If χ_0 is the principal character, we set $S_k(q) = S_k(\Gamma_0(q), \chi_0)$, and we have
  $$\dim_{\mathbb{C}} S_k(q) \sim \frac{k-1}{12} q \prod_{p|q} \left(1 + \frac{1}{p}\right).$$
- A cusp form f has a Fourier series expansion
  $$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} q^n, \qquad \text{where } q = e^{2\pi i z}.$$
  It is called normalized if $\lambda_f(1) = 1$.

- Linear operators called Hecke operators act on $S_k(\Gamma_0(q), \chi)$.
- A Hecke eigenform is a cusp form that is a simultaneous eigenvector for all the Hecke operators.
- $S_k(\Gamma_0(q), \chi)$ has an orthogonal basis of normalized primitive eigenforms - $H_k(\Gamma_0(q), \chi)$.

Modular L-functions

For $f \in H_k(\Gamma_0(q), \chi)$, the L-function attached to f is defined as:

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi(p)}{p^{2s}}\right)^{-1}.$$

- $L(s, f)$ converges absolutely for $\Re(s) > 1$,
- admits an analytic continuation to $\mathbb{C}$ and
- satisfies a functional equation: $\Lambda(s, f) = \epsilon_f \Lambda(1 - s, \overline{f})$ with $|\epsilon_f| = 1$.

Moments of $L(\frac{1}{2}, f)$

We consider harmonic averages of the form

$$\sum_f^h \alpha_f = \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_f \frac{\alpha_f}{\langle f, f\rangle}.$$

For $f \in S_k(\Gamma_0(q), \chi)$, we define the ℓ-th moment of $L(\frac{1}{2}, f)$ as

$$M_\ell^h(q,\chi) = \sum_{f \in H_k(\Gamma_0(q),\chi)}^h |L(\tfrac{1}{2}, f)|^\ell.$$

First and Second Moments in the Level Aspect

Consider

$$M_\ell^h(q) = \sum_{f \in H_k(q)}^h L(\tfrac{1}{2}, f)^\ell \quad \text{as } q \to \infty.$$

Theorem (Duke, 1995)

For $k = 2$ and q prime, we have

$$M_1^h(q) = \sum_{f \in H_2(q)}^h L(\tfrac{1}{2}, f) = 1 + O(q^{-\frac{1}{2}} \log q),$$

and

$$M_2^h(q) = \sum_{f \in H_2(q)}^h L(\tfrac{1}{2}, f)^2 = \log q + O(q^{-\frac{1}{2}} \log q).$$

For any fixed even k and q squarefree

- Iwaniec-Sarnak (2000): $M_1^h(q) \sim 1$ and $M_2^h(q) \sim \log q$

4th Moments in the Level Aspect

Theorem (Kowalski-Michel-Vanderkam, 2000)

For $k=2$, as $q \to \infty$ through prime numbers

$$M_4^h(q) = P(\log q) + O_\epsilon(q^{-\frac{1}{12}+\epsilon}),$$

where P is a degree 6 polynomial with leading coefficient $\frac{1}{60\pi^2}$.

- Balkanova-Frolenkov (2017): improved error term for $M_4^h(q)$ of size $O_\epsilon(q^{-\frac{25}{228}+\epsilon})$.
- Balkanova (2016): k is a fixed even integer and $q = p^v$ for a fixed prime p as $v \to \infty$. We have

$$M_4^h(q) = R(\log q) + O_{\epsilon,k,p}(q^{-\frac{1}{4}+\epsilon} + q^{-\frac{k-1-2\theta}{8-8\theta}+\epsilon}),$$

  where R is a degree 6 polynomial .

Higher Moments

$$\mathcal{M}_\ell^h(q) = \frac{2}{\phi(q)} \sum_{\substack{\chi \mod q \\ \chi(-1)=(-1)^k}} \sum_{f \in H_k(\Gamma_0(q),\chi)}^h |L(\tfrac{1}{2}, f)|^\ell.$$

For fixed odd integer $k \geq 3$, as $q \to \infty$ through the primes

- Djanković (2011): $\mathcal{M}_6^h(q) \ll q^\epsilon$.
- Stucky (2021): $\mathcal{M}_6^h(q) \ll (\log q)^9$.
- Chandee-Li (2017):

$$\frac{2}{\phi(q)} \sum_{\chi \mod q} \sum_{f \in H_k(\Gamma_0(q),\chi)}^h \int_{-\infty}^{\infty} \left|\Gamma\left(\tfrac{k}{2}+it\right) L\left(\tfrac{1}{2}+it, f\right)\right|^6 \, dt$$
$$\sim \frac{42}{9!} b_3 (\log q)^9 \int_{-\infty}^{\infty} \left|\Gamma\left(\tfrac{k}{2}+it\right)\right|^6 \, dt.$$

- Chandee-Li (2020): $\mathcal{M}_8^h(q) \ll q^\epsilon$.

Weight Aspect Results

Consider

$$M_\ell^h(k) = \sum_{f \in H_{2k}(1)}^{h} L(\tfrac{1}{2}, f)^\ell \quad \text{as } k \to \infty.$$

- Balkanova-Frolenkov (2021): as $k \to \infty$

$$M_1^h(k) = 1 + i^{2k} + O\left((2\pi e/k)^k\right),$$

$$M_2^h(k) = 2\log(k/2\pi) + 2\gamma + O(k^{-\frac{1}{2}}).$$

- Frolenkov (2020): $M_3^h(k) \ll (\log k)^{\frac{9}{2}}$.
- Spectral large sieve $\implies \displaystyle\sum_{K \leq k \leq 2K} M_4^h(k) \ll K^{1+\epsilon}$.
- Khan (2020): $\displaystyle\sum_k h\left(\frac{2k-1}{K}\right) M_5^h(k) \ll K^{1+2\theta+\epsilon}$.

Rankin-Selberg Convolutions of Modular Forms

Assume $(q, N) = 1$, and let $f \in H_k(q)$ and $g \in H_r(N)$ be eigenforms.

$$L(s, f \otimes g) = \zeta^{qN}(2s) \sum_{m \geq 1} \frac{a_f(m) a_g(m)}{m^s}, \qquad \text{for } \Re(s) > 1.$$

- analytic continuation to $\mathbb{C}$ unless $f = g$.
- functional equation $\Lambda(s, f \otimes g) = \Lambda(1 - s, f \otimes g)$.

Moments in the Level Aspect

Let $g \in H_r(N)$. For a fixed even $k < 12$, consider

$$M_\ell^h(q;g) = \sum_{f \in H_k(q)}^{h} \left(L\left(\tfrac{1}{2}, f \otimes g\right)\right)^\ell .$$

As $q \to \infty$ through primes

- Luo (1999): $M_1^h(q;g) = \prod_{p|N}(1 - p^{-1}) \log q + O_g(1)$.
- Kowalski-Michel-Vanderkam (2000):

$$M_2^h(q;g) = P(\log q) + O_g\left(q^{-\frac{1}{12}+\epsilon}\right), \quad \deg(P) = 3.$$

Moments in the Weight Aspect

Let $g \in H_r(N)$. Consider

$$M_\ell^h(k;g) = \sum_{f \in H_k(1)}^{h} \left(L\left(\tfrac{1}{2}, f \otimes g\right)\right)^\ell.$$

For fixed r and $N = 1$, as $k \to \infty$ we have

- Ganguly-Hoffstein-Sengupta: $M_1^h(k;g) = \log k + O_g(1)$.
- Blomer-Harcos (2012): Asymptotic formula for the second moment of $L(\frac{1}{2} + it, f \otimes g)$ over $t \asymp T$ and $k \asymp K$ with $K^{\frac{3}{4}+\epsilon} \leq T \leq K^{\frac{5}{4}-\epsilon}$.
- Sarnak (2000): For any $\epsilon > 0$ and $K^{\frac{151}{165}} \leq M \leq K^{1-\epsilon}$, we have $\sum_{|k-K| \leq M} \sum_{f \in H_{2k}(1)} |L(\frac{1}{2} + it, f \otimes g)|^2 \ll_{\epsilon,t,g} (KM)^{1+\epsilon}$.

For $r = k$, we have

- Hamieh-Tanabe (2021): $M_2^h(k;g) \ll (\log k)^c$.
- Hamieh-Tanabe (work in progress): Asymptotic formula for $M_2^h(k;g)$ over $k \asymp K$ as $K \to \infty$.

Multiple Dirichlet Series

Diaconu, Goldfeld, and Hoffstein introduced the multi-variable complex functions

$$Z(s_1, \ldots, s_{2k}, w) = \int_1^{\infty} \zeta(s_1 + \varepsilon_1 it) \cdots \zeta(s_{2k} + \varepsilon_{2k} it) \left(\frac{2\pi e}{t}\right)^{kit} t^{-w} \qquad (1)$$

where

$$w, s_1, s_2, \ldots, s_{2k} \in \mathbb{C}, \varepsilon_j = \pm 1, 1 \leq j \leq 2k.$$

- They show that $Z(s_1, \ldots, s_{2k}, w)$ satisfies certain quasi functional equations.
- Assuming certain meromorphicity conjectures for $Z(s_1, \ldots, s_{2k}, w)$ they deduce the Keating-Snaith conjecture:

$$I_k(T) \sim \frac{g_k}{(k^2)!} a_k T (\log T)^{k^2}.$$

Motohashi's exact formula

Let ω be a smooth weight function.

$$\int_{-\infty}^{\infty} |\zeta(\tfrac{1}{2}+it)|^4 \omega(t)\,dt = \mathrm{Mainterm}(\omega) + \sum_{j=1}^{\infty} \alpha_j L(\tfrac{1}{2}, f_j)^3 \tilde{\omega}(j)$$
$$+\,\pi \int_{-\infty}^{\infty} \frac{|\zeta(\frac{1}{2}+it)|^6}{|\zeta(1+2it)|^2} \widehat{\omega}(t)\,dt + \sum_{k=1}^{\infty} \Big(\sum_{f\in H_{2k}(\Gamma)} L(\tfrac{1}{2}, f)^3 C(k,f,\omega) \Big). \qquad (2)$$

- If $\omega \approx \mathbf{1}_{[T,2T]}$, then $\mathrm{Mainterm}(\omega) \sim \frac{T}{2\pi^2}(\log T)^4$.
- $\{L(s,f_j)\}$ ranges through Maass form L-functions attached to full modular group.
- $\{L(s,f)\}$ ranges through a modular L-functions attached to a basis of Hecke eigenforms of $S_{2k}(\Gamma)$.
- $\tilde{\omega}(j)$ and $\widehat{\omega}(t)$ are certain integral transforms of ω.
- Zavorotnyi (1989) deduces: $O(T^{\frac{2}{3}+\varepsilon})$ for $I_2(T)$.

Moments of $\zeta'(\rho)$

$$J_k(T) = \sum_{0<\gamma<T} |\zeta'(\rho)|^{2k}$$

applications: $M(x)$, simple zeros, gaps between zeros of zeta

Conjecture (Hughes-Keating-O'Connell, 2001)

For $k > -\frac{3}{2}$,

$$J_k(T) \sim C_k \cdot a_k \cdot N(T) \cdot \left(\log \tfrac{T}{2\pi}\right)^{k(k+2)}$$

- $C_k = \frac{G^2(k+2)}{G(2k+3)}$ where G is Barnes' function.
- $N(T) = \#\{\rho \mid \zeta(\rho) = 0, 0 < \Im(\rho) \leq T\}$.
- Gonek-Hejhal (1989), $J_k(T) \asymp T(\log T)^{(k+1)^2}$ (all k)
- $k = 0$. von-Mangoldt/Riemann $J_0(T) = N(T) \sim \frac{T}{2\pi} \log T$

Results on $J_k(T)$

- Gonek (1984) RH implies

$$J_1(T) = \frac{T}{24\pi}(\log T)^4 + O((\log T)^3).$$

Milinovich (unpublished), main term$+O(T^{\frac{1}{2}+\varepsilon})$.

- Ng (2004) RH implies

$$c_1 T(\log T)^9 \leq J_2(T) \leq c_2 T(\log T)^9.$$

Garunkstis and Steuding (2005) improve c_2.

- Milinovich-Ng (2014). Let $k \in \mathbb{N}$. GRH implies

$$J_k(T) \gg T(\log T)^{(k+1)^2}.$$

Gao (2022) extends this to all $k > 0$ on GRH.
GRH (2022+) GRH replaced by RH (Benli, Elma, Ng).

- Kirila (2020). Let $k \in \mathbb{N}$. RH implies

$$J_k(T) \ll T(\log T)^{(k+1)^2}.$$

Milinovich (2007), extra factor $(\log T)^{\varepsilon}$.

- Milinovich-Ng (2012). RH + SZ implies

$$J_{-1}(T) \geq \Big(\frac{3}{2\pi^3} + o(1)\Big)T.$$

Gonek (1989), $J_{-1}(T) \gg T$.

- Heap-Li-Zhao (2022). Let $k \in \mathbb{Q}_{<0}$. RH +SZ implies

$$J_k(T) \gg T(\log T)^{(k+1)^2}.$$

Gao-Zhao (2022) generalize this to $k < 0$.

Open problems

- Evaluate asymptotically $\sum_{n\le x} d_3(n)d_3(n+1)$.
- Find an asymptotic formula for $\sum_{n\le x} d(n)d(n+1)d(n+2)$.
- Omega theorem for $\sum_{n\le x} d_k(n)d(n+h)$.
- Connection to Elliot-Halberstam type conjectures for d_k as in Nguyen (2022).

  Let $\epsilon > 0$. Then, for any $k \ge 1$, we have, uniformly in $1 \le h \le X^{\frac{k-1}{k}}$, the upper bound

$$\sum_{q\le X^{\frac{k-1}{k}}} \left| \sum_{\substack{n\le x \\ n\equiv h (\text{mod}) q}} d_k(n) - \frac{1}{\phi(\frac{q}{(h,q)})} \sum_{\substack{n\le x \\ (n,\frac{q}{(h,q)})=1}} d_k(n) \right| \ll_\epsilon X^{\frac{1}{2}+\varepsilon}$$

  as $X \to \infty$.
- Do such conjectures imply asymptotic for $I_3(T)$?

- Improve Zavorotnyi's bound $O(T^{\frac{2}{3}+\varepsilon})$ for $I_2(T)$.
- Smoothed fourth moment, show error term is $O(T^{\frac{1}{2}+\theta+\varepsilon})$, θ is upper bound in Ramanujan's conjecture.
- Omega theorem for $I_3(T)$ assuming ternary additive divisor conjecture.
- Full main term for $I_4(T)$ and the shifted version.
- A suitable conjecture for the error term $E_k(T) = I_k(T) -$ main term .
- Conjecture: $E_1(T) = O(T^{\frac{1}{4}+\varepsilon})$, $E_2(T) = O(T^{\frac{1}{2}+\varepsilon})$.
- Conjecture (Ivic-Motohashi) $E_3(T) = O(T^{\frac{3}{4}+\varepsilon})$??
  What does MDS method say? Recent work of Baluyot-Cech.
- For some small $\epsilon_0 \geq 0$ evaluate asymptotically

$$\int_0^T \Big| \sum_{n \leq T^{2+\epsilon_0}} \frac{d_k(n)}{n^{\frac{1}{2}+it}} \Big|^2 dt.$$

- Improve error term for $\sum_{0<d\leq X} L(\frac{1}{2}, \chi_d)$. Conjecture $O(X^{\frac{1}{4}+\varepsilon})$
  work of Florea, Goldfeld-Hoffstein, Young
- Full main term for $\sum_{0<d\leq X} L(\frac{1}{2}, \chi_d)^4$ (degree 10 polynomial)
  works of Shen-Stucky, Florea, Diaconu-Pasol-Popa
- Connection between MDS method and AFE method.
  works of Diaconu-Whitehead, Diaconu-Twiss, Diaconu-Pasol-Popa. Patnaik-Puskas,
- Full main term for $\sum_{0<d\leq X} L(\frac{1}{2}, f\otimes\chi_d)^2$
- Asymptotic for $\sum_{0<d\leq X} L(\frac{1}{2}, f\otimes\chi_d)L(\frac{1}{2}, g\otimes\chi_d)$
  work of Xiannan Li

- Compute asymptotic formula for the twisted second moments

$$\sum_{f\in H_k(p^v)} \lambda_f(r)L(\tfrac{1}{2}+\mu, f\otimes g)^2$$

  for a fixed prime p as $v\to\infty$.
- Establish non-vanishing results for the above family (work of Balkanova-Frolenkov)
- Establish upper bounds for higher moments of this family (work of Chandee-Li)
- Establish weight aspect estimates for higher moments of Rankin-Selberg convolutions in the weight aspect (work of Khan and Humphries-Khan).
- Study shifted convolution sums of coefficients of holomorphic cusp forms in the weight aspect (recent work of Hoffstein-Lee).