# Uncommon Knowledge in Multiparty Auctions

## Subtitle

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# Overview

# First-price Sealed-bid Auction

## First-price Sealed-bid Auction

- Bidders participate in the auction simultaneously, and the highest bidder wins.
- No bidder knows how much the other auction participants have bid.
- Each bidder knows his own value for the object, and that value is not influenced by the other bidders. (Independence)

Assuming all bidders seek to maximize their expected utilities, our goal is to find the "best strategy".

# Pure Strategy Nash Equilibrium Solution

## Assumptions

For simplicity (but not essential), we assume that everyone's utility function is linear in money. The Pure Strategy Nash Equilibrium solution (PSNE) models the $n$th bidder's private value as a random variable $V_n$ that is independently drawn from his type distribution $G_n(v_n)$.

- the $G_n$ have common and compact support $[\underline{v}, \bar{v}]$;
- the $G_n$ are continuous with strictly positive densities $g_n(v_n) > 0$ on the support;
- the $G_n$ are common knowledge and known to be mutual, but actual realizations $v_n$ are private information.

# Pure Strategy Nash Equilibrium Solution

## Objective

The goal is to use the type distributions to determine each participant's bidding function $\beta_n(v_n)$. This bidding function prescribes his optimal bid $b_n$ given his value $v_n$ and his knowledge of the type distributions of all of his opponents and the type distribution they attribute to him.

## PSNE solutions

Let $\beta_{-n}$ be the collection of optimal bidding functions except that of the $n$th bidder. A PSNE is the $N$-tuple of bidding functions such that for all $n$, $\beta_n$ is a best response function to every other strategy in the collection $\beta_{-n}$.

# Pure Strategy Nash Equilibrium Solution

## Assumptions

The first two assumptions are easily weakened, but not the third. However, this common knowledge assumption is strong and often unrealistic. We think it is entirely reasonable for two bidders to have different beliefs about the type distribution of a third bidder.

# Symmetric Auctions

## Symmetric Auctions

In a symmetric auction, all $G_n(v_n)$ are equal to $G(v_n)$, such that all $N$ bidding functions are the same, and we denote that function by $\beta(v)$.

Each bidder's expected profit is the difference between his value $v$ and his bid $\beta(v)$ times the probability that all the $N-1$ other bids are less than $\beta(v)$. The probability of winning with a bid of $\beta(v)$ is $G(v)^{N-1}$.

The bidding function $\beta(v)$ solves

$$\beta(v) = \text{argmax}_{w \in R^+}[v - \beta(w)]G(w)^{N-1}$$

for $w$ in some open ball around $v$.

# Symmetric Auctions

## Solutions to a symmetric auction

$$b(v) = v - \frac{\int_{\underline{v}}^{v} G(z)^{N-1}\, dz}{G(v)^{N-1}}$$

which usually requires numerical solution.

# Asymmetric Auctions

## Asymmetric Auctions

In an asymmetric auction, each bidder has his/her own $G_n(v_n)$ and bidding function $\beta_n(v)$.
The goal is to solve the following set of equations

$$\beta_i(v_i) = \text{argmax}_{w \in R^+}[v_i - \beta_i(w)]\mathcal{P}(\beta_j(v_j) < \beta_i(w) \quad \forall j \neq i)$$

for each participant $i$.

## Notation

We denote $F_i(w) = \mathcal{P}(\beta_j(v_j) < \beta_i(w) \quad \forall j \neq i)$.

# Mirroring Equilibrium

## Mirroring Equilibrium

Instead of assuming common knowledge, we take the adversarial risk analysis (ARA) perspective. A *mirroring equilibrium* is the solution that each bidder believes his/her opponents are modeling all participants as seeking a PSNE.

# Mirroring Equilibrium

**Proposition 1.**

In an asymmetric auction with common knowledge, if all type distributions are atomless, continuous and differentiable except at a finite number of points, then there is a unique solution such that all the bidding functions are continuous.

# Mirroring Equilibrium

**Proof:** Let $(G_1, G_2, \ldots, G_N)$ denote a set of continuous type distributions that are differentiable except at $K_n$ points. For any point $x_i$ at which $G_n$ is not differentiable, there is an open interval $(x_i - \delta, x_i + \delta)$ such that $G_n$ is differentiable at all points in that interval except $x_i$. Replace the value of $G_n$ on those intervals by a monotonic increasing differentiable function that is also differentiable at $x_i - \epsilon$ and $x_i + \delta$.

For $m = 1, 2, \ldots$, let $G_{nm}$ be the constructed distributions described above for the intervals $(x_i - \delta/m, x_i + \delta/m)$. Then, for $m' > m$,

$$\|G_{nm} - G_{n,m'}\| = \int_{\underline{v}}^{\bar{v}} (G_n m(x) - G_{nm'}(x))^2 \, dx$$

$$= \sum_{i=1}^{K_n} \int_{x_i - \delta/m}^{x_i + \delta/m} (G_n m(x) - G_{nm'}(x))^2 \, dx \leq 2K_n \delta/m$$

# Mirroring Equilibrium

**Proof (Continued):** since there are $K_n$ regions of integration where the functions are not equal, and since the value on that region has to be less than or equal to the width of the interval $(2\delta)$ times the maximum possible discrepancy (which is 1, since these are distribution functions). For any $\epsilon > 0$, $\|G_{nm} - G_{nm'}\| < \epsilon$ when $m > 2K_n/\epsilon$. By Cauchy's convergence criterion, $\lim G_{nm}$ exists, and this holds for all $G_n$.
Each $G_{nm}$ is differentiable, and Lebrun (1999) proved that the corresponding bidding distribution $F_{nm}$ exists, is unique, and is differentiable.

# Mirroring Equilibrium

**Proof (Continued):**
Lebrun (1999) proved the existence of unique differentiable monotone bidding functions $\beta = (\beta_1, \ldots \beta_N)^\top$ for the case when all type distributions are differentiable. Thus, for the type distributions $(G_{1k}, G_{2k}, \ldots, G_{Nk})$, there exist differentiable bidding functions $(\beta_{1k}, \beta_{2k}, \ldots, \beta_{Nk})$.
Each $\beta_{nk}$ is nonnegative and bounded above by $\bar{v}$. Monotonicity on a compact interval implies that the derivatives are uniformly bounded and thus equicontinous. The Arzelà-Ascoli theorem ensures that the sequence $(\beta_{1k}, \beta_{2k}, \ldots \beta_{Nk})$ contains a convergent subsequence, and its limit must be unique and a.e. differentiable. $\square$

# Mirroring Equilibrium

**Proposition 2.**

For type distributions that are a.e. differentiable, the mirroring equilibrium problem has a unique solution that is a.e. differentiable.

# Mirroring Equilibrium

**Proof:** From Proposition 1, a common knowledge auction has a unique PSNE solution that is continuous and a.e. differentiable.
Bidder 1 models each opponent's analysis of the auction as a common knowledge auction, although different opponents may be modeled as believing different common knowledge. Thus, Bidder 1 finds a unique, continuous, almost everywhere differentiable distribution for the bid of each opponent. Since this is an independent private value auction, the product of those distributions determines Bidder 1's bidding function, and the product is continuous and almost everywhere differentiable.

# Level-$k$ Thinking

## Level-$k$ Thinking

Level-$k$ thinking is an alternative to the PSNE proposed by Stahl and Wilson (1994, 1995). It supposes that opponents reason in an "I think that you think that I think ..." fashion to a fixed depth of $k$ plies. A level-0 thinker is non-strategic, perhaps acting at random. A level-1 thinker views his opponent as a level-0 thinker, and makes the best response to that model. A level-2 thinker models his opponent as a level-1 thinker, and so forth.

# Level-$k$ Thinking

## Two-player Auction

To illustrate level-$k$ thinking, we start with a two-player auction. Bidder 1 might model Bidder 2 as nonstrategic. Bidder 1 does not know that fraction nor the value, but he has subjective distributions with densities $h(p)$ and $g(v)$ for them. Then Bidder 1 calculates his distribution $F(x)$ for Bidder 2's bid:

$$F(x) = [PV \leq x] = \int_0^\infty \int_0^{x/v} h(p)g(v)\, dpdv.$$

Then Bidder 1 bids $x^*$ such that

$$x^* = \text{argmax}_{x \in^+} (x_0 - x)F(x)$$

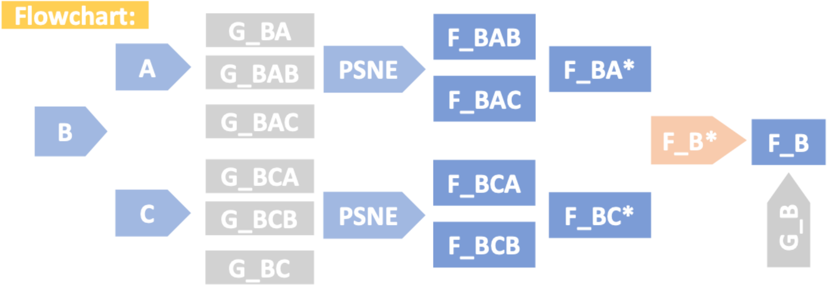
where $x_0$ is Bidder 1's true value.

# Level-$k$ Thinking

## Three-player Auction

Without lose of generality, we stand with Bidder 1. Now suppose that Bidder 1 is a level-2 thinker who models all of the other participants as level-1 thinkers.
Bidder 1 considers Bidders 2 and 3 separately. He assumes Bidder 2 calculates $F_{21}$ and $F_{23}$, and makes the bid that maximizes his expected utility against the maximum of the two corresponding random variables, which has distribution $F_2^*$. Bidder 1 does the same calculation for Bidder 3 and finds the analogous $F_3^*$. Finally, Bidder 1 makes the bid that maximizes his expected utility against the maximum of two independent bids, one from $F_2^*$ and one from $F_3^*$.

# Level-$k$ Thinking



# Level-$k$ Thinking

## Numerical Example

To illustrate for the case of three bidders, assume Bidder 1 believes that all participants think that all the types are triangular distributions with support on $[0, 1]$. Suppose Bidder 1 thinks Bidder 2 thinks that his type has its peak at $1/7$ and that Bidder 3's type has its peak at $2/7$. Similarly, he believes that Bidder 3 thinks Bidder 1's type has its peak at $3/7$ and Bidder 2's type has its peak at $4/7$. Finally, Bidder 1 thinks Bidder 2's type has its peak at $5/7$ and Bidder 3's type has its peak at $6/7$. Additionally, Bidder 1 thinks that Bidder 2 believes people bid a random fraction of the their true value, and that fraction has the Beta(9, 1) distribution. Similarly, he thinks Bidder 3 believes that the random fraction people bid has the Beta(8, 1) distribution.

# Level-$k$ Thinking

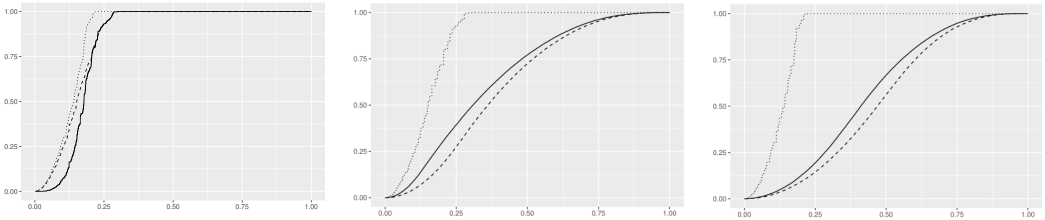



Figure: **Left**: The solid line is $F_1^*$, the dashed line is $F_{12}$, and the dotted line is $F_{13}$.
**Middle**: The solid line is $F_{21}$, the dashed line is $F_{23}$, and the dotted line is $F_2$.
**Right**: The solid line is $F_{31}$, the dashed line is $F_{32}$, and the dotted line is $F_3$.