Method of moving frames and computing generalized Casimir operators

Roman Popovych

University of Vienna, Austria
&
Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine

Joint work with Vyacheslav Boyko & Jiri Patera

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Coadjoint representation

Consider a Lie algebra $\mathfrak{g}$ of dimension $\dim \mathfrak{g} = n < \infty$ over the complex or real field $\mathbb{F}$ (either $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$) and the corresponding connected Lie group G.

$\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ denotes the usual adjoint representation of G on $\mathfrak{g}$.
The image Ad_G of G under Ad is the inner automorphism group $\mathrm{Inn}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$.

Let $\mathfrak{g}^*$ be the dual space of the vector space $\mathfrak{g}$.

Definition

The map $\mathrm{Ad}^*\colon G \to \mathrm{GL}(\mathfrak{g}^*)$ defined for any $g \in G$ by the relation

$$\langle \mathrm{Ad}^*_g x, u\rangle = \langle x, \mathrm{Ad}_{g^{-1}} u\rangle \quad \text{for all } x \in \mathfrak{g}^* \text{ and } u \in \mathfrak{g}$$

is called the *coadjoint representation* of the Lie group G.

The image of G under Ad^* is a subgroup of $\mathrm{GL}(\mathfrak{g}^*)$ and is denoted by Ad^*_G.

The maximal dimension of orbits of Ad^*_G is called the *rank of the coadjoint representation* of G (and of $\mathfrak{g}$) and denoted by $\mathrm{rank}\,\mathrm{Ad}^*_G$.

It is a basis independent characteristic of the algebra $\mathfrak{g}$.

Orbits of dimension $\mathrm{rank}\,\mathrm{Ad}^*_G$ are called *regular* ones.

Definition

A function $F \in C^\infty(\Omega)$, where Ω is a domain in $\mathfrak{g}^*$, is called a (global in Ω) *invariant* of Ad^*_G if

$$F(\mathrm{Ad}^*_g x) = F(x) \text{ for all } g \in G \text{ and } x \in \Omega \text{ such that } \mathrm{Ad}^*_g x \in \Omega.$$

The set of invariants of Ad^*_G on Ω is denoted by $\mathrm{Inv}(\mathrm{Ad}^*_G)$ without an explicit indication of the domain Ω.

$\mathrm{Inv}(\mathrm{Ad}^*_G)$ can be considered as "the set of invariants of $\mathfrak{g}$" and denoted by $\mathrm{Inv}(\mathfrak{g})$.

Let below Ω is a neighborhood of a point from a regular orbit.

It can always be chosen in such a way that the group Ad^*_G acts regularly on Ω.

Then the maximal number $N_\mathfrak{g}$ of functionally independent invariants in $\mathrm{Inv}(\mathrm{Ad}^*_G)$ coincides with the codimension of the regular orbits of Ad^*_G,

$$N_\mathfrak{g} = \dim \mathfrak{g} - \mathrm{rank}\,\mathrm{Ad}^*_G.$$

Functionally independent invariants $F^l(x)$, $l = 1, \dots, N_\mathfrak{g}$, forms a *functional basis* of $\mathrm{Inv}(\mathrm{Ad}^*_G)$ (or *fundamental invariant* of Ad^*_G) since any element from $\mathrm{Inv}(\mathrm{Ad}^*_G)$ can be (uniquely) represented as a function of these invariants.

Fixing a basis

To calculate the invariants explicitly, one should fix a basis of the algebra $\mathfrak{g}$,

$$\mathcal{E} = (e_1, \dots, e_n).$$

It leads to fixing the dual basis

$$\mathcal{E}^* = (e_1^*, \dots, e_n^*)$$

in the dual space $\mathfrak{g}^*$ and to the identification of $\mathrm{Inn}(\mathfrak{g})$ and Ad_G^* with the associated matrix groups.

The basis elements $e_1, \dots, e_n$ satisfy the commutation relations

$$[e_i, e_j] = c_{ij}^k e_k,$$

where c_{ij}^k are components of the tensor of structure constants of $\mathfrak{g}$ in the basis $\mathcal{E}$.

Here and in what follows the indices i, j and k run from 1 to n and the summation convention over repeated indices is used.

Let

$$\mathfrak{g}^* \ni x \to \check{x} = (x_1, \dots, x_n) \in \mathbb{F}^n$$

be the coordinates in $\mathfrak{g}^*$ associated with $\mathcal{E}^*$.

It is well known that there exists a bijection between

- elements of the center of the universal enveloping algebra (i.e., *Casimir operators*) of $\mathfrak{g}$ and
- polynomial invariants of $\mathfrak{g}$ (which can be assumed defined globally on $\mathfrak{g}^*$).

Such a bijection is established, e.g., by the natural association $e_i \leftrightarrow x_i$ and the symmetrization operator Sym which acts on monomials by the formula

$$\mathrm{Sym}(e_{i_1} \cdots e_{i_r}) = \frac{1}{r!} \sum_{\sigma \in \mathrm{S}_r} e_{i_{\sigma_1}} \cdots e_{i_{\sigma_r}},$$

where $i_1, \ldots, i_r$ take values from 1 to n, $r \in \mathbb{N}$. The symbol S_r denotes the symmetric group on r letters.

[The symmetrization also can be correctly defined for rational invariants. If $\mathrm{Inn}(\mathrm{Ad}^*_G)$ has no a functional basis consisting of only rational invariants, the correctness of the symmetrization needs an additional investigation for each fixed algebra $\mathfrak{g}$ since general results on this subject do not exist.]

This is why elements of $\mathrm{Inv}(\mathrm{Ad}^*_G) = \mathrm{Inv}(\mathfrak{g})$ are naturally called *generalized Casimir operators* of $\mathfrak{g}$.

- **Specific methods for certain classes of Lie algebras**

Simple Lie algebras, Lie algebras with nontrivial Levi decomposition, ...

- **Infinitesimal approach**

Invariants $F(x_1,\dots,x_n)$ of Ad^*_G are found as solutions of the linear system of first-order partial differential equations

$$X_iF = c_{ij}^k x_k F_{x_j} = 0,$$

where $X_i = c_{ij}^k x_k \partial_{x_j}$ is the infinitesimal generator of the local 1-parameter group $\{\mathrm{Ad}^*_{\exp(\varepsilon e_i)}\}$ corresponding to e_i, the parameter ε runs through a neighborhood of zero in $\mathbb{F}$.

The mapping $e_i \to X_i$ gives a representation of the Lie algebra $\mathfrak{g}$.

- **Method of moving frames**

Let us recall some facts on the Cartan's method of moving frames in Fels–Olver version, simultaneously adapting them to the particular case of the coadjoint action of G on $\mathfrak{g}^*$.

Let $\mathcal{G} = \mathrm{Ad}^*_G \times \mathfrak{g}^*$ denote the trivial left principal Ad^*_G-bundle over $\mathfrak{g}^*$.

The right regularization $\widehat{R}$ of the coadjoint action of G on $\mathfrak{g}^*$ is the diagonal action of Ad^*_G on $\mathcal{G} = \mathrm{Ad}^*_G \times \mathfrak{g}^*$,

$$\widehat{R}_g(\mathrm{Ad}^*_h, x) = (\mathrm{Ad}^*_h \cdot \mathrm{Ad}^*_{g^{-1}}, \mathrm{Ad}^*_g x), \quad g, h \in G, \quad x \in \mathfrak{g}^*.$$

The action $\widehat{R}$ on the bundle $\mathcal{G} = \mathrm{Ad}^*_G \times \mathfrak{g}^*$ is regular and free.

We call $\widehat{R}$ the *lifted coadjoint action* of G. It projects back to the coadjoint action on $\mathfrak{g}^*$ via the Ad^*_G-equivariant projection $\pi_{\mathfrak{g}^*} : \mathcal{G} \to \mathfrak{g}^*$.

A *lifted invariant* of Ad^*_G is a (locally defined) smooth function from $\mathcal{G}$ to a manifold, which is invariant with respect to $\widehat{R}$.

The function $\mathcal{I}\colon \mathcal{G} \to \mathfrak{g}^*$ given by $\mathcal{I}(\mathrm{Ad}^*_g, x) = \mathrm{Ad}^*_g x$ is the *fundamental lifted invariant* of Ad^*_G, i.e., $\mathcal{I}$ is a lifted invariant and any lifted invariant can be locally written as a function of $\mathcal{I}$ in a unique way.

Using an arbitrary function $F(x)$ on $\mathfrak{g}^*$, we can produce the lifted invariant $F \circ \mathcal{I}$ of Ad^*_G by replacing x with $\mathrm{Ad}^*_g x$ in the expression for F.

Ordinary invariants are particular cases of lifted invariants, where one identifies any invariant formed as its composition with the standard projection $\pi_{\mathfrak{g}^*}$.

Therefore, ordinary invariants are particular functional combinations of lifted ones that happen to be independent of the group parameters of Ad^*_G.

The essence of the normalization procedure by Fels and Olver can be presented in the form of on the following statement.

Proposition

Suppose that

- $\mathcal{I} = (\mathcal{I}_1, \ldots, \mathcal{I}_n)$ is a fundamental lifted invariant,
- for the lifted invariants $\mathcal{I}_{j_1}$, ..., $\mathcal{I}_{j_\rho}$ and some constants c_1, ..., c_ρ the system $\mathcal{I}_{j_1} = c_1$, ..., $\mathcal{I}_{j_\rho} = c_\rho$ is solvable with respect to the parameters θ_{k_1}, ..., θ_{k_ρ}, and
- the substitution of the found values of θ_{k_1}, ..., θ_{k_ρ} into the other lifted invariants results in $m = n - \rho$ expressions $\hat{\mathcal{I}}_l$, $l = 1, \ldots, m$, depending only on x's.

Then $\rho = \mathrm{rank}\,\mathrm{Ad}^*_G$, $m = N_{\mathfrak{g}}$ and $\hat{\mathcal{I}}_1$, ..., $\hat{\mathcal{I}}_m$ form a basis of $\mathrm{Inv}(\mathrm{Ad}^*_G)$.

- *Construction of the generic matrix $B(\theta)$ of Ad^*_G.*

  $B(\theta)$ is the matrix of an inner automorphism of $\mathfrak{g}$ in $(e_1, \dots, e_n)$,
  $\theta = (\theta_1, \dots, \theta_r)$ is a complete tuple of group parameters (coordinates) of $\mathrm{Inn}(\mathfrak{g})$,
  $r = \dim \mathrm{Ad}^*_G = \dim \mathrm{Inn}(\mathfrak{g}) = n - \dim \mathrm{Z}(\mathfrak{g})$, where $\mathrm{Z}(\mathfrak{g})$ is the center of $\mathfrak{g}$.

- *Representation of the fundamental lifted invariant.*
  The explicit form of the fundamental lifted invariant $\mathcal{I} = (\mathcal{I}_1, \dots, \mathcal{I}_n)$ of Ad^*_G in the chosen coordinates $(\theta, \check{x})$ in $\mathrm{Ad}^*_G \times \mathfrak{g}^*$ is $\mathcal{I} = \check{x} \cdot B(\theta)$, i.e.,

$$(\mathcal{I}_1, \dots, \mathcal{I}_n) = (x_1, \dots, x_n) \cdot B(\theta_1, \dots, \theta_r).$$

  3\. *Elimination of parameters by normalization*. We choose the maximum possible number ρ of lifted invariants $\mathcal{I}_{j_1}, \dots, \mathcal{I}_{j_\rho}$, constants $c_1, \dots, c_\rho$ and group parameters $\theta_{k_1}, \dots, \theta_{k_\rho}$ such that the equations

$$\mathcal{I}_{j_1} = c_1, \quad \dots, \quad \mathcal{I}_{j_\rho} = c_\rho$$

  are solvable with respect to $\theta_{k_1}, \dots, \theta_{k_\rho}$. After substituting the found values of $\theta_{k_1}, \dots, \theta_{k_\rho}$ into the other lifted invariants, we obtain $N_\mathfrak{g} = n - \rho$ expressions $F^l(x_1, \dots, x_n)$ without θ's.

The functions $F^l(x_1, \dots, x_n)$ necessarily form a basis of $\mathrm{Inv}(\mathrm{Ad}^*_G)$.

Our experience on the calculation of invariants of a wide range of Lie algebras shows that the version of the algebraic method, which is based on the normalization procedure, is the most efficient.

In particular, it provides finding the cardinality of the invariant basis in the process of construction of the invariants.

The algorithm can in fact involve

- various kinds of coordinate in the inner automorphism groups (the first canonical, the second canonical or special one) and
- various techniques of eliminating parameters (empiric techniques, with additional combining of lifted invariants, using a floating system of normalization equations etc).

Thus, solving a linear system of first-order PDEs under the conventional infinitesimal approach, is replaced here by the construction of the matrix $B(\theta)$ of inner automorphisms and by excluding the parameters θ from the fundamental lifted invariant $\mathcal{I} = \check{x} \cdot B(\theta)$ in some way.

Illustrative example. $\mathfrak{g}^b_{4.8}$

$\mathfrak{g}=\mathfrak{g}^b_{4.8}$: $[e_2,e_3]=e_1,\quad [e_1,e_4]=(1{+}b)e_1,\quad [e_2,e_4]=e_2,\quad [e_3,e_4]=be_3,\quad |b|\le 1.$

[Infinitesimal approach: Patera & Sharp & Winternitz & Zassenhaus (1976)]

$$\hat{\mathrm{ad}}_{e_1}=\begin{pmatrix}0&0&0&1+b\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad \hat{\mathrm{ad}}_{e_2}=\begin{pmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

$$\hat{\mathrm{ad}}_{e_3}=\begin{pmatrix}0&-1&0&0\\0&0&0&0\\0&0&0&b\\0&0&0&0\end{pmatrix},\quad \hat{\mathrm{ad}}_{e_4}=\begin{pmatrix}-1-b&0&0&0\\0&-1&0&0\\0&0&-b&0\\0&0&0&0\end{pmatrix}.$$

The inner automorphisms of $\mathfrak{g}^b_{4.8}$ are then described by the triangular matrix

$$B(\theta)=\prod_{i=1}^{3}\exp(\theta_i\hat{\mathrm{ad}}_{e_i})\cdot\exp(-\theta_4\hat{\mathrm{ad}}_{e_4})$$

$$=\begin{pmatrix}e^{(1+b)\theta_4}&-\theta_3e^{\theta_4}&\theta_2e^{b\theta_4}&b\theta_2\theta_3+(1+b)\theta_1\\0&e^{\theta_4}&0&\theta_2\\0&0&e^{b\theta_4}&b\theta_3\\0&0&0&1\end{pmatrix}.$$

Therefore, a functional basis of lifted invariants is formed by

$$\mathcal{I}_1 = e^{(1+b)\theta_4}x_1,$$
$$\mathcal{I}_2 = e^{\theta_4}(-\theta_3 x_1 + x_2),$$
$$\mathcal{I}_3 = e^{b\theta_4}(\theta_2 x_1 + x_3),$$
$$\mathcal{I}_4 = (b\theta_2\theta_3 + (1+b)\theta_1)x_1 + \theta_2 x_2 + b\theta_3 x_3 + x_4.$$

- $b \neq -1$.

  The system $\mathcal{I}_1 = 1$, $\mathcal{I}_2 = \mathcal{I}_3 = \mathcal{I}_4 = 0$ is solvable w.r.t. the whole tuple of the parameters θ. $\Rightarrow$ $N_{\mathfrak{g}} = 0$.

- $b = -1$, "the diamond algebra".

  $\mathrm{Z}(\mathfrak{g}) = \langle e_1 \rangle$ $\Rightarrow$ x_1 is an invariant. [$\mathcal{I}_1 = x_1$ does not depend on θ at all.]

  Another invariant is $\mathcal{I}_1\mathcal{I}_4 - \mathcal{I}_2\mathcal{I}_3 = x_1x_4 - x_2x_3$.

  It can also be constructed by the normalization technique. We solve $\mathcal{I}_2 = \mathcal{I}_3 = 0$ w.r.t. (θ_2, θ_3) and substitute the expressions into $\mathcal{I}_4$. $\Rightarrow$ $x_4 - x_2x_3/x_1$

Review of obtained results, 2006–2008

Using the moving frames approach, we recalculated invariant bases and, in a number of cases, enhanced their representation for the following Lie algebras (in additional brackets we cite the papers where invariants bases of the same algebras were computed by the infinitesimal method):

- the complex and real Lie algebras up to dimension 6 [Patera & Sharp & Winternitz & Zassenhaus (1976), Ndogmo (2000), Campoamor-Stursberg (2005)];
- the complex and real Lie algebras with Abelian nilradicals of codimension one [Snobl& Winternitz(2005)];
- the complex indecomposable solvable Lie algebras with the nilradicals isomorphic to $\mathfrak{J}_0^n$, $n = 3, 4, \ldots$ (the nonzero commutation relations between the basis elements $e_1, \ldots, e_n$ of $\mathfrak{J}_0^n$ are exhausted by $[e_k, e_n] = e_{k-1}$, $k = 2, \ldots, n-1$) [Ndogmo & Wintenitz (1994)];
- the nilpotent Lie algebra $\mathfrak{t}_0(n)$ of $n \times n$ strictly upper triangular matrices [Tremblay & Winternitz(2001)];
- the solvable Lie algebra $\mathfrak{t}(n)$ of $n \times n$ upper triangular matrices and the solvable Lie algebras $\mathfrak{st}(n)$ of $n \times n$ special upper triangular matrices [Tremblay & Winternitz (2001)];
- the solvable Lie algebras with nilradicals isomorphic to $\mathfrak{t}_0(n)$ and diagonal nilindependent elements [Tremblay & Winternitz (2001)].

[Review of results within the framework of the infinitesimal approach: Snobl& Winternitz(2014)] solvable Lie algebra $\mathfrak{t}_\gamma(n)$:

- the nilradical $\mathrm{NR}(\mathfrak{t}_\gamma(n)) \simeq \mathfrak{t}_0(n)$,
- s nilindependent elements f_p, $p = 1, \dots, s$, which act on elements of the nilradical in the way as the matrices $\Gamma_p = \mathrm{diag}(\gamma_{p1}, \dots, \gamma_{pn})$ act on strictly upper triangular matrices.

The matrices Γ_p, $p = 1, \dots, s$, and the identity matrix are jointly linear independent since otherwise $\mathrm{NR}(\mathfrak{t}_\gamma(n)) \neq \mathfrak{t}_0(n)$. Therefore, the algebra $\mathfrak{t}_\gamma(n)$ is naturally embedded into $\mathfrak{t}(n)$ as an ideal under identification of $\mathrm{NR}(\mathfrak{t}_\gamma(n))$ with $\mathfrak{t}_0(n)$ and of f_p with Γ_p.

We choose the concatenation of the canonical basis of $\mathrm{NR}(\mathfrak{t}_\gamma(n))$ and of the s-element tuple $(f_p,\ p = 1, \dots, s)$ as the canonical basis of $\mathfrak{t}_\gamma(n)$.

In the basis of $\mathrm{NR}(\mathfrak{t}_\gamma(n))$ we use 'matrix' enumeration of basis elements e_{ij}, $i < j$, with the 'increasing' pair of indices similarly to the canonical basis $(E^n_{ij},\ i < j)$ of the isomorphic matrix algebra $\mathfrak{t}_0(n)$.

$$e_{ij} \sim E^n_{ij},\ i < j,\ f_p \sim \textstyle\sum_i \gamma_{pi} E^n_{ii}\colon \quad [e_{ij}, e_{i'j'}] = \delta_{i'j} e_{ij'} - \delta_{ij'} e_{i'j}, \quad [f_p, e_{ij}] = (\gamma_{pi} - \gamma_{pj}) e_{ij}$$

The Lie algebra $\mathfrak{t}_\gamma(n)$ can be considered as the Lie algebra of the Lie subgroup

$$\mathrm{T}_\gamma(n) = \{B \in \mathrm{T}(n) \mid \exists\, \varepsilon_p \in \mathbb{F}\colon\ b_{ii} = e^{\gamma_{pi}\varepsilon_p}\}$$

of the Lie group $\mathrm{T}(n)$ of non-singular upper triangular $n \times n$ matrices.

Proposition

$\mathfrak{t}_\gamma(n)$ and $\mathfrak{t}_{\gamma'}(n)$ are isomorphic iff $\exists\ \lambda \in \mathrm{GL}(s,\mathbb{F})$ and $\exists\ \mu \in \mathbb{F}^s$ s.t. either

$$\gamma'_{pi} = \lambda_{pp'}\gamma_{p'i} + \mu_p, \quad p = 1,\dots,s,\ i = 1,\dots,n,$$

or

$$\gamma'_{pi} = \lambda_{pp'}\gamma_{p',n-i+1} + \mu_p, \quad p = 1,\dots,s,\ i = 1,\dots,n.$$

Proposition

Up to the equivalence relation on algebra parameters, the following conditions can be assumed satisfied

$$\exists\, s' \in \left\{0,\dots,\min\left(s,\left[\frac{n}{2}\right]\right)\right\},$$

$$\exists\, k_q,\ q = 1,\dots,s', \quad 1 \leqslant k_1 < k_2 < \dots < k_{s'} \leqslant \left[\frac{n}{2}\right]:$$

$$\gamma_{qk} = \gamma_{q\varkappa},\ k < k_q, \quad \gamma_{q\varkappa_q} - \gamma_{qk_q} = 1, \quad \gamma_{pk_q} = \gamma_{p\varkappa_q},\ p \neq q, \quad q = 1,\dots,s',$$

$$\gamma_{pk} = \gamma_{p\varkappa},\ p > s',\ k = 1,\dots,\left[\frac{n}{2}\right],$$

where $\varkappa := n - k + 1$, $\varkappa_q := n - k_q + 1$.

Representation of the coadjoint action for $\mathfrak{t}_\gamma(n)$

Let e^*_{ji}, x_{ji} and y_{ij} denote the basis element and the coordinate function in the dual space $\mathfrak{t}^*_\gamma(n)$ and the coordinate function in $\mathfrak{t}_\gamma(n)$, which correspond to the basis element e_{ij}, $i<j$.

In particular, $\langle e^*_{j'i'}, e_{ij}\rangle = \delta_{ii'}\delta_{jj'}$.

f^*_p, x_{p0} and y_{p0} denote similar objects corresponding to the basis element f_p.

We additionally set $y_{ii}=\gamma_{pi}y_{p0}$ and then complete the collections of x_{ji} and of y_{ij} to the matrices X and Y with zeros.

$\Rightarrow$ X is a strictly lower triangular matrix and Y is a non-strictly upper triangular one.

Lemma

A complete set of functionally independent lifted invariants of $\mathrm{Ad}^*_{\mathrm{T}_\gamma(n)}$ is exhausted by the expressions

$$\mathcal{I}_{ij}=\sum_{i\leqslant i',\, j'\leqslant j} b_{ii'}\widehat{b}_{j'j}x_{i'j'},\quad j<i,\qquad \mathcal{I}_{p0}=x_{p0}+\sum_{j<i}\sum_{j\leqslant l\leqslant i}\gamma_{pl}b_{li}\widehat{b}_{jl}x_{ij},$$

where $B=(b_{ij})$ is an arbitrary matrix from $\mathrm{T}_\gamma(n)$, $B^{-1}=(\widehat{b}_{ij})$ is the inverse of B.

Invariants of the coadjoint action for $\mathfrak{t}_\gamma(n)$

Below $A^{i_1,i_2}_{j_1,j_2}$, where $i_1 \leqslant i_2$, $j_1 \leqslant j_2$, denotes the submatrix $(a_{ij})^{i=i_1,\dots,i_2}_{j=j_1,\dots,j_2}$ of $A = (a_{ij})$.

The standard notation $|A| = \det A$ is used.

The conjugate values of k's with respect to n is denoted by respective $\varkappa$'s, i.e.,

$$\varkappa := n - k + 1, \quad \varkappa_q := n - k_q + 1, \quad \varkappa' := n - k' + 1.$$

Theorem

Let the parameter matrix γ be of a reduced form. A basis of $\mathrm{Inv}(\mathrm{Ad}^*_{\mathrm{T}_\gamma(n)})$ is formed by

$$|X^{\varkappa,n}_{1,k}|\prod_{q=1}^{s'}|X^{\varkappa_q,n}_{1,k_q}|^{\beta_{qk}}, \quad k \in \{1,\dots,[n/2]\} \setminus \{k_1,\dots,k_{s'}\},$$

$$x_{p0} + \sum_{k=1}^{[\frac{n}{2}]} \frac{(-1)^{k+1}}{|X^{\varkappa,n}_{1,k}|}(\gamma_{pk} - \gamma_{p,k+1}) \sum_{k<i<\varkappa} \begin{vmatrix} X^{i,i}_{1,k} & 0 \\ X^{\varkappa,n}_{1,k} & X^{\varkappa,n}_{i,i} \end{vmatrix}, \quad p = s'+1,\dots,s,$$

where $\beta_{qk} = -\Delta_{qk}/\Delta$, $\Delta = \det(\alpha_{q'k_{q''}})_{q',q''=1,\dots,s'} = (-1)^{s'}$, Δ_{qk} is the determinant obtained from Δ with change of $(\alpha_{q'k_q})_{q'=1,\dots,s'}$ by $(\alpha_{q'k})_{q'=1,\dots,s'}$,

$$\alpha_{qk} := -\textstyle\sum_{k'=1}^{k}(\gamma_{q\varkappa'} - \gamma_{qk'}) = -\sum_{k'=k_q}^{k}(\gamma_{q\varkappa'} - \gamma_{qk'}).$$

Sketch of the proof

Basic normalization constraints on the lifted invariants $\mathcal{I}_{ij}$, $j<i$:

$$\mathcal{I}_{ij}=0 \quad \text{if} \quad j<i,\ (i,j)\neq(n-j'+1,j'),\ j'=1,\dots,\left[\frac{n}{2}\right].$$

The decision on what to do with $\mathcal{I}_{p0}$'s and $\mathcal{I}_{n-k+1,k}$, $k=1,\dots,[n/2]$, should be taken in the course of the proof since it turns out that necessity of imposing normalization conditions on them depends on values of γ.

The final normalization in all the cases provides satisfying the conditions of the normalization procedure and, therefore, is correct.

In view of (triangular) structure of the matrices B and X the formula $\mathcal{I}=BXB^{-1}$ determining the matrix part of lifted invariants implies that

$$BX=\mathcal{I}B.$$

This matrix equality is also significant only for the matrix entries of its left- and right-hand sides BX and $\mathcal{I}B$ that underlie the respective main diagonals, i.e., we have the system

$$e^{\gamma_{pi}\varepsilon_p}x_{ij}+\sum_{i<i'}b_{ii'}x_{i'j}=\mathcal{I}_{ij}e^{\gamma_{pj}\varepsilon_p}+\sum_{j'<j}\mathcal{I}_{ij'}b_{j'j},\quad j<i.$$

Corollary

$|X^{\varkappa,n}_{1,k}|$, $k = 1, \dots, [n/2]$, are functionally independent relative invariants of $\mathrm{Ad}^*_{\mathrm{T}_\gamma(n)}$ for any admissible value of γ.

Thank you for your attention!