A Dynamical Shafarevich Theorem for Endomorphisms of $\mathbb{P}^N$

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Motivation

K : a number field
S : a finite set of places of K including all archimedean places
$\mathcal{O}_S$: ring of S-integers of K

Theorem (Faltings)

There are only finitely many K-isomorphism classes of principally polarized abelian varieties A/K of a given dimension having good reduction at all primes outside S.

Question

Is there a dynamical analogue of this theorem?

Good Reduction for morphisms

Definition

A morphism $f : \mathbb{P}^N_K \to \mathbb{P}^N_K$ is said to have *good reduction outside* S if there exists a $\mathcal{O}_S$-morphism $\mathbb{P}^N_{\mathcal{O}_S} \to \mathbb{P}^N_{\mathcal{O}_S}$ whose generic fiber is $\mathrm{PGL}_{N+1}(K)$-conjugate to f.

Example

The morphism $f(X : Y) = [pX^2 : Y^2]$ of $\mathbb{P}^1_{\mathbb{Q}}$ has good reduction outside S if $p \in S$.

Note: If f has good reduction outside S and $\phi \in \mathrm{PGL}_{N+1}(\mathcal{O}_S)$ then $f^{\phi} = \phi^{-1} \circ f \circ \phi$ also has good reduction outside S.

Naive dynamical Shafarevich finiteness

For given K, N, d, S, are there finitely many $\mathrm{PGL}_{N+1}(\mathcal{O}_S)$-conjugacy classes of morphisms of $\mathbb{P}^N_K$ with good reduction outside S?

Naive dynamical Shafarevich finiteness

For given K, N, d, S, are there finitely many $\mathrm{PGL}_{N+1}(\mathcal{O}_S)$-conjugacy classes of morphisms of $\mathbb{P}^N_K$ with good reduction outside S? **NO!**

Example

Let $S = S_\infty$. The polynomials

$$[X^d + a_1 X^{d-1} Y + \cdots + a_d Y^d : Y^d]$$

with $a_i \in \mathcal{O}_K$ all have good reduction outside S.

There are infinitely many $\mathrm{PGL}_2(\bar{K})$-conjugacy classes of these polynomials.

So the most naive version of dynamical Shafarevich fails!

We need more structure...

An abelian variety is really a variety together with a marked point, without the marked point Shafarevich finiteness fails. (Petsche - Stout, 2015)

So to produce a Shafarevich finiteness result it is natural to add level structure.

Good reduction of a set in $\mathbb{P}^1$

Definition
Write $x \in \mathbb{P}^N(K)$ as $x = [x_0 : \cdots : x_N]$, where the x_i are normalized so that they are $\mathfrak{p}$-integral for all i, and at least one x_i is a $\mathfrak{p}$-adic unit. Then the *reduction of x modulo $\mathfrak{p}$* is the point $\widetilde{x} = [\widetilde{x_0} : \cdots : \widetilde{x_N}] \in \mathbb{P}^N(k_{\mathfrak{p}})$, where $\widetilde{x_i}$ are the reductions of x_i modulo $\mathfrak{p}$. If $X \subseteq \mathbb{P}^N(K)$, then $\widetilde{X} := \{\widetilde{x} : x \in X\}$ is the *reduction of X modulo $\mathfrak{p}$*.

Definition
Let $X = \{P_1, P_2, \ldots, P_n\} \subseteq \mathbb{P}^1(\bar{K})$ be a finite $\mathrm{Gal}(\bar{K}/K)$-invariant set. We say X has *good reduction outside S* if for every prime $\mathfrak{p} \notin S$, and every $\mathfrak{P}$ of $K(P_1, \ldots, P_n)$ lying over $\mathfrak{p}$, the map

$$X \to \widetilde{X} \mod \mathfrak{P}$$

is injective.

Silverman's dynamical Shafarevich theorem for $N = 1$

For $d \geq 2$ and $n \geq 1$, define $\mathcal{GR}_d^1[n](K, S)$ to be the set of triples (f, Y, X) where

- $f : \mathbb{P}^1_K \to \mathbb{P}^1_K$ is a degree d morphism defined over K
- $Y \subseteq \mathbb{P}^1(\bar{K})$ is a finite set
- $X = Y \cup f(Y)$ and is $\mathrm{Gal}(\bar{K}/K)$-invariant
- f and X have good reduction outside S
- $\sum_{P \in Y} e_f(P) = n$, where $e_f(P)$ is the ram. index at P.

Note: We have an action of $\mathrm{PGL}_2(\mathcal{O}_S)$ on $\mathcal{GR}_d^1[n](K, S)$ given by

$$\phi \cdot (f, Y, X) := (\phi^{-1} \circ f \circ \phi, \phi^{-1}(Y), \phi^{-1}(X))$$

Silverman's dynamical Shafarevich theorem for $N = 1$

Theorem (Silverman, 2017)

For all $n \geq 2d + 1$, the set

$$\mathcal{GR}_d^1[n](K, S)/\operatorname{PGL}_2(\mathcal{O}_S)$$

is finite.

Silverman's dynamical Shafarevich theorem for $N = 1$

Theorem (Silverman, 2017)

For all $n \geq 2d + 1$, the set

$$\mathcal{GR}^1_d[n](K, S)/\operatorname{PGL}_2(\mathcal{O}_S)$$

is finite.

Various definitions of good reduction and finiteness theorems have been explored by other authors in the case $N = 1$ including Morton - Silverman, Petsche - Stout, Szpiro - Tucker, Szpiro - Tucker - West, Canci - Peruginelli - Tossici, etc.

The naive generalization of Silverman's theorem fails for $N \geq 2$

Example (Silverman)

Fix n. Consider

- S is the set of primes dividing $2\prod_{i=1}^{n}(2^i - 1)$
- $f_{a,b} : \mathbb{P}^2 \to \mathbb{P}^2$ by

$$f[X : Y : Z] = [aXZ + X^2 : bYZ + Y^2 : Z^2] \text{ with } a, b \in \mathcal{O}_S$$

- $X_n = \{[1 : 2^i : 0] : 0 \leq i \leq n\}$.

Then $(f_{a,b}, X_{n-1}, X_n)$ in $\mathcal{GR}_2^2[n](\mathbb{Q}, S)/\operatorname{PGL}_3(\bar{K})$ gives infinitely many inequivalent triples.

Good reduction of a set in general linear position

Definition
We say that a finite set of points $X \subseteq \mathbb{P}^N(F)$ with $|X| \geq N+1$ is in *general linear position* if no hyperplane contains $N+1$ points of X.

Definition
If $H_a : a_N x_N + \cdots + a_0 x_0 = 0$ is a hyperplane in $\mathbb{P}^N_K$ corresponding to $a = [a_0 : \cdots : a_N] \in \mathbb{P}^N(K)$, then we write $\widetilde{H_a} := H_{\tilde{a}}$ for the reduction of H_a in $\mathbb{P}^N_{k_\mathfrak{p}}$.

Definition
Let $X \subseteq \mathbb{P}^N(\bar{K})$ be a finite $\mathrm{Gal}(\overline{K}/K)$-invariant subset of size at least $N+1$ with field of definition $K(X)$. We say X has *good reduction outside* S if for all primes $\mathfrak{p} \notin S$ and all primes $\mathfrak{P}$ in $K(X)$ lying over $\mathfrak{p}$, the reduction modulo $\mathfrak{P}$ of X is in general linear position.

Example

Fix $n \geq 4$. Consider

- $f_c = [x_0^2 : c(x_0^2 + x_1^2 - x_2^2) + x_0^2 + x_1^2 : x_0^2 + x_1^2 - x_2^2]$ for $c \in \mathbb{Z}$
- $Y \subseteq \mathbb{P}^2(\mathbb{Q})$ be any set of order n in general linear position contained in $x_0^2 + x_1^2 - x_2^2 = 0$
- S be finite set of places such that $X = Y \cup f_c(Y)$ has good reduction outside of S (note this is independent of c)

There are infinitely many $\mathrm{PGL}_3(\bar{K})$ conjugacy classes in $\{(f_c, X, Y) : c \in \mathbb{Z}\}$.

Denote by $\mathcal{R}_{d,N}[m](K,S)$ the set of triples (f,X,Y) so that:

- f is a degree $d \geq 2$ morphism $\mathbb{P}^N \to \mathbb{P}^N$ defined over K,
- $Y \subseteq \mathbb{P}^1(\bar{K})$ with $|Y| = m$,
- $X = Y \cup f(Y)$ is a $\mathrm{Gal}(\bar{K}/K)$-invariant,
- f and X have good reduction outside S,
- Y is not contained in a hypersurface of degree at most $2d$.

The last condition implies $|Y| \geq \binom{N+2d}{2d} - 1$.

Shafarevich finiteness theorem

Theorem (J. - Krieger - Looper - Mavraki, 2021)

$\mathcal{R}_{d,N}[m](K,S)/\mathrm{PL}_{N+1}(\mathcal{O}_S)$ *is finite.*

Shafarevich finiteness theorem

Theorem (J. - Krieger - Looper - Mavraki, 2021)

$$\mathcal{R}_{d,N}[m](K,S)/\mathrm{PL}_{N+1}(\mathcal{O}_S) \textit{ is finite.}$$

For a ring R,

$$\mathrm{PL}_{N+1}(R) := \mathrm{GL}_{N+1}(R)/R^\times \hookrightarrow \mathrm{PGL}_{N+1}(R).$$

When $\mathrm{Pic}(R)$ is trivial, $\mathrm{PL}_{N+1}(R) = \mathrm{PGL}_{N+1}(R)$(Faber, 2020).

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Action of $\mathrm{PL}_{n+1}(\mathcal{O}_S)$ on $\mathcal{R}_{d,N}[m](K,S)$ is given by

$$(f,X,Y)^{\phi} := (\phi^{-1}\circ f\circ\phi, \phi^{-1}(X), \phi^{-1}(Y)).$$

Sketch of proof

Define $\mathcal{X}[n](K,S)$ be the collection of sets $X \subseteq \mathbb{P}^N(\bar{K})$ such that

- $|X| = n$
- X is $\mathrm{Gal}(\bar{K}/K)$-invariant
- X has good reduction outside S.

1. Prove the natural map

   $$\mathcal{X}[n](K,S)/\mathrm{PL}_{N+1}(\mathcal{O}_S) \to \{X \subseteq \mathbb{P}^N(\overline{K}) : \#X = n\}/\mathrm{PGL}_{N+1}(\overline{K})$$

   is finite-to-one.
2. The image of the map from Step 1 is finite.
3. For $X \in \mathcal{X}[n](K,S)$ there are finitely many choices for f, Y with $(f,X,Y) \in \mathcal{R}_{d,N}[n](K,S)$.

Step 1: $\mathcal{X}[n](K,S)/\mathrm{PL}_{N+1}(\mathcal{O}_S) \to \{X \subseteq \mathbb{P}^N(\overline{K}) : \#X = n\}/\mathrm{PGL}_{N+1}(\overline{K})$ is finite-to-one

Key Ingredients:

- All $X \in \mathcal{X}[n](K,S)$ are defined over a finite extension L of K. (Hermite-Minkowski)
- Let T a finite set of primes of L containing primes over S, large enough so $\mathcal{O}_T$ is a PID so we can normalize coordinates.
- Fix $X_0 \in \mathcal{X}[n](K,S)$, define a map from $\{\phi \in \mathrm{PGL}_{N+1}(\bar{K}) : \phi(X_0) \in \mathcal{X}[n](K,S)\}$ to a finite set $(\mathrm{Map}_{\mathrm{Set}}(\mathrm{Gal}(L/K), S_{X_0}))$ and show if two elements of this set have the same image, they differ by an element of $\mathrm{PL}_{N+1}(\mathcal{O}_S)$.

This means it suffices to prove the theorem for (L, T).

Step 2: X is PGL equivalent to an element of a finite set

Since X is in general linear position, we can change coordinates so that $[1:0:\cdots:0]$, $[0:1:\cdots:0]$, $[0:0:\cdots:1]$, $[1:1:\cdots:1] \in X$.

Define $\mathcal{H} := \left(\bigcup_{0 \le i \ne j \le N} \{z_i - z_j = 0\} \right) \cup \left(\bigcup_{i=0}^{N} \{z_i = 0\} \right)$

Note: If $x = [x_0 : x_1 : \cdots : x_n] \in X \setminus \mathcal{H}$, then good reduction implies $x_i, x_i - x_j$ are S-units for all $i, j \in \{0, \ldots, n\}$ with $i \ne j$. For each i we can write the following S-unit equation

$$\frac{x_0 - x_i}{x_0} + \frac{-x_i}{x_0} = 1$$

So there is a finite set Π_0 with $\frac{x_i}{x_0} \in \Pi_0$. Hence, there are finitely many choices for the points in $X \setminus \mathcal{H}$.

Step 3: f is determined by it action on Y

Given X there are finitely many choices for $Y, f(Y), f|_Y$.

Since f and g are degree d morphisms the subvariety given by $f = g$ is contained in a hypersurface of degree $2d$. We have assumed that Y is not contained in any such hypersurface.

Thank you