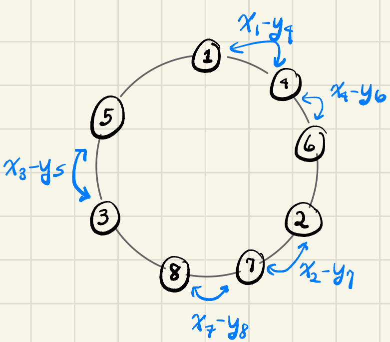
45 min.

The TASEP on a ring, Schubert polynomials, & evil-avoiding permutations

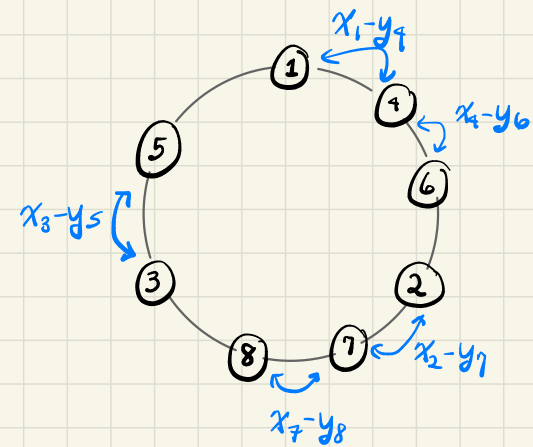


Lauren Williams

joint work w/ Donghyun Kim

arXiv: 2106.13378

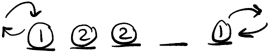
Outline

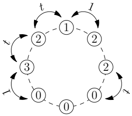


- Exclusion process on a ring
- Stationary dist & Schubert polynomials
- Monomial factor conjecture
- Evil-avoiding permutations

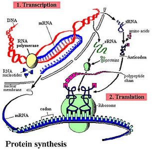
I. The asymmetric simple exclusion process (ASEP)

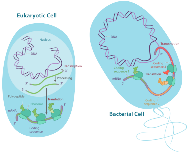
- Introduced by biologists (MacDonald, Gibbs, Pipkin) in 1968, and independently by a mathematician (Spitzer) in 1970.
- Particles hop on a 1D lattice; at most one particle per site. Particles may have different *weights*, which affect their hopping rate.





- Lattice could be a line with open boundaries or a *ring*...
- Cited as a model for traffic flow and for translation in protein synthesis





- Over 1000 papers on the exclusion process on the arXiv: Liggett, Derrida, Evans, Hakim, Pasquier, Spohn, Sasamoto, Yau, Borodin, Corwin, Ferrari, Seppalainen, Tracy-Widom, ...

ASEP / TASEP on ring: works by

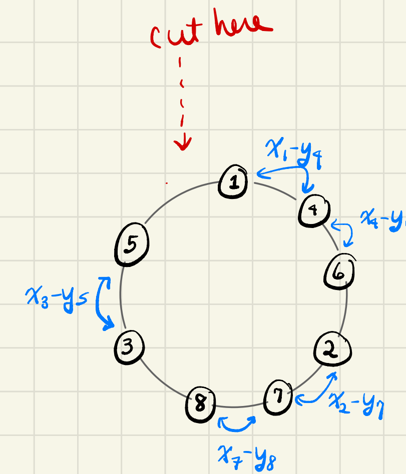
Angel, Amir, Arita, Aas, Ayyer

Cantini, Corteel, de Gier, Derrida, Evans,

Ferrari, Kim, Lam, Linusson, Martin, Mallick, Mandelshtam

Prolhac, De Sarkar, Sjostrand, Valko, Wheeler, others ...

Inhomogeneous TASEP on ring



Fix lattice of n sites on a ring, filled w/ particles of weights $1, 2, \ldots, n$.

Markov chain:

$n!$ states (by cutting open circle, identify state w/ permutation)

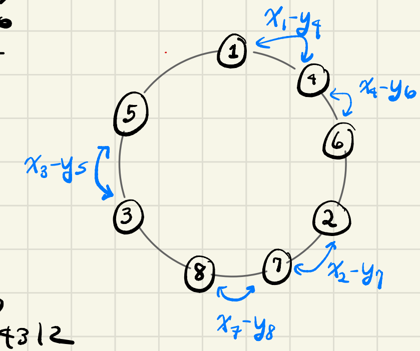
Identify state above w/ $(1, 4, 6, 2, 7, 8, 3, 5)$

Transitions:

If $i<j$, prob $\Bigg($  $\Bigg) = x_i - y_j$.

If $i>j$, prob $($ $) = 0$.

What are the steady state probabilities?



- For $w \in S_n$, let Ψ_w be (unnormalized) steady state prob of being in state w.
- Circular symmetry in model $\Rightarrow$ e.g. $\Psi_{3124} = \Psi_{1243} = \Psi_{2431} = \Psi_{4312}$
- So can restrict attention to $w \in S_n$ with $w_1 = 1$.

State w	Probability ψ_w (we set $y_i = 0$ for all i)
1234	$x_1^3 x_2$
1324	$x_1 \mathfrak{S}_{1432}$
1342	$x_1 x_2 \mathfrak{S}_{1423}$
1423	$x_1^2 x_2 \mathfrak{S}_{1243}$
1243	$x_1^2 \mathfrak{S}_{1342}$
1432	$\mathfrak{S}_{1423} \mathfrak{S}_{1342}$

- Here $\mathfrak{S}_w$ is the Schubert polynomial associated to permutation w.

- For $w \in S_n$, let ψ_w be (unnormalized) steady state prob of being in state w.
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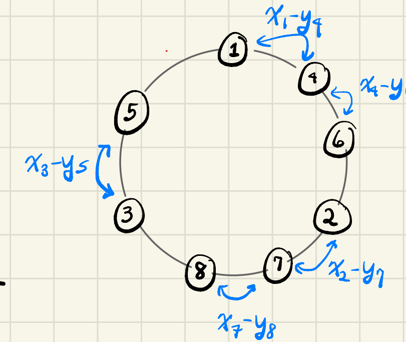


Ex: $n = 4$

State w	Probability ψ_w (we set $y_i = 0$ for all i)
1234	$x_1^3 x_2$
1324	$x_1 \mathfrak{S}_{1432}$
1342	$x_1 x_2 \mathfrak{S}_{1423}$
1423	$x_1^2 x_2 \mathfrak{S}_{1243}$
1243	$x_1^2 \mathfrak{S}_{1342}$
1432	$\mathfrak{S}_{1423} \mathfrak{S}_{1342}$

- Here $\mathfrak{S}_w$ is the Schubert polynomial associated to permutation w.

Note: ψ_w appears to be monomial times prod of Schuberts.

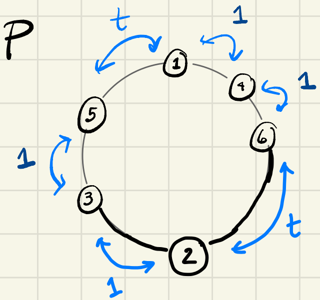
Main questions :

- Find combinatorial formula for steady state prob's
  (using multiline queues of Ferrari-Martin)

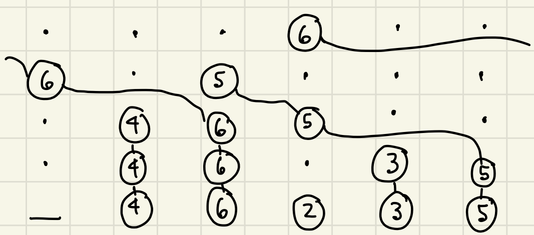
- Figure out those monomial factors
  + relationship to Schubert polynomials

Asymmetric exclusion process on a ring: two versions

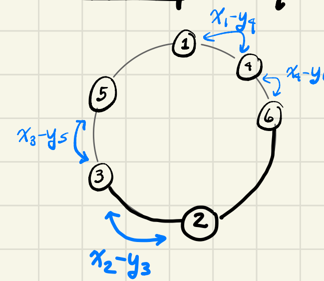
"usual" multispecies ASEP



Formula for steady state prob's in terms of multiline queues
(Martin, Corteel-Mandelshtam-W.)

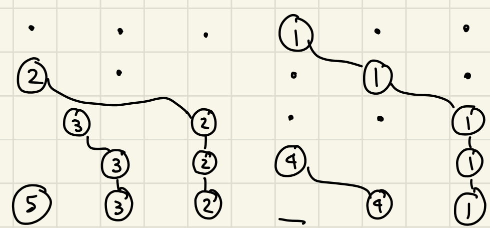


inhomogeneous, **totally** asymmetric (TASEP)

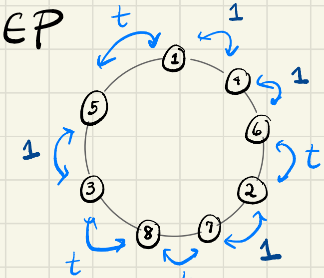


When $y_i = 0$, have formula for prob's in terms of multiline queues
(Ayyer-Linusson (conj), Arita-Mallick (proved))

[No formula in general case w/ $y_i \neq 0$]



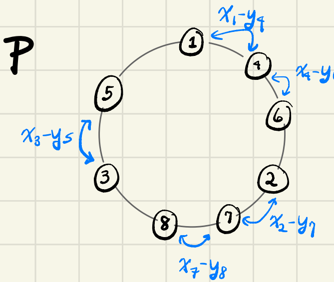
"usual" multispecies ASEP



Connected to Macdonald polynomials

- Partition function equals Macdonald poly $P_\lambda(x_1,\dots,x_n;q,t)$ specialized at $x_i=1$, $q=1$, where λ obtained by sorting particle weights in weakly decreasing order (Cantini - de Gier - Wheeler)
- Each steady state prob is a specialization of a permuted basement Macdonald poly (Corteel - Mandelshtam - W.)
- Can use multiline queues to give formulas for (permuted) Macdonald poly's (CMW)

inhomogeneous TASEP



Connected to (double) Schubert polynomials

- 2012: conj in $y_i=0$ case (Lam-W.)
- 2016: n of the $n!$ states have prob's Ψ_w $\sim$ prod's of double Schubert poly's (Cantini)
- 2021: same as above, but for $\sim \frac{(2+\sqrt{2})^{n-1}}{2}$ out of $n!$ states (Kim-W.)

Conj all other Ψ_w = nontrivial sum of Schuberts.

<u>Rk</u>: When $y_i=0$, $Z_n=\prod_{i=1}^{n} h_{n-i}(x_1,x_2,\dots,x_{i-1},x_i,x_i)$.
Has $\prod_{i=0}^{n}\binom{n}{i}$ terms.

(Double) Schubert polynomials

- For $1 \leq i < n$, the *divided difference operator* ∂_i acts on polynomials $P(x_1, \ldots, x_n)$ as follows:

$$(\partial_i P)(x_1, \ldots, x_n) = \frac{P(\ldots, x_i, x_{i+1}, \ldots) - P(\ldots, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}}.$$

- If $s_{i_1} \ldots s_{i_m}$ is a reduced expression for a permutation $w \in S_n$, then $\partial_{i_1} \ldots \partial_{i_m}$ depends only on w; we denote this operator by ∂_w.
- Let $\mathrm{x} = (x_1, \ldots, x_n)$ and $\mathrm{y} = (y_1, \ldots, y_n)$ be two sets of variables;

$$\text{Let } \Delta(\mathrm{x}, \mathrm{y}) = \prod_{i+j \leq n} (x_i - y_j).$$

- To each $w \in S_n$ we associate the *double Schubert polynomial* (Lascoux-Schutzenberger)

$$\mathfrak{S}_w(\mathrm{x}, \mathrm{y}) = \partial_{w^{-1} w_0} \Delta(\mathrm{x}, \mathrm{y}),$$

where the *divided difference operator* acts on the x-variables.

(Double) Schubert polynomials, cont.

- Ordinary Schubert polynomials $\mathfrak{S}_w(\mathrm{x})$ obtained from $\mathfrak{S}_w(\mathrm{x}, \mathrm{y})$ when all $y_i = 0$.
- When $w \in S_n$ is a Grassmannian permutation, $\mathfrak{S}_w(\mathrm{x})$ is a Schur polynomial. In particular, both Macdonald polynomials and Schubert polynomials generalize Schur polynomials.
- Geometric significance: Schubert polynomials represent cohomology classes of Schubert varieties in the cohomology ring of the complete flag variety $\mathcal{F}\ell_n$.
- Double Schubert polynomials represent Schubert classes in equivariant cohomology for the Borel group action on $\mathcal{F}\ell_n$.
- Many combinatorial formulas for Schubert polynomials, starting with Billey-Jokusch-Stanley '93, Fomin-Kirillov '96, Kohnert, etc.

Steady State Probabilities when $n=5$

State w	Probability ψ_w (we set $y_i = 0$ for all i)
12345	$x^{(6,3,1)}$
12354	$x^{(5,2,0)}\mathfrak{S}_{13452}$
12435	$x^{(4,1,0)}\mathfrak{S}_{14532}$
12453	$x^{(4,1,1)}\mathfrak{S}_{14523}$
12534	$x^{(5,2,1)}\mathfrak{S}_{12453}$
12543	$x^{(3,0,0)}\mathfrak{S}_{14523}\mathfrak{S}_{13452}$
13245	$x^{(3,1,1)}\mathfrak{S}_{15423}$
13254	$x^{(2,0,0)}\mathfrak{S}_{15423}\mathfrak{S}_{13452}$
13425	$x^{(3,2,1)}\mathfrak{S}_{15243}$
13452	$x^{(3,3,1)}\mathfrak{S}_{15234}$
13524	$x^{(2,1,0)}(\mathfrak{S}_{164325} + \mathfrak{S}_{25431})$
13542	$x^{(2,2,0)}\mathfrak{S}_{15234}\mathfrak{S}_{13452}$
14235	$x^{(4,2,0)}\mathfrak{S}_{13542}$
14253	$x^{(4,2,1)}\mathfrak{S}_{12543}$
14325	$x^{(1,0,0)}(\mathfrak{S}_{1753246} + \mathfrak{S}_{265314} + \mathfrak{S}_{2743156} + \mathfrak{S}_{356214} + \mathfrak{S}_{364215} + \mathfrak{S}_{365124})$
14352	$x^{(1,1,0)}\mathfrak{S}_{15234}\mathfrak{S}_{14532}$
14523	$x^{(4,3,1)}\mathfrak{S}_{12534}$
14532	$x^{(1,1,1)}\mathfrak{S}_{15234}\mathfrak{S}_{14523}$
15234	$x^{(5,3,1)}\mathfrak{S}_{12354}$
15243	$x^{(3,1,0)}(\mathfrak{S}_{146325} + \mathfrak{S}_{24531})$
15324	$x^{(2,1,1)}(\mathfrak{S}_{15432} + \mathfrak{S}_{164235})$
15342	$x^{(2,2,1)}\mathfrak{S}_{15234}\mathfrak{S}_{12453}$
15423	$x^{(3,2,0)}\mathfrak{S}_{12534}\mathfrak{S}_{13452}$
15432	$\mathfrak{S}_{15234}\mathfrak{S}_{14523}\mathfrak{S}_{13452}$

$x^{(5,2,0)} = x_1^5 x_2^2 x_3^0$

Most ψ_w here are product of monomial & some Schub poly's

Probabilities seem to have nice expressions in terms of Schubert polynomials. Often they are products of Schubert polynomials!

Monomial Factor Conjecture (2012 conj from Lam-W)

Def: Let $w=(w_1,\dots,w_n)\in S_n$. Let $r=w^{-1}(i+1)$ and $s=w^{-1}(i)$.

Let $\alpha_i(w) :=$ # integers greater than $i+1$ among $\{\overset{i+1}{\overset{\shortparallel}{w_r}}, w_{r+1}, w_{r+2}, \dots, \overset{i}{\overset{\shortparallel}{w_s}}\}$.

Theorem (Kim-W): Consider inhomog TASEP on S_n (where $y_i=0\ \forall i$).

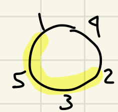
For $w\in S_n$, Let $\eta(w)$ be the largest monomial that can be factored out of Ψ_w.

Then
$$\eta(w)=\prod_{i=1}^{n-2} x_i^{\alpha_i(w)+\dots+\alpha_{n-2}(w)}.$$

$n-1 \longrightarrow n-2 \longrightarrow n-3 \longrightarrow \dots \longrightarrow i$

If two states $w, w'\in S_n$ have $\eta(w)=\eta(w')$, then w & w' are cyclically equivalent.

Ex: If $w=(1,4,2,3,5)$,



$\alpha_1=2$, $\alpha_2=2$, $\alpha_3=0$ so

$$\eta(w)=x_1^{\alpha_1+\alpha_2+\alpha_3}\,x_2^{\alpha_2+\alpha_3}\,x_3^{\alpha_3}=x_1^4x_2^2$$

(cf previous page)

Pattern Avoidance

- We say a permutation $w = (w_1, \ldots, w_n)$ *avoids the pattern* 2413 if there are not four positions $1 \leq i < j < k < l \leq n$ such that w_i, w_j, w_k, w_l are in relative order 2413.
- Example: $w = (3, 4, 6, 1, 2, 5)$ contains the pattern 2413 while $w = (1, 2, 6, 4, 3, 5)$ avoids the pattern 2413.

Def: Say permutation w is evil-avoiding if it avoids patterns 2413 and also patterns coming from

$$\begin{array}{cccc} 2 & 4 & 1 & 3 \\ \shortparallel & \shortparallel & \shortparallel & \shortparallel \\ e & v & i & l \end{array}$$

these anagrams of evil:

vile $= 4132$

veil $= 4213$

leiv $= 3214$

Theorem (Kim-W.) Let $w \in S_n$ be an evil-avoiding permutation representing state of TASEP. WLOG $w_1 = 1$. Then the (unnormalized) probability Ψ_w is proportional to product of k (double) Schubert poly's, where $k = \#$ descents of w^{-1}.

evil = 2413

vile = 4132

veil = 4213

leiv = 3214

	State w	Probability ψ_w (we set $y_i = 0$ for all i)
	12345	$x^{(6,3,1)}$
	12354	$x^{(5,2,0)}\mathfrak{S}_{13452}$
	12435	$x^{(4,1,0)}\mathfrak{S}_{14532}$
	12453	$x^{(4,1,1)}\mathfrak{S}_{14523}$
	12534	$x^{(5,2,1)}\mathfrak{S}_{12453}$
	12543	$x^{(3,0,0)}\mathfrak{S}_{14523}\mathfrak{S}_{13452}$
	13245	$x^{(3,1,1)}\mathfrak{S}_{15423}$
	13254	$x^{(2,0,0)}\mathfrak{S}_{15423}\mathfrak{S}_{13452}$
	13425	$x^{(3,2,1)}\mathfrak{S}_{15243}$
	13452	$x^{(3,3,1)}\mathfrak{S}_{15234}$
pattern 2413	13524	$x^{(2,1,0)}(\mathfrak{S}_{164325} + \mathfrak{S}_{25431})$
	13542	$x^{(2,2,0)}\mathfrak{S}_{15234}\mathfrak{S}_{13452}$
	14235	$x^{(4,2,0)}\mathfrak{S}_{13542}$
	14253	$x^{(4,2,1)}\mathfrak{S}_{12543}$
3214	14325	$x^{(1,0,0)}(\mathfrak{S}_{1753246} + \mathfrak{S}_{265314} + \mathfrak{S}_{2743156} + \mathfrak{S}_{356214} + \mathfrak{S}_{364215} + \mathfrak{S}_{365124})$
	14352	$x^{(1,1,0)}\mathfrak{S}_{15234}\mathfrak{S}_{14532}$
	14523	$x^{(4,3,1)}\mathfrak{S}_{12534}$
	14532	$x^{(1,1,1)}\mathfrak{S}_{15234}\mathfrak{S}_{14523}$
	15234	$x^{(5,3,1)}\mathfrak{S}_{12354}$
4132	15243	$x^{(3,1,0)}(\mathfrak{S}_{146325} + \mathfrak{S}_{24531})$
4213	15324	$x^{(2,1,1)}(\mathfrak{S}_{15432} + \mathfrak{S}_{164235})$
	15342	$x^{(2,2,1)}\mathfrak{S}_{15234}\mathfrak{S}_{12453}$
	15423	$x^{(3,2,0)}\mathfrak{S}_{12534}\mathfrak{S}_{13452}$
	15432 inverse has 3 descents	$\mathfrak{S}_{15234}\mathfrak{S}_{14523}\mathfrak{S}_{13452}$

Combinatorics of evil-avoiding permutations

evil = 2413
vile = 4132
veil = 4213
leiv = 3214

- Let $e(n) = \#$ evil-avoiding perms in S_n.

  Then $e(n) = \dfrac{(2+\sqrt{2})^{n-1} + (2-\sqrt{2})^{n-1}}{2}$

  (Sloane A006012)

- Apparently $e(n)$ also counts
  - rectangular perms on $[n]$, those avoiding 2413, 2431, 4213, 4231
  - $\{\pi \in S_n \mid$ for each j there's at most one $i \leq j$ with $\pi(i) > j\}$
  - (several other objects)

Bijective proof?

Theorem (Kim–W.) Let $w \in S_n$ be an evil-avoiding permutation representing state of TASEP w/ $\mathrm{des}(w^{-1}) = k$. WLOG $w_1 = 1$.
Then there are partitions $\lambda^1 \dots \lambda^k$ and $\mu := \left(\binom{n-1}{2}, \binom{n-2}{2}, \dots, \binom{2}{2}\right) - \sum_{i=1}^{k} \lambda^i$
such that

$$\Psi_w = x^{\mu} \prod_{i=1}^{k} \underbrace{s_{\lambda^i}(x_{n-\lambda_1^i}, x_{n-\lambda_2^i}, \dots)}_{\text{flagged Schur poly of shape } \lambda^i \text{ w/ entries in row } j \text{ bounded above by } n-\lambda_j^i}$$

To construct $\lambda^1, \dots, \lambda^k$:

Let $c = (c_1, \dots, c_n)$ be Lehmer code of w^{-1}.

Let $a_0 = 0$, and $a_1 \dots a_k$ be descent positions of $c_1 \dots c_n$.

Define $\lambda^i = (n-a_i)^{a_i} - (\underbrace{0, \dots, 0}_{a_{i-1}}, c_{a_{i-1}+1}, c_{a_{i-1}+2}, \dots, c_{a_i})$

Rem: I stated theorem in case all y_i's $= 0$, but we have version when y_i's $\neq 0$ using double Schub poly's.

Ex: If $w = 14253$, $\mathrm{code}(w^{-1}) = (\underline{0,1,2},0,0)$, $a_0 = 0$, $a_1 = 3$,

$\lambda^1 = (5-3)^3 - (\underline{0,1,2}) = (2,2,2) - (0,1,2) = (2,1,0)$.

descents of w^{-1}

k	$w \in \mathrm{St}(5,k)$	$\Psi(w)$	probability ψ_w	$s(w)$
0	12345	$\varnothing$	$\mathbf{x}^{(6,3,1)}$	(0)
1	12354	$(1,1,1)$	$\mathbf{x}^{(5,2,0)}\mathfrak{S}_{13452}$	(0)
1	12435	$(2,2,1)$	$\mathbf{x}^{(4,1,0)}\mathfrak{S}_{14532}$	(0)
1	12453	$(2,2)$	$\mathbf{x}^{(4,1,1)}\mathfrak{S}_{14523}$	(0)
1	12534	$(1,1)$	$\mathbf{x}^{(5,2,1)}\mathfrak{S}_{12453}$	(0)
1	13245	$(3,2)$	$\mathbf{x}^{(3,1,1)}\mathfrak{S}_{15423}$	(0)
1	13425	$(3,1)$	$\mathbf{x}^{(3,2,1)}\mathfrak{S}_{15243}$	(0)
1	13452	(3)	$\mathbf{x}^{(3,3,1)}\mathfrak{S}_{15234}$	(0)
1	14235	$(2,1,1)$	$\mathbf{x}^{(4,2,0)}\mathfrak{S}_{13542}$	(0)
1	14253	$(2,1)$	$\mathbf{x}^{(4,2,1)}\mathfrak{S}_{12543}$	(0)
1	14523	(2)	$\mathbf{x}^{(4,3,1)}\mathfrak{S}_{12534}$	(0)
1	15234	(1)	$\mathbf{x}^{(5,3,1)}\mathfrak{S}_{12354}$	(0)
2	12543	$(2,2),(1,1,1)$	$\mathbf{x}^{(3,0,0)}\mathfrak{S}_{14523}\mathfrak{S}_{13452}$	$(0,-1)$
2	13254	$(3,2),(1,1,1)$	$\mathbf{x}^{(2,0,0)}\mathfrak{S}_{15423}\mathfrak{S}_{13452}$	$(0,0)$
2	13542	$(3),(1,1,1)$	$\mathbf{x}^{(2,2,0)}\mathfrak{S}_{15234}\mathfrak{S}_{13452}$	$(0,-1)$
2	14352	$(3),(2,2,1)$	$\mathbf{x}^{(1,1,0)}\mathfrak{S}_{15234}\mathfrak{S}_{14532}$	$(0,-1)$
2	14532	$(3),(2,2)$	$\mathbf{x}^{(1,1,1)}\mathfrak{S}_{15234}\mathfrak{S}_{14523}$	$(0,-1)$
2	15342	$(3),(1,1)$	$\mathbf{x}^{(2,2,1)}\mathfrak{S}_{15234}\mathfrak{S}_{12453}$	$(0,-1)$
2	15423	$(2),(1,1,1)$	$\mathbf{x}^{(3,2,0)}\mathfrak{S}_{12534}\mathfrak{S}_{13452}$	$(0,-2)$
3	15432	$(3),(2,2),(1,1,1)$	$\mathfrak{S}_{15234}\mathfrak{S}_{14523}\mathfrak{S}_{13452}$	$(0,-1,-2)$

sum of these vectors is constant

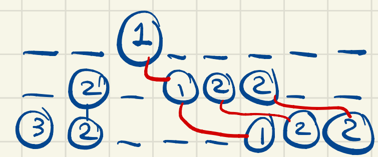
$(2,2)+(1,1,1)+(3,0,0)$
$=(6,3,1)$

sequence of partitions $\lambda^1 \ldots \lambda^k$

<u>Theorem (Kim-W.)</u> Let $w \in S_n$ be an evil-avoiding permutation representing state of TASEP w/ $\mathrm{des}(w^{-1}) = k$. WLOG $w_1 = 1$.
Then there are partitions $\lambda^1 \dots \lambda^k$ and $\mu := \left(\binom{n-1}{2}, \binom{n-2}{2}, \dots, \binom{2}{2}\right) - \sum_{i=1}^{k} \lambda^i$
such that

$$\Psi_w = x^{\mu} \prod_{i=1}^{k} s_{\lambda^i}\left(X_{n-\lambda^i_1}, X_{n-\lambda^i_2}, \dots\right)$$

<u>Note:</u> If y_i's $= 0$, there is combinatorial formula for stationary dist in terms of multiline queues (Ayyer-Linusson and Arita-Mallick)
In this case & when $k=1$, can use it to prove Thm,
by mapping MLQ's to semistandard tableaux.



But to prove theorem for general y_i's & k,
need another technique...

To prove our main result connecting steady state prob's $\Psi_w(x_1 \ldots x_n, y_1 \ldots y_n)$ to double Schubert poly's, use theorem of Cantini that adds auxiliary parameters $z_1 \ldots z_n$ into the picture.

Gives recursion for the Ψ_w.

To prove our main result connecting steady state prob's $\Psi_w(x_1 \dots x_n, y_1 \dots y_n)$ to double Schubert poly's, use theorem of Cantini that adds auxiliary parameters $z_1 \dots z_n$ into the picture.

Gives recursion for the Ψ_w.

Theorem (Cantini)

For each state $w \in S_n$ we define the quantity $\psi_w(z) = \psi_w(z_1, \dots, z_n)$ via:

$$\psi_{(1,2,\cdots,n)}(z) = \prod_{1 \le i < j \le n} (x_i - y_j)^{j-i-1} \prod_{i=1}^{n} \Big(\prod_{j=1}^{i-1} (z_i - x_j) \prod_{j=i+1}^{n} (z_i - y_j) \Big),$$

$$\psi_{s_\ell w}(z) = \pi_\ell(w_\ell, w_{\ell+1}; n) \psi_w(z) \quad \text{if } w_\ell > w_{\ell+1},$$

where $\pi_\ell(\beta, \alpha; n)$ is the *isobaric divided difference operator* defined by

$$\pi_\ell(\beta, \alpha; n) G(\mathbf{z}) = \frac{(z_\ell - y_\beta)(z_{\ell+1} - x_\alpha)}{x_\alpha - y_\beta} \frac{G(\mathbf{z}) - s_\ell G(\mathbf{z})}{z_\ell - z_{\ell+1}}.$$

Then the leading coefficient $\mathrm{LC}_z(\psi_w(z))$ w/ respect to z equals the steady state prob ψ_w.

Then need to show that for evil-avoiding w,

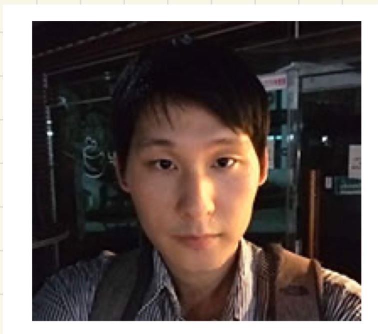
$LC_z(\Psi_w(z)) =$ product of Schuberts.

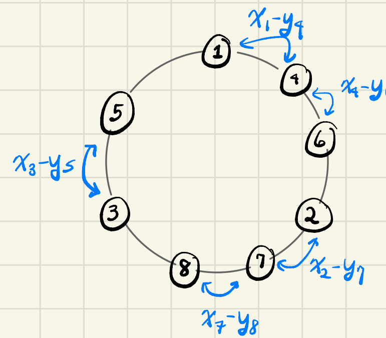
Conj: If w not evil-avoiding, can write Ψ_w as monomial $\cdot$ (pos sum of Schubert poly's)

State w	Probability ψ_w (we set $y_i = 0$ for all i)
12345	$x^{(6,3,1)}$
12354	$x^{(5,2,0)}\mathfrak{S}_{13452}$
12435	$x^{(4,1,0)}\mathfrak{S}_{14532}$
12453	$x^{(4,1,1)}\mathfrak{S}_{14523}$
12534	$x^{(5,2,1)}\mathfrak{S}_{12453}$
12543	$x^{(3,0,0)}\mathfrak{S}_{14523}\mathfrak{S}_{13452}$
13245	$x^{(3,1,1)}\mathfrak{S}_{15423}$
13254	$x^{(2,0,0)}\mathfrak{S}_{15423}\mathfrak{S}_{13452}$
13425	$x^{(3,2,1)}\mathfrak{S}_{15243}$
13452	$x^{(3,3,1)}\mathfrak{S}_{15234}$
13524	$x^{(2,1,0)}(\mathfrak{S}_{164325} + \mathfrak{S}_{25431})$
13542	$x^{(2,2,0)}\mathfrak{S}_{15234}\mathfrak{S}_{13452}$
14235	$x^{(4,2,0)}\mathfrak{S}_{13542}$
14253	$x^{(4,2,1)}\mathfrak{S}_{12543}$
14325	$x^{(1,0,0)}(\mathfrak{S}_{1753246} + \mathfrak{S}_{265314} + \mathfrak{S}_{2743156} + \mathfrak{S}_{356214} + \mathfrak{S}_{364215} + \mathfrak{S}_{365124})$
14352	$x^{(1,1,0)}\mathfrak{S}_{15234}\mathfrak{S}_{14532}$
14523	$x^{(4,3,1)}\mathfrak{S}_{12534}$
14532	$x^{(1,1,1)}\mathfrak{S}_{15234}\mathfrak{S}_{14523}$
15234	$x^{(5,3,1)}\mathfrak{S}_{12354}$
15243	$x^{(3,1,0)}(\mathfrak{S}_{146325} + \mathfrak{S}_{24531})$
15324	$x^{(2,1,1)}(\mathfrak{S}_{15432} + \mathfrak{S}_{164235})$
15342	$x^{(2,2,1)}\mathfrak{S}_{15234}\mathfrak{S}_{12453}$
15423	$x^{(3,2,0)}\mathfrak{S}_{12534}\mathfrak{S}_{13452}$
15432	$\mathfrak{S}_{15234}\mathfrak{S}_{14523}\mathfrak{S}_{13452}$

Open: Is there some geometric interpretation of steady state prob's?

Thank you!





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