The McKay Correspondance via VGIT

Work-in-progress with Tariq Ablelgadir

McKay: Klenian surface singularities

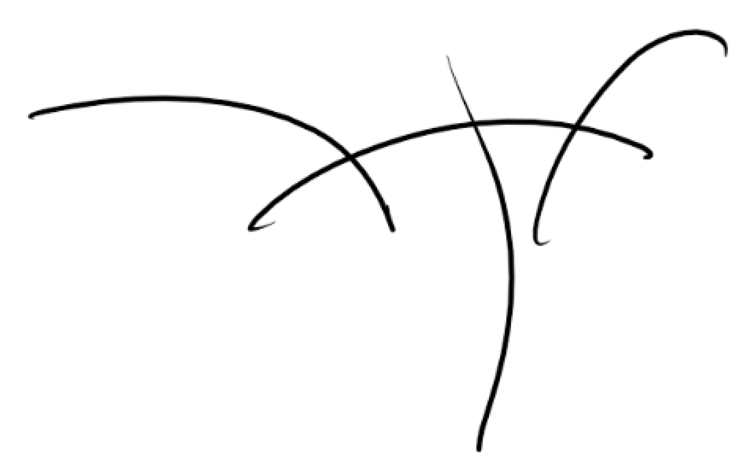
$$\mathbb{C}^2 \simeq V/G = X, \quad G \subset SL_2$$

$$\begin{tikzcd} X \\ \tilde{X} \arrow[u] \end{tikzcd}$$

has min res

ADE classification.

Exceptional locus is a tree of $\mathbb{P}^1$'s.

rep thy of $G \longleftrightarrow$ 

Modern formulation:

X can be 'resolved' by

$$\begin{tikzcd} {[V/G]} \arrow[r] & V/G \\ \tilde{X} \arrow[u, dashed, leftrightarrow, "\text{birational}"] \arrow[ur] & \end{tikzcd}$$

"flop"

both sides are CY2.

$$D^b([V/G]) \simeq D^b(\tilde{X})$$

[Kapranov-Vasserot, ...]

VGIT $\to$ birational equivalences.

$\exists$ VGIT giving $[V/G] \dashrightarrow \tilde{X}$?

Type A - easy / toric.

Type D + E - not until now. - Abdelgadir

$\exists$ Key idea : Tamaka duality.

<u>Plan</u>: 1) More background, why don't existing approaches work?

2) Tamaka idea.

3) D_4 case.

4) Potential applications?

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(1) Kleinian / du Val surface sing's

$X = V/G$, $V = \mathbb{C}^2$
$G \subset_{\text{finite}} SL_2$

$SL_2 \xrightarrow{2:1} SO(3)$

A: Cyclic $\to$ Cyclic

D: Binary dihedral $\to$ Dihedral

E: exceptional $\to$ Platonic solids

Orbifold $[V/G]$.

$Coh([V/G]) \cong Coh_G(V) \cong A\text{-mod}$

for a nc. algebra A.

i) Tilting: any rep of G gives a vect. bundle on $[V/G]$. Set $T = \bigoplus_{\text{irreps}} U_i$

$A = End(T)$

$$\mathrm{Coh}_G(V) \xrightarrow[\sim]{\mathrm{Hom}(T,-)} A\text{-mod}$$

ii) $\mathrm{Coh}_G(V) = \mathbb{C}[x,y] \rtimes \mathbb{C}[G]\text{-mod}$

"twisted group ring"

remove repeated irreps ⇝ A ← Morita equiv.

"basic"

e.g: Type A_1, $G = \mathbb{Z}_2 \curvearrowright \mathbb{C}^2$

$T = \mathcal{O} \oplus \mathcal{O}(1)$

$$A = \begin{tikzcd} \mathcal{O} \arrow[r, bend left, shift left, "x_0, y_0"] \arrow[r, bend left] & \mathcal{O}(1) \arrow[l, bend left, shift left, "x_1, y_1"] \arrow[l, bend left] \end{tikzcd}$$

"quiver alg"

$x_0 y_1 y_0 = y_0 y_1 x_0$ ... etc.

"preprojective alg" (of type A_1).

Claim: Given A, can reconstruct $\tilde{X}$.

$[V \setminus 0 / G] = X \setminus 0 = \tilde{X} \setminus$ exceptional locus.

points in $\tilde{X}$ $\longleftrightarrow$ points in X

$\updownarrow$

point sheaves in $\mathrm{Coh}(\tilde{X})$ $\longleftrightarrow$ orbits in $V \setminus 0$

$\updownarrow$

G-eq sheaves $\bigoplus_{g \in G} \mathcal{O}_{gx}$

$\longleftrightarrow$

A-modules

or Q-reps with dimension vector

$d_i = \dim V_i$

Set up GIT problem for these A-modules.

Set of Q-reps with dim $\underset{\sim}{d}$ is a vector space $\supseteq R =$ subvariety where relations hold.

Acted on by $GL(\underset{\sim}{d}) = \prod GL(d_i)$

<u>Fact</u>: $R /\!/ GL(\underset{\sim}{d}) = \tilde{X}$

(for the right stability condition)

But $[V/G]$ <u>cannot</u> arise as a

GIT quotient here.

Modules never have finite stab!

<u>Type</u> A_1: $\tilde{X} = \mathrm{Tot}(\mathcal{O}(-2)_{\mathbb{P}^1})$

$$\begin{array}{c}\tilde{X} \\ \downarrow \\ X = ODP.\end{array}$$

$\tilde{X}$ is toric, get as

$$\mathbb{C}^3 /\!\!/ \mathbb{C}^*_{1,1,-2}.$$

Other quotient is $[\mathbb{C}^2/\mathbb{Z}_2]$

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(2) $\tilde{X}$ is a GIT quotient of Z

by $GL(\underline{d}) = \prod GL(d_i)$

Could $[V/G]$ arise in this way?

$BG = [pt/G]$ " " ?

<u>Tannaka Ansatz</u>:

Think about rep ring of G.

(i) Find generators: $U_1, \dots, U_k$ irreps
($\dim d_1, \dots, d_k$.)

(ii) Find relations, (enough)

e.g. $\mathrm{Sym}^2 U_1 \underset{B_2}{\overset{B_1}{\cong}} U_2 \oplus U_3$

(iii) Find relations between these B_i's.

e.g. $B_1 B_2 = id = B_2 B_1$

Set up a GIT problem.

Pick VS's $V_1, \dots, V_k$ (of dim d_i).

Form the VS corresponding to rels (ii)

e.g. $\mathrm{Hom}(\mathrm{Sym}^2 V_1, V_2 \oplus V_3) \oplus \cdots$

$$M = \mathrm{Hom}(\mathrm{Sym}^2 V_1, V_2 \oplus V_3) \oplus \cdots$$

$$\cup$$

$$Z = \{\text{rels (iii) hold}\}$$

$$\circlearrowright$$

$$GL(\underline{d}).$$

Tannaka: a generic point in Z has stabilizer G.

Hope: generic = stable.

Really take $Z \times V_i$ $\Leftarrow$ the 2d irrep we started with.

Hope: $Z \times V_i /\!/ GL(\underset{\sim}{d}) = [V_i / G]$

Problems: • stability

• $GL(\underset{\sim}{d})$ transitive on generic points?

__e.g__ Type A_n: $G = \mathbb{Z}_{n+1}$

(i) Rep ring generated by U_1

(ii) rel $U_1^{\otimes n+1} = \mathbb{C}$

(iii) nothing.

GIT: Take L 1dim.

Consider $\mathrm{Hom}(L^{\otimes n+1}, \mathbb{C}) \curvearrowleft GL(L$

$$\mathbb{C} \curvearrowleft \mathbb{C}^*$$
wt $-n-1$

For 1 stab condition get $B\mathbb{Z}_{n+1}$.

Or $\mathrm{Hom}(L^{\otimes n+1}, \mathbb{C}) \oplus L \oplus L^{\otimes n}$

is $\mathbb{C}^3 \curvearrowleft \mathbb{C}^*$

$1, \ n, \ -n-1$

One quotient is $[\mathbb{C}^2/\mathbb{Z}_{n+1}]$

Other is $\mathrm{Tot}(K_{\mathbb{P}^1_{1:n}})$

More irreps $\rightsquigarrow$ bigger GIT

All irreps $\rightsquigarrow$ get $\tilde{X}$.

$L_1, L_2,$ $\quad L_1^2 \xrightarrow{\sim} L_2$

$L_2^2 \xrightarrow{\sim} L_1$

$\mathrm{Hom}(L_1^2, L_2) \oplus \mathrm{Hom}(L_2^2, L_1) \oplus L_1 \oplus L_2$

(is toric construction) $\;\circlearrowleft GL(L_1) \times GL(L_2)$

$\underline{D_4}$: $\quad \mathbb{Z}_2 \to BD_2 \to D_2$

$BD_2 =$ Quaternions. $\neq D_4$.

$D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$

4 1d irreps.

1 2d irrep.

Pick L_1, L_2, L_3 & V

$\leftarrow$ 1d $\longrightarrow$ 2d.

find rels ....

$L_i^2 \xrightarrow{\alpha_i} \Lambda^2 V$, $\quad (\Lambda^2 V)^2 \overset{\beta}{=} L_1 L_2 L_3$

$\omega = \beta^2 \alpha_1 \alpha_2 \alpha_3 \in (\Lambda^2 V)^{-1}$ (Cartan matrix $\begin{pmatrix} -2 & 0 & 0 & 1 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & -2 & 1 \\ 1 & 1 & 1 & -2 \end{pmatrix}$

$$\mathrm{Sym}^2 V \xrightarrow{B} L_1 \oplus L_2 \oplus L_3$$

GIT problem is $4 + 9 = 13$ dim vector space $\curvearrowleft GL_1^3 \times GL_2$

(iii) $\mathrm{Sym}^2 V$ pairs to $(\Lambda^2 V)^2 \xrightarrow{\omega} \Lambda^2 V$

$L_1 \oplus L_2 \oplus L_2$ pairs to $\Lambda^2 V$

Rel is "B an isometry"

really 3 sets of eqn's, each of which reduce to this if $\det B \neq 0$.