Singularities of General Systems of Differential Equations

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SYMBIONT PROJECT

What is this all about?

Many forms of **singular behaviour** appear in differential equations:

- (derivatives of) **solutions** become singular $\rightsquigarrow$ "blow-up", "shock"
- **stationary points** or **equilibria** of vector fields
- **bifurcations** in parameter dependent systems
- **singular integrals** (additional solutions not contained in the "general integral")
- **multi-valued solutions** (like "breaking waves")
- ...

here: singularities as "special" points on a geometric model of (general systems of) differential equations

What is this all about?

Our goals:

- use **commutative** and **differential algebra** to obtain **effective differential topological** framework for **defining** and **detecting** singularities of **arbitrary systems** of ordinary or partial differential equations
- **analyse local solution behaviour**, i. e. study singular initial value problems

Today: only definition and detection of singularities

What is a Differential Equation?
Setting the Arena

Choose base field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and consider (local) smooth/analytic functions $\mathbf{s}\colon \Omega \subseteq \mathbb{K}^n \to \mathbb{K}^m$

ℓ-th order **jet bundle** $\mathcal{J}_\ell$ $\rightsquigarrow$ set of all equivalence classes $[\mathbf{s}]^{(\ell)}_{\bar{\mathbf{x}}}$ containing all functions with the same Taylor polynomial of degree ℓ at $\bar{\mathbf{x}} \in \mathbb{K}^n$ as function $\mathbf{s}$

- manifold diffeomorphic to **affine space** $\mathbb{K}^{d_\ell}$ with $d_\ell = n + m\binom{n+\ell}{\ell}$
- local coordinates: **independent variables** $x^1, \dots, x^n$, **dependent variables** $u^1, \dots, u^m$, **derivatives** u^α_μ with $|\mu| \leq \ell$
- possesses **natural fibrations** $\pi^\ell_k\colon \mathcal{J}_\ell \to \mathcal{J}_k$ for $0 \leq k < \ell$ and $\pi^\ell\colon \mathcal{J}_\ell \to \mathbb{K}^n$

What is a Differential Equation?
Setting the Arena

Different roles of different types of local coordinates encoded in **contact structure** of jet bundle $\mathcal{J}_\ell$

distribution $\mathcal{C}_\ell \subset T\mathcal{J}_\ell$ generated by n π^ℓ**-transversal** vector fields

$$C_i^{(\ell)} = \partial_{z^i} + \sum_\alpha \sum_{0 \leq \ell|\mu| < \ell} u^\alpha_{\mu+1_i} \partial_{u^\alpha_\mu} \qquad 1 \leq i \leq n$$

and $m\binom{n+\ell-1}{\ell}$ π^ℓ**-vertical** vector fields

$$C^\mu_\alpha = \partial_{u^\alpha_\mu} \qquad 1 \leq \alpha \leq m, \ \ |\mu| = \ell$$

What is a Differential Equation?
The Classical Geometric Answer

Definition

Differential equation of order ℓ $\rightsquigarrow$ fibred submanifold $\mathcal{R}_\ell \subseteq \mathcal{J}_\ell$ such that restricted projection $\pi^\ell : \mathcal{R}_\ell \to \mathbb{K}^n$ surjective submersion

- No distinction between **scalar equations** and **systems**
- Only of limited use for **effective computations**
- Rather **strict** conditions, often not met in **applications**
  $xu' = 1, \quad (u')^2 + u^2 + x^2 = 1, \quad \dots$

Proscribes all kinds of singularities discussed here!

What is a Differential Equation?

A More Relaxed and Algebraic Answer

Definition

- **Algebraic jet set** (of order ℓ) $\rightsquigarrow$ locally Zariski closed subset $\mathcal{R}_\ell \subseteq \mathcal{J}_\ell$ (i. e. difference of two varieties)
- **Algebraic differential equation** (of order ℓ) $\rightsquigarrow$ algebraic jet set $\mathcal{R}_\ell$ such that Euclidean closure of $\pi^\ell(\mathcal{R}_\ell)$ is $\mathbb{K}^n$

- globally described by equations and **inequations**
- restricts to equations with **polynomial** nonlinearities
- **admits** equations like $xu' = 1$
- still **excludes** equations like $x = 0$

But We Want Systems!

The Algebraic Case

Polynomial ring $\mathbb{K}[x^1, \dots, x^n]$ with total ordering on variables

- **leader** $\operatorname{ld} p$ $\rightsquigarrow$ largest variable in polynomial p
- consider p as **univariate** polynomial in $\operatorname{ld} p$
  - **initial** $\operatorname{init} p$ $\rightsquigarrow$ leading coefficient of p
  - **separant** $\operatorname{sep} p$ $\rightsquigarrow$ $\partial p / \partial(\operatorname{ld} p)$

Definition

- **Algebraic system** $\rightsquigarrow$ polynomial equations and *in*equations

$$\mathcal{S} = \{p_1 = 0, \dots, p_s = 0,\ q_1 \neq 0, \dots, q_t \neq 0\}$$

- **Solution set** (locally closed wrt Zariski topology)

$$\operatorname{Sol} \mathcal{S} = \{\mathbf{x} \in \mathbb{K}^n \mid p_i(\mathbf{x}) = 0,\ q_j(\mathbf{x}) \neq 0\}$$

But We Want Systems!

The Algebraic Case

Definition

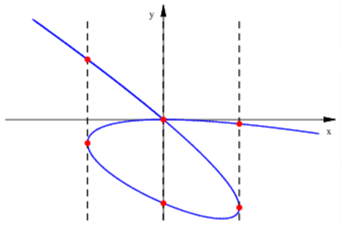
- S **simple** algebraic system $\rightsquigarrow$
  - **triangular:** $\left|\{\operatorname{ld} p_i,\ \operatorname{ld} q_j\} \setminus \{1\}\right| = s + t$
  - **constant degree:** no equation $\operatorname{init} p_i = 0$ or $\operatorname{init} q_j = 0$ possesses solution in $\operatorname{Sol} \mathcal{S}$
  - **square-free:** dito for separants
- **Thomas decomposition** of arbitrary algebraic system $\mathcal{S}$ $\rightsquigarrow$ construction of finitely many **simple** systems $\mathcal{S}_1, \dots, \mathcal{S}_k$ such that $\operatorname{Sol} \mathcal{S}$ **disjoint** union of all $\operatorname{Sol} \mathcal{S}_i$

Thomas decomposition always **exists** over *algebraically closed* field (Thomas 1937), can be determined **algorithmically** (very expensive) and is **implemented** in `Maple` (Bächler, Gerdt, Lange-Hegermann, Robertz 2012)

But We Want Systems!

The Algebraic Case

consider $S = \{ p = y^3 + (3x+1)y^2 + (3x^2+2x)y + x^3 = 0 \}$
(non-simple algebraic "system")



Thomas decomposition

- $S_1 = \{p = 0,\ 27x^3 - 4x \neq 0\}$
- $S_2 = \{6y^2 - (27x^2 - 12x - 6)y - 3x^2 + 2x = 0,\ 27x^3 - 4x = 0\}$

(**Caution:** real picture, but decomposition over complex numbers!)

But We Want Systems!
The Differential Case

Ring of differential polynomials

- $\mathbb{F} = \mathbb{K}(x^1, \dots, x^n)$ differential field of rational functions with derivations $\delta_i = \partial/\partial x^i$
- finitely many differential unknowns: $U = \{u^1, \dots, u^m\}$
  $\rightsquigarrow$ **jet variables** $u^\alpha_\mu = \delta^\mu u^\alpha$
- $\mathbb{F}\{U\} = \mathbb{F}\left[u^\alpha_\mu \mid 1 \leq \alpha \leq m, \mu \in \mathbb{N}_0^n\right]$ (∞ many variables!)
  derivations can be extended: $\delta_i u^\alpha_\mu = u^\alpha_{\mu+1_i}$
- distinguish:
  - **algebraic ideal:** $\langle p_1, \dots, p_s \rangle$
  - **differential ideal:** $\langle p_1, \dots, p_s \rangle_\Delta$ (closed under derivations)
- set $\mathcal{D} = \mathbb{K}\left[x^i, u^\alpha_\mu\right] \subset \mathbb{F}\{U\}$ $\rightsquigarrow$ $\mathcal{D}_\ell = \mathbb{K}\left[x^i, u^\alpha_\mu \mid |\mu| \leq \ell\right]$
- **jet bundle** $\mathcal{J}_\ell$ $\rightsquigarrow$ affine space $\mathbb{K}^{d_\ell}$ with coordinate ring $\mathcal{D}_\ell$

But We Want Systems!

The Differential Case

Ranking on $\mathbb{F}\{U\}$

- **total ordering** $\prec$ of jet variables
- $u^\alpha \prec \delta_i u^\alpha$
- $u^\alpha_\mu \prec u^\beta_\nu \quad \Longrightarrow \quad \delta_i u^\alpha_\mu \prec \delta_i u^\beta_\nu$

allows to extend concepts like **leader**, **initial** or **separant**

Definition

- **Differential system** $\leadsto$ finite set of differential polynomial equations and *in*equations

$$\mathcal{S} = \{p_1 = 0, \dots, p_s = 0,\ q_1 \neq 0, \dots, q_t \neq 0\}$$

- **Solution set** $\leadsto$ consider formal power series solutions (different function spaces possible)

But We Want Systems!

The Differential Case

Definition

- S **simple** differential system $\rightsquigarrow$
  - S **simple algebraic system** in the finitely many effectively occuring jet variables
  - equations **involutive** for Janet division
  - no leader of inequation derivative of leader of equation
- **Thomas decomposition** of arbitrary differential system $\mathcal{S}$ $\rightsquigarrow$ construction of finitely many **simple** systems $\mathcal{S}_1, \dots, \mathcal{S}_k$ such that $\operatorname{Sol}\mathcal{S}$ **disjoint** union of all $\operatorname{Sol}\mathcal{S}_i$

Thomas decomposition always **exists** over algebraically closed field (Thomas 1937), **algorithmically** computable via combination of algebraic Thomas decomposition and Janet–Riquier theory, **implemented** in `Maple` (Bächler, Gerdt, Lange-Hegermann, Robertz 2012)

Starting point in applications: **differential system** S

Naive construction of associated **algebraic jet set** in order ℓ:

1. Differential ideal $\hat{\mathcal{I}}_{\text{diff}}(S) = \langle p_1, \dots, p_s \rangle_\Delta \subseteq \mathcal{D}$
2. Algebraic ideal $\hat{\mathcal{I}}_\ell(S) = \hat{\mathcal{I}}_{\text{diff}}(S) \cap \mathcal{D}_\ell$
3. Algebraic ideal $\mathcal{K}_\ell(S) = \langle \prod_{\text{ord}(q_j) \leq \ell} q_j \rangle_{\mathcal{D}_\ell}$
4. Algebraic jet set $\hat{\mathcal{R}}_\ell(S) = \text{Sol}\big(\hat{\mathcal{I}}_\ell(S)\big) \setminus \text{Sol}\big(\mathcal{K}_\ell(S)\big)$

Starting point in applications: **differential system** S

Naive construction of associated **algebraic jet set** in order ℓ:

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Construction leads to many **problems**:

- Ideals $\hat{\mathcal{I}}_\ell(S)$ too small (not radical)
- Do not necessarily obtain algebraic differential equation
- Effective determination of $\hat{\mathcal{I}}_\ell(S)$ difficult
- Jet set $\hat{\mathcal{R}}_\ell(S)$ possibly too small (inequations too strong)

Better: assume S **simple** differential system

1. Differential ideal $\mathcal{I}_{\mathrm{diff}}(S) = \hat{\mathcal{I}}_{\mathrm{diff}}(S) : \left(\prod_j \operatorname{init}(p_j) \operatorname{sep}(p_j)\right)^{\infty}$
2. Algebraic ideal $\mathcal{I}_\ell(S) = \mathcal{I}_{\mathrm{diff}}(S) \cap \mathcal{D}_\ell$
3. Algebraic ideal $\mathcal{K}_\ell(S) = \langle \prod_{\mathrm{ord}(q_j) \leq \ell} q_j \rangle_{\mathcal{D}_\ell}$
4. Algebraic jet set $\mathcal{R}_\ell(S) = \mathrm{Sol}\big(\mathcal{I}_\ell(S)\big) \setminus \mathrm{Sol}\big(\mathcal{K}_\ell(S)\big)$

Proposition

- *$\mathcal{I}_{\mathrm{diff}}(S)$ (and thus $\mathcal{I}_\ell(S)$) **radical*** *(Robertz 2014)*
- *$\mathcal{I}_\ell(S)$ easily computable*
- $\forall\, k > 0 : \pi^{\ell+k}_{\ell}\big(\mathcal{R}_{\ell+k}(S)\big) = \mathcal{R}_\ell(S)$

Bridging the Gap

Locally Integrable Equations

Definition

Algebraic differential equation $\mathcal{R}_\ell$ **locally integrable** $\rightsquigarrow$
$\exists$ Zariski open and dense subset $\mathcal{L}_\ell \subseteq \mathcal{R}_\ell$ such that at least one solution goes through each point $\rho \in \mathcal{L}_\ell$

Proposition

S simple differential system $\Longrightarrow$ *Zariski closure* $\overline{\mathcal{R}_\ell(S)} = \mathrm{Sol}\big(\mathcal{I}_\ell(S)\big)$ *locally integrable algebraic differential equation*

(Essentially a consequence of Riquier's Theorem)

What is a Singularity?
Some Useful Structures

Definition

Given point ρ on algebraic jet set $\mathcal{R}_\ell \subseteq \mathcal{J}_\ell$

- **Vessiot cone** $\mathcal{V}_\rho[\mathcal{R}_\ell] = \mathsf{C}_\rho\mathcal{R}_\ell \cap \mathsf{C}_\ell|_\rho$
- **Symbol cone** $\mathcal{N}_\rho[\mathcal{R}_\ell] = \mathcal{V}_\rho[\mathcal{R}_\ell] \cap \mathsf{V}_\rho\pi^\ell_{\ell-1}$

- At smooth points ρ cones **linear spaces** $\rightsquigarrow$ computable as solution spaces of linear systems of equations
- Vessiot cone at ρ contains all potential **infinitesimal solutions (integral elements)**
- Use tangent cone as it contains all **limits of secants**
- **Dimensions** and **orientations** of cones depend generally on ρ

What is a Singularity?

Finally the Definitions…

Definition

$\mathcal{R}_\ell \subseteq \mathcal{J}_\ell$ locally integrable algebraic differential equation

1. $\rho \in \mathcal{R}_\ell$ **algebraic singularity** $\rightsquigarrow$ ρ **non-smooth** point of $\mathcal{R}_\ell$
2. $\rho \in \mathcal{R}_\ell$ **regular** $\rightsquigarrow$ ρ smooth and $\exists$ Euclidean open neighbourhood $\rho \in \mathcal{U} \subseteq \mathcal{R}_\ell$ such that $\mathcal{V}[\mathcal{R}_\ell]|_\mathcal{U}$ **regular** and decomposable as $\mathcal{V}[\mathcal{R}_\ell]|_\mathcal{U} = \mathcal{N}[\mathcal{R}_\ell]|_\mathcal{U} \oplus \mathcal{H}$ with ***n*-dimensional**, transversal, **involutive**, smooth distribution $\mathcal{H} \subseteq T\mathcal{U}$
3. $\rho \in \mathcal{R}_\ell$ **regular singular** $\rightsquigarrow$ ρ smooth and $\exists$ Euclidean open neighbourhood $\rho \in \mathcal{U} \subseteq \mathcal{R}_\ell$ such that $\mathcal{V}[\mathcal{R}_\ell]|_\mathcal{U}$ **regular** but $\dim \mathcal{V}_\rho[\mathcal{R}_\ell] - \dim \mathcal{N}_\rho[\mathcal{R}_\ell] < n$
4. $\rho \in \mathcal{R}_\ell$ **irregular singular** $\rightsquigarrow$ ρ smooth, but $\nexists$ Euclidean open neighbourhood $\rho \in \mathcal{U} \subseteq \mathcal{R}_\ell$ such that $\mathcal{V}[\mathcal{R}_\ell]|_\mathcal{U}$ regular

What is a Singularity?
Finally the Definitions…

- Definition **relative** to $\mathcal{R}_\ell$
- Algebraic singularities detectable with **Jacobian criterion** $\rightsquigarrow$ linear algebra
- Items 3 and 4 $\rightsquigarrow$ **geometric singularities** (critical points for restriction of canonical projection map $\pi^\ell : \mathcal{R}_\ell \to \mathbb{K}^n$)
- For **equations of finite type**, no neighbourhoods necessary $\rightsquigarrow$ **pointwise** criteria possible, as “right” dimensions a priori known ($\Longrightarrow$ involutivity no issue!)
- Except for **involutivity condition**, distinction between items 2–4 corresponds to analysis of linear system of equations
- Involutivity condition difficult to analyse $\rightsquigarrow$ taxonomy possibly **incomplete** for **partial** differential equations

What is a Singularity?

… and an Example

Consider equation $\mathcal{R}_2$ for unknown function $u = u(x, y)$ given by

$$x^2 u_{xx} + x u_x + (x-1)^2 u = 0\,, \qquad (1-y^2) u_{yy} + 2y u_y + 2u = 0$$

Seven distinct cases arise:

1. $x \neq 0 \wedge y^2 - 1 \neq 0 \quad \rightsquigarrow$ **regular**: $\dim \mathcal{V}_\rho[\mathcal{R}_2] = 3$, $\dim \mathcal{H}_\rho = 2$
2. $x = 0 \wedge y^2 - 1 \neq 0 \wedge (u_x \neq 0 \vee u_y \neq 0) \quad \rightsquigarrow$ **regular singular**: $\dim \mathcal{V}_\rho[\mathcal{R}_2] = 3$, $\dim \mathcal{H}_\rho = 1$
3. $x \neq 0 \wedge y^2 - 1 = 0 \wedge (y u_x + u_{xy} \neq 0 \vee u \neq 0) \quad \rightsquigarrow$ as 2.
4. $x = 0 \wedge y^2 - 1 = 0 \wedge (y u_x + u_{xy} \neq 0 \vee u_x \neq 0) \quad \rightsquigarrow$ **irregular singular**: $\dim \mathcal{V}_\rho[\mathcal{R}_2] = 4$, $\dim \mathcal{H}_\rho = 1$
5. $x = 0 \wedge y^2 - 1 \neq 0 \wedge u_x = 0 \wedge u_y = 0 \quad \rightsquigarrow$ **(purely) irregular singular**: $\dim \mathcal{V}_\rho[\mathcal{R}_2] = 4$, $\dim \mathcal{H}_\rho = 2$
6. $x \neq 0 \wedge y^2 - 1 = 0 \wedge u_x = 0 \wedge u_y = 0 \quad \rightsquigarrow$ as 5.
7. $x = 0 \wedge y^2 - 1 = 0 \wedge u_x = 0 \wedge u_y = 0 \quad \rightsquigarrow$ as 5. but with $\dim \mathcal{V}_\rho[\mathcal{R}_2] = 5$

What is a Singularity?

Singular! So what?

Some possible effects of singularities for **real** ordinary differential equations:

- Only **one-sided** solutions either starting or ending in singularity (generically the case at regular singularities)
- **Multiple** solutions (even infinitely many ones)
- Solutions of **finite regularity**

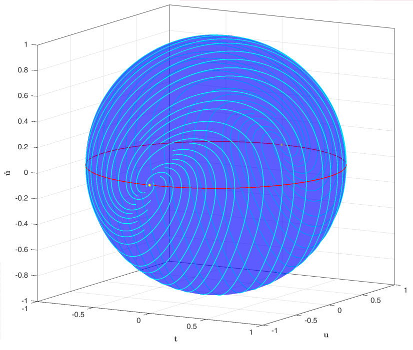
Example

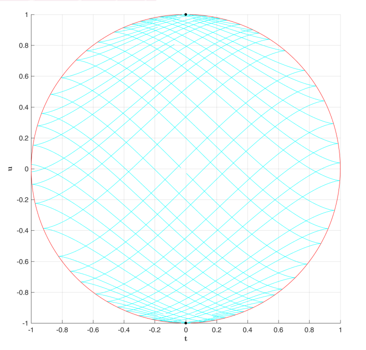
Consider $xu'' = (u')^2 + x - k^2/4$ at $\rho = (0, c, k/2)$ for $k > 0$

- $k \in \mathbb{N}$ $\rightsquigarrow$ infinitely many solutions all in $\mathcal{C}^k \setminus \mathcal{C}^{k+1}$
- otherwise $\rightsquigarrow$ one smooth solution and infinitely many solutions in $\mathcal{C}^\ell \setminus \mathcal{C}^{\ell+1}$ with $\ell = \lceil k \rceil$

$\rho = (0, c, -k/2)$ $\rightsquigarrow$ unique smooth solution

Example: $(u')^2 + u^2 + x^2 = 1$

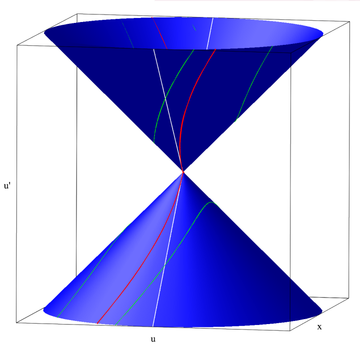


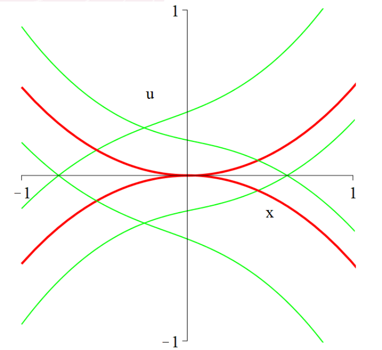


equator $\rightsquigarrow$ **regular** singularities
“east” and “west pole” $\rightsquigarrow$
two **irregular** singularities

two solutions either begin or end at each **regular** singularity; infinitely many **approach irregular** singularity

Example: $(u')^2 - u^2 - x^2 = 0$





vertex of cone $\rightsquigarrow$ **algebraic** singularity (Vessiot cone in white)

four solutions through origin $\rightsquigarrow$ two $\mathcal{C}^\omega$, two $\mathcal{C}^1$

What makes a Point Regular?
The Quest for a Deeper Meaning

- $\mathcal{R}_\ell$ locally integrable equation of **finite type** (i. e. $\mathcal{N}_\rho[\mathcal{R}_\ell] = 0$ for all points ρ on Zariski dense subset) $\rightsquigarrow$ $\rho \in \mathcal{R}_\ell$ regular, if and only if sufficiently small neighbourhood $\rho \in U \subseteq \mathcal{R}_\ell$ **uniquely foliated** by prolonged solutions
- $\mathcal{R}_\ell$ locally integrable equation of **infinite type** $\rightsquigarrow$ **infinitely many foliations** by prolonged solutions exist (each Vessiot connection induces one) $\rightsquigarrow$ further types of singular behaviour possible???

Singularities are Singular!

First Main Result

Theorem

*S **simple** differential system with no (in)equation of order* $> \ell \implies$
regular points in $\overline{\mathcal{R}_\ell(S)}$ contain Zariski open and dense subset

Proof: (easy for equations of finite type via **Riquier's theorem**)

- S Janet basis $\implies$ $\exists$ Zariski open and dense subset $\mathcal{F}_\ell \subseteq \mathcal{R}_\ell(S)$ defining **formally integrable** differential equation
- S induces Janet basis of **symbol module** $\mathcal{M}_\rho[\mathcal{R}_\ell(S)]$ for $\rho \in \mathcal{F}_\ell$
- this implies $\ell \geq \operatorname{reg} \mathcal{M}_\rho[\mathcal{R}_\ell(S)]$ (WMS 2002)
- this implies $\mathcal{F}_\ell$ **involutive** differential equation (WMS 2002)
- **Vessiot theory** $\implies$ existence of involutive complement $\mathcal{H}$ (Fesser 2008)

You Prefer Regular?

The Regularity Decomposition

Definition

Algebraic jet set $\mathcal{R}_\ell \subseteq \mathcal{J}_\ell$ **regular** $\rightsquigarrow$

- $\mathcal{R}_\ell$ smooth **manifold**
- Vessiot spaces define smooth **vector bundle** over $\mathcal{R}_\ell$
- Symbol spaces define smooth **vector bundle** over $\mathcal{R}_\ell$

You Prefer Regular?

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- Symbol spaces define smooth **vector bundle** over $\mathcal{R}_\ell$

S **simple** differential system, $\overline{\mathcal{R}_\ell(S)}$ associated algebraic jet set for ℓ sufficiently high with unique decomposition into **irreducible components** $\overline{\mathcal{R}_\ell(S)} = \mathcal{R}_{\ell,1} \cup \cdots \cup \mathcal{R}_{\ell,t}$

Definition

Regularity decomposition of $\mathcal{R}_{\ell,k}$ $\rightsquigarrow$ representation as **disjoint** union of finitely many **regular** algebraic jet sets $\mathcal{R}^{(i)}_{\ell,k}$ and of set $\mathrm{ASing}(\mathcal{R}_{\ell,k})$ of algebraic singularities

You Prefer Regular?
The Second Main Result

Theorem

*Regularity decompositions exist for **any** differential system for a sufficiently high order ℓ and can be algorithmically computed.*

Basic idea:

1. Use **differential** Thomas decomposition to split into simple differential systems
2. Construct corresponding algebraic jet sets
3. Apply on each jet set **algebraic** Thomas decomposition to linear systems describing tangent and Vessiot spaces
4. Analyse results

Notion of *"regular differential equation"* in geometric theory requires **prolongations** $\rightsquigarrow$ need to associate **differential system** $S(\mathcal{R}_\ell)$ with given algebraic jet set $\mathcal{R}_\ell \subseteq \mathcal{J}_\ell$

Construction of $S(\mathcal{R}_\ell)$

$\mathcal{R}_\ell$ locally Zariski closed subset of $\mathcal{J}_\ell$ $\Longrightarrow$ $\mathcal{R}_\ell$ solution set of **algebraic system** $S(\mathcal{R}_\ell) \subset \mathcal{D}_\ell$ $\rightsquigarrow$ consider $S(\mathcal{R}_\ell) \subset \mathcal{D}$ as **differential system**!

Caveat: $S(\mathcal{R}_\ell)$ not necessarily **differentially consistent** and generally even $\mathcal{R}_\ell(S(\mathcal{R}_\ell)) \subsetneq \mathcal{R}_\ell$

When is a Differential Equation Regular?

Attempt of a Rigorous Definition

Definition

Algebraic differential equation $\mathcal{R}_\ell \subseteq \mathcal{J}_\ell$ **regular** $\rightsquigarrow$ associated differential system $S(\mathcal{R}_\ell)$ satisfies

1. $\mathcal{I}_{\text{diff}}\big(S(\mathcal{R}_\ell)\big)$ prime differential ideal
2. $\mathcal{R}_\ell\big(S(\mathcal{R}_\ell)\big) = \mathcal{R}_\ell$
3. $\forall k \geq 0 : \mathcal{R}_{\ell+k}\big(S(\mathcal{R}_\ell)\big)$ regular algebraic jet set

Note: item 3 comprises **infinitely many** conditions and thus unclear how to verify **effectively** whether or not given algebraic differential equation is regular

When is a Differential Equation Regular?

The Third Main Result

Theorem

*For each prime component $\mathcal{I}_{\ell,k}(S)$ of **simple** differential system S, output of our algorithm contains unique simple algebraic system $S^{\mathrm{gen}}_{\ell,k}$ such that its solution space $\mathrm{Sol}(S^{\mathrm{gen}}_{\ell,k})$ is*

1. ***regular differential equation***
2. ***Zariski dense** in $\mathrm{Sol}\big(\mathcal{I}_{\ell,k}(S)\big)$*
3. *entirely composed of **regular points***

Where Do We Go From Here?

- Effective theory for **real** differential equations $\rightsquigarrow$ extend to **semi-algebraic equations**, replace Thomas decomposition by **quantifier elimination**, ...
- **Complete** taxonomy even for equations of **infinite** type?
- What happens at **algebraic singularities**? $\rightsquigarrow$ so far only individual cases considered, no general results known
- **Local solution behaviour** at singularities of **ordinary** differential equations as analysable via **dynamical systems theory** $\rightsquigarrow$ extension of **partial** differential equations of **finite type** seems possible via **theory of singular foliations** $\rightsquigarrow$ extension to **partial** differential equations of **infinite type** unclear

All the Gory Details...

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