Recent developments in singularity formation of nonlinear waves.

Joachim Krieger (EPFL)

Banff, September 2021

Bubbling off dynamics

- Consider an evolution equation of either wave or Schrodinger type

$$-u_{tt} + \triangle u = F(u, \nabla_{t,x} u)$$
$$iu_t + \triangle u = G(u)$$

We are interested in solutions of a 'bubbling type' of essentially the following form

$$u(t,x) = \lambda^{\alpha}(t) Q(\lambda(t)x) + \epsilon(t,x)$$

where $\lambda(t)$ blows up either in finite or infinite time, while $\epsilon(t,x)$ stays 'regular' and bounded.

Bubbling off dynamics

- Consider an evolution equation of either wave or Schrodinger type

$$-u_{tt} + \triangle u = F(u, \nabla_{t,x} u)$$
$$iu_t + \triangle u = G(u)$$

  We are interested in solutions of a 'bubbling type' of essentially the following form

$$u(t,x) = \lambda^{\alpha}(t) Q(\lambda(t)x) + \epsilon(t,x)$$

  where $\lambda(t)$ blows up either in finite or infinite time, while $\epsilon(t,x)$ stays 'regular' and bounded.
- Usually the bulk profile $Q(x)$ is either a stationary or even static solution of the problem.

Bubbling off dynamics

- Consider an evolution equation of either wave or Schrodinger type

$$-u_{tt} + \triangle u = F(u, \nabla_{t,x} u)$$
$$iu_t + \triangle u = G(u)$$

We are interested in solutions of a 'bubbling type' of essentially the following form

$$u(t,x) = \lambda^{\alpha}(t) Q(\lambda(t)x) + \epsilon(t,x)$$

where $\lambda(t)$ blows up either in finite or infinite time, while $\epsilon(t,x)$ stays 'regular' and bounded.

- Usually the bulk profile $Q(x)$ is either a stationary or even static solution of the problem.
- Smoothness of solution before blow up : only require smoothness H^s-class in which problem is strongly locally well-posed, i. e. not just C^∞-data.

L^2-Critical NLS

- L^2-critical NLS, for example in one spatial dimension given by

$$iu_t + u_{xx} = -|u|^4 u,$$

admits static solution $Q(x) = \frac{(\frac{3}{2})^{\frac{1}{4}}}{\cosh^{\frac{1}{2}}(\frac{x}{2})}$. Application of suitable pseudo-conformal transformation leads to

$$u(t,x) = t^{-\frac{1}{2}} e^{\frac{x^2}{4it}} Q(\frac{x}{t})$$

L^2-Critical NLS

- L^2-critical NLS, for example in one spatial dimension given by

$$iu_t + u_{xx} = -|u|^4 u,$$

admits static solution $Q(x) = \frac{(\frac{3}{2})^{\frac{1}{4}}}{\cosh^{\frac{1}{2}}(\frac{x}{2})}$. Application of suitable pseudo-conformal transformation leads to

$$u(t,x) = t^{-\frac{1}{2}} e^{\frac{x^2}{4it}} Q(\frac{x}{t})$$

- Application of inherent symmetry leads to a very rigid blow up type (precisely one blow up rate).

L^2-Critical NLS

- L^2-critical NLS, for example in one spatial dimension given by
$$iu_t + u_{xx} = -|u|^4 u,$$
admits static solution $Q(x) = \frac{(\frac{3}{2})^{\frac{1}{4}}}{\cosh^{\frac{1}{2}}(\frac{x}{2})}$. Application of suitable pseudo-conformal transformation leads to
$$u(t,x) = t^{-\frac{1}{2}} e^{\frac{x^2}{4it}} Q(\frac{x}{t})$$
- Application of inherent symmetry leads to a very rigid blow up type (precisely one blow up rate).
- Most 'natural problems' don't admit such inherent algebraic symmetries to infer bubbling off blow up. Nonetheless, the latter is quite ubiquitous.

Models for Bubbling off dynamics

- Key examples which are typically Hamiltonian and also critical :

Critical Wave Maps : $-u_{tt}+u_{rr}+\frac{1}{r}u_r=\frac{\sin 2u}{2r^2}$

Critical focussing NLW on $\mathbf{R}^{3+1}$: $-u_{tt}+\triangle u=-u^5$

Critical Yang-Mills : $-u_{tt}+\triangle u=-\frac{2}{r^2}u(1-u^2)$

critical Schrodinger Maps : $u_t=u\times\triangle u$

Energy critical NLS on $\mathbf{R}^{3+1}$: $iu_t+\triangle u=-|u|^4u$

Models for Bubbling off dynamics

- Key examples which are typically Hamiltonian and also critical :

  Critical Wave Maps : $-u_{tt}+u_{rr}+\frac{1}{r}u_r=\frac{\sin 2u}{2r^2}$

  Critical focussing NLW on $\mathbf{R}^{3+1}$: $-u_{tt}+\triangle u=-u^5$

  Critical Yang-Mills : $-u_{tt}+\triangle u=-\frac{2}{r^2}u(1-u^2)$

  critical Schrodinger Maps : $u_t=u\times\triangle u$

  Energy critical NLS on $\mathbf{R}^{3+1}$: $iu_t+\triangle u=-|u|^4u$

- Outlier example :

  Hyperbolic Vanishing mean curvature flow :

$$\sum_{\alpha=0}^{n}\partial_\alpha\Big(\frac{\partial^\alpha u}{\sqrt{1+\partial_\alpha u\partial^\alpha u}}\Big)=0,\ n=8. \tag{1}$$

Bubbling off blow up for WM I

- Specific example : co-rotational critical Wave Maps $\phi : \mathbf{R}^{2+1} \longrightarrow S^2$:

$$-\phi_{tt} + \triangle\phi = \phi(|\phi_t|^2 - |\nabla_x \phi|^2),\ \phi \in S^2 \hookrightarrow \mathbf{R}^3.$$

$$\phi(t,x) = \begin{pmatrix} \cos\theta \sin u \\ \sin\theta \sin u \\ \cos u \end{pmatrix},\ u = u(t,r)\ r = |x|.$$

implies the equation

$$-u_{tt} + u_{rr} + \frac{1}{r}u_r = \frac{\sin 2u}{2r^2}. \qquad (2)$$

Bubbling off blow up for WM I

- Specific example : co-rotational critical Wave Maps $\phi : \mathbf{R}^{2+1} \longrightarrow S^2$:

$$-\phi_{tt} + \triangle\phi = \phi(|\phi_t|^2 - |\nabla_x\phi|^2),\ \phi \in S^2 \hookrightarrow \mathbf{R}^3.$$

$$\phi(t,x) = \begin{pmatrix} \cos\theta\sin u \\ \sin\theta\sin u \\ \cos u \end{pmatrix},\ u = u(t,r)\ r = |x|.$$

implies the equation

$$-u_{tt} + u_{rr} + \frac{1}{r}u_r = \frac{\sin 2u}{2r^2}. \tag{2}$$

- Model admits non-trivial finite energy static solution $Q(r) = 2\arctan r$, corresponding to stereographic projection.

Bubbling off blow up for WM II

- Two approaches to building finite time bubbling off blow up.

Bubbling off blow up for WM II

- Two approaches to building finite time bubbling off blow up.
- Raphael-Rodnianski('09) approach : exhibits open data set within sufficiently smooth class of (co-rotational) data resulting in solutions of the form

$$u(t,r) = Q(\lambda(t)r) + \epsilon(t,r),\ \lambda(t) = (T-t)^{-1}e^{\sqrt{\log(T-t)}}.$$

The result implies the same blow up rate for an open data set, but the topology is important. The following appears a natural conjecture :

Conjecture : Stable finite time blow up solutions of (2) with C^{∞}-data are of the preceding form.

Bubbling off blow up for WM II

- Two approaches to building finite time bubbling off blow up.
- Raphael-Rodnianski('09) approach : exhibits open data set within sufficiently smooth class of (co-rotational) data resulting in solutions of the form

$$u(t,r) = Q(\lambda(t)r) + \epsilon(t,r),\ \lambda(t) = (T-t)^{-1} e^{\sqrt{\log(T-t)}}.$$

  The result implies the same blow up rate for an open data set, but the topology is important. The following appears a natural conjecture :
  Conjecture : Stable finite time blow up solutions of (2) with C^{∞}-data are of the preceding form.
- One may also conjecture quantized set of blow up rates correspoonding to sufficiently smooth data and unstable blow up.

Bubbling off blow up for WM III

- Important to note : the Raphael-Rodnianski solutions live in a strictly sub-critical topology, namely H^2, while the problem is actually strongly locally well-posed in H^1.

Bubbling off blow up for WM III

- Important to note : the Raphael-Rodnianski solutions live in a strictly sub-critical topology, namely H^2, while the problem is actually strongly locally well-posed in H^1.
- Can be contrasted with the following theorem of Donninger('16) for the ODE-type blow up solutions $u(t,x) = c(T-t)^{-\frac{1}{2}}$ for the energy-critical NLW on $\mathbf{R}^{3+1}$

$$-u_{tt} + \triangle u = -u^5.$$

Theorem(Donninger'16) : The ODE blow up solutions are stable under radial H^1-perturbations

Bubbling off blow up for WM III

- Important to note : the Raphael-Rodnianski solutions live in a strictly sub-critical topology, namely H^2, while the problem is actually strongly locally well-posed in H^1.
- Can be contrasted with the following theorem of Donninger('16) for the ODE-type blow up solutions $u(t,x) = c(T-t)^{-\frac{1}{2}}$ for the energy-critical NLW on $\mathbf{R}^{3+1}$

$$-u_{tt} + \triangle u = -u^5.$$

Theorem(Donninger'16) : The ODE blow up solutions are stable under radial H^1-perturbations

- This is the strongest stability statement one can hope for since H^1 is the largest natural space in which the problem is locally well-posed.

The phenomenon of a continuum of blow up rates

- Back to co-rotational critical Wave Maps into S^2, another approach for finite time bubbling off blow up due to K.-Schlag-Tataru('06) : exhibits solutions of the form

$$u(t,r) = Q(\lambda(t)r) + \epsilon(t,r),\ \lambda(t) = t^{-1-\nu},\ \epsilon \in C^{\nu+\frac{1}{2}-} \cap H^{1+\nu-}.$$

Any $\nu > 0$ is admissible. Original result gave no stability, even of conditional type.

The phenomenon of a continuum of blow up rates

- Back to co-rotational critical Wave Maps into S^2, another approach for finite time bubbling off blow up due to K.-Schlag-Tataru('06) : exhibits solutions of the form

$$u(t,r) = Q(\lambda(t)r) + \epsilon(t,r),\ \lambda(t) = t^{-1-\nu},\ \epsilon \in C^{\nu+\frac{1}{2}-} \cap H^{1+\nu-}.$$

  Any $\nu > 0$ is admissible. Original result gave no stability, even of conditional type.

- Two peculiar features : (1) continuum of blow up rates. (2) The solutions are only of finite regularity, depending on the blow up rate.

The phenomenon of a continuum of blow up rates

- Back to co-rotational critical Wave Maps into S^2, another approach for finite time bubbling off blow up due to K.-Schlag-Tataru('06) : exhibits solutions of the form

$$u(t,r) = Q(\lambda(t)r) + \epsilon(t,r),\ \lambda(t) = t^{-1-\nu},\ \epsilon \in C^{\nu+\frac{1}{2}-} \cap H^{1+\nu-}.$$

  Any $\nu > 0$ is admissible. Original result gave no stability, even of conditional type.

- Two peculiar features : (1) continuum of blow up rates. (2) The solutions are only of finite regularity, depending on the blow up rate.
- More precisely, the solutions are of class C^∞ in the inside of light cone $|x| < |t|$ centered at singularity, but experience a shock on the light cone $|x| = |t|$.

Remarks on KST('06) construction

- There are two key steps in the construction : (1) construction of an approximate solution u_{approx}. Here the shock on light cone already manifests itself. (2) Completion of approximate solution to an exact one via spectral methods.

Remarks on KST('06) construction

- There are two key steps in the construction : (1) construction of an approximate solution u_{approx}. Here the shock on light cone already manifests itself. (2) Completion of approximate solution to an exact one via spectral methods.
- Approximate solution : distinguish the regions $r \ll |t|$ (elliptic region) and $r \sim |t|$ (wave region).

Remarks on KST('06) construction

- There are two key steps in the construction : (1) construction of an approximate solution u_{approx}. Here the shock on light cone already manifests itself. (2) Completion of approximate solution to an exact one via spectral methods.
- Approximate solution : distinguish the regions $r \ll |t|$ (elliptic region) and $r \sim |t|$ (wave region).
- In the elliptic region, one can make a formal power series ansatz

$$u_{approx} = \sum_{j \geq 0} t^{\nu j} f_j(R), \; R = \lambda(t) r.$$

Remarks on KST('06) construction

- There are two key steps in the construction : (1) construction of an approximate solution u_{approx}. Here the shock on light cone already manifests itself. (2) Completion of approximate solution to an exact one via spectral methods.
- Approximate solution : distinguish the regions $r \ll |t|$ (elliptic region) and $r \sim |t|$ (wave region).
- In the elliptic region, one can make a formal power series ansatz

$$u_{approx} = \sum_{j \geq 0} t^{\nu j} f_j(R), \; R = \lambda(t) r.$$

- In the wave region introduce $a = \frac{r}{t}$, and write

$$u_{approx} = \sum_{j \geq 0} t^{\nu j} g_j(R, a)$$

where the g_j admit suitable Puiseux type expansion in a reflecting the shock across the light cone.

Continuum of blow up rates for other models

- All of the models mentioned at the beginning admit analoga of the KST blow ups.

Continuum of blow up rates for other models

- All of the models mentioned at the beginning admit analoga of the KST blow ups.
- **Theorem**(Gal. Perelman('12)) : The critical Schrodinger Maps $u : \mathbf{R}^{2+1} \longrightarrow S^2$ admits finite time blow up solutions of the form

$$u(t,x) = Q(\lambda(t)x) + \zeta(t,x),\ \lambda(t) = t^{-\frac{1}{2}-\nu}, \nu > 1.$$

Continuum of blow up rates for other models

- All of the models mentioned at the beginning admit analoga of the KST blow ups.
- **Theorem**(Gal. Perelman('12)) : The critical Schrodinger Maps $u : \mathbf{R}^{2+1} \longrightarrow S^2$ admits finite time blow up solutions of the form

$$u(t,x) = Q(\lambda(t)x) + \zeta(t,x),\ \lambda(t) = t^{-\frac{1}{2}-\nu}, \nu > 1.$$

- The restriction $\nu > 1$ can probably be relaxed to $\nu > 0$.

Continuum of blow up rates for other models

- All of the models mentioned at the beginning admit analoga of the KST blow ups.
- **Theorem**(Gal. Perelman('12)) : The critical Schrodinger Maps $u : \mathbf{R}^{2+1} \longrightarrow S^2$ admits finite time blow up solutions of the form

$$u(t,x) = Q(\lambda(t)x) + \zeta(t,x),\ \lambda(t) = t^{-\frac{1}{2}-\nu}, \nu > 1.$$

- The restriction $\nu > 1$ can probably be relaxed to $\nu > 0$.
- To build approximate solution, one needs to distinguish between elliptic region $r \ll t^{\frac{1}{2}}$, the Schrodinger wave region $r \sim t^{\frac{1}{2}}$ and the far region $r \gg t^{\frac{1}{2}}$.

Continuum of blow up rates for other models

- All of the models mentioned at the beginning admit analoga of the KST blow ups.
- **Theorem**(Gal. Perelman('12)) : The critical Schrodinger Maps $u : \mathbf{R}^{2+1} \longrightarrow S^2$ admits finite time blow up solutions of the form

$$u(t,x) = Q(\lambda(t)x) + \zeta(t,x),\ \lambda(t) = t^{-\frac{1}{2}-\nu}, \nu > 1.$$

- The restriction $\nu > 1$ can probably be relaxed to $\nu > 0$.
- To build approximate solution, one needs to distinguish between elliptic region $r \ll t^{\frac{1}{2}}$, the Schrodinger wave region $r \sim t^{\frac{1}{2}}$ and the far region $r \gg t^{\frac{1}{2}}$.
- By analyzing the approximate solution in far region, Perelman can extract the leading radiation part that is left over at the singularity formation.

A further natural candidate which blends wave and Schrodinger

- The critical Zakharov system on $\mathbf{R}^{4+1}$:

$$i\partial_t u + \triangle u = -nu,$$
$$(-\partial_{tt} + \triangle)u = \triangle(|u|^2)$$

A further natural candidate which blends wave and Schrodinger

- The critical Zakharov system on $\mathbf{R}^{4+1}$:

$$i\partial_t u + \triangle u = -nu,$$
$$(-\partial_{tt} + \triangle)u = \triangle(|u|^2)$$

- Admits the static solution $(u, n) = (W, -W^2)$ where (same as for energy critical NLS on $\mathbf{R}^{4+1}$)

$$W(x) = \frac{1}{1 + \frac{|x|^2}{8}}.$$

A further natural candidate which blends wave and Schrodinger

- The critical Zakharov system on $\mathbf{R}^{4+1}$:

$$i\partial_t u + \triangle u = -nu,$$
$$(-\partial_{tt} + \triangle)u = \triangle(|u|^2)$$

- Admits the static solution $(u, n) = (W, -W^2)$ where (same as for energy critical NLS on $\mathbf{R}^{4+1}$)

$$W(x) = \frac{1}{1 + \frac{|x|^2}{8}}.$$

- **Conjecture** : Zakharov admits a finite time bubbling off blow up where

$$u(t, x) = e^{i\alpha(t)}\lambda(t)W(\lambda(t)x) + \zeta(t, x),\ \lambda(t) = t^{-\frac{1}{2}-\nu},$$

and $\nu > \nu_* > 0$

How to get a coherent picture of all these dynamics ?

- The issue of stability : the following appears reasonable but non-trivial since due to a nonlinear instability : **Conjecture** : A KST type blow up solution with $\lambda(t) = t^{-1-\nu}$ and ν large is unstable, but stable along a manifold of finite co-dimension in a sufficiently smooth class of perturbations.

How to get a coherent picture of all these dynamics ?

- The issue of stability : the following appears reasonable but non-trivial since due to a nonlinear instability : **Conjecture** : A KST type blow up solution with $\lambda(t) = t^{-1-\nu}$ and ν large is unstable, but stable along a manifold of finite co-dimension in a sufficiently smooth class of perturbations.
- Recent progress on the question of stability for KST blowup in context of critical Wave Maps into S^2. The case of $\lambda(t) = t^{-1-\nu}$ with $\nu > 0$ small was considered.

How to get a coherent picture of all these dynamics ?

- The issue of stability : the following appears reasonable but non-trivial since due to a nonlinear instability : **Conjecture** : A KST type blow up solution with $\lambda(t) = t^{-1-\nu}$ and ν large is unstable, but stable along a manifold of finite co-dimension in a sufficiently smooth class of perturbations.
- Recent progress on the question of stability for KST blowup in context of critical Wave Maps into S^2. The case of $\lambda(t) = t^{-1-\nu}$ with $\nu > 0$ small was considered.
- Recall that these solutions experience shock across light cone of the form $(1-a)^{\frac{1}{2}+\nu}\log(1-a)$, $a = \frac{r}{|t|}$.

How to get a coherent picture of all these dynamics ?

- The issue of stability : the following appears reasonable but non-trivial since due to a nonlinear instability : **Conjecture** : A KST type blow up solution with $\lambda(t) = t^{-1-\nu}$ and ν large is unstable, but stable along a manifold of finite co-dimension in a sufficiently smooth class of perturbations.
- Recent progress on the question of stability for KST blowup in context of critical Wave Maps into S^2. The case of $\lambda(t) = t^{-1-\nu}$ with $\nu > 0$ small was considered.
- Recall that these solutions experience shock across light cone of the form $(1-a)^{\frac{1}{2}+\nu} \log(1-a)$, $a = \frac{r}{|t|}$.
- Displacing this shock 'costs a lot', i. e. requires a rough perturbation of the data. Hence natural to consider smooth perturbations which 'cannot displace' the shock.

Stability of KST blow up for critical WM into S^2

- **Theorem**(K.-Miao '19) The KST finite time blow up solutions for critical co-rotational wave maps $u : \mathbf{R}^{2+1} \longrightarrow S^2$ are stable under sufficiently smooth and small *co-rotational* perturbations, provided $\nu > 0$ is sufficiently small. More precisely if $\nu > 0$ is sufficiently small and $u_\nu(t, x)$ a KST blow up solution with $\lambda(t) = t^{-1-\nu}$, constructed on some interval $[t_0, 0)$, and if (ϵ_0, ϵ_1) is sufficiently small in the $H^4 \times H^3$-norm, then the data

$$u_\nu[t_0] + (\epsilon_0, \epsilon_1)$$

lead to a finite time blow up solution of the form

$$u(t, r) = Q(\lambda(t)r) + \epsilon(t, r)$$

with $\epsilon \in H^{1+\nu-}$. In particular, the perturbed solution blows up in the same space-time location (*rigidity of blow up).*

Comments on result

- One key difficulty in proof has to do with the low regularity (just H^{1+}) of the solution u_ν being perturbed. On the other hand, the low regularity is solely linked to the shock along the light cone.

Comments on result

- One key difficulty in proof has to do with the low regularity (just H^{1+}) of the solution u_ν being perturbed. On the other hand, the low regularity is solely linked to the shock along the light cone.
- Remarkable feature of co-rotational reduction : no derivatives in nonlinearity :

$$\Box u = \frac{\sin 2u}{2r^2} \text{ versus } \Box u = u(|u_t|^2 - |\nabla_x u|^2).$$

Comments on result

- One key difficulty in proof has to do with the low regularity (just H^{1+}) of the solution u_ν being perturbed. On the other hand, the low regularity is solely linked to the shock along the light cone.
- Remarkable feature of co-rotational reduction : no derivatives in nonlinearity :

$$\Box u = \frac{\sin 2u}{2r^2} \text{ versus } \Box u = u(|u_t|^2 - |\nabla_x u|^2).$$

- Applying Duhamel parametrix to source term $\frac{\sin 2u}{2r^2}$ leads to terms of regularity H^{2+}, which gives a key boost in regularity.

Stability of KST blow up under general (non-equivariant) perturbations I

- Up until recently, the stability of either the KST type blow up or the Raphael-Rodnianski blow up for the co-rotational critical Wave Maps into S^2 and under *generic, non-equivariant* perturbations has been completely open. In fact, for the (Ra-Ro) solutions it is conjectured that they are unstable.

Stability of KST blow up under general (non-equivariant) perturbations I

- Up until recently, the stability of either the KST type blow up or the Raphael-Rodnianski blow up for the co-rotational critical Wave Maps into S^2 and under *generic, non-equivariant* perturbations has been completely open. In fact, for the (Ra-Ro) solutions it is conjectured that they are unstable.
- For the KST blow up u_ν at low ν (where $\lambda(t) = t^{-1-\nu}$ and solutions are at regularity $H^{1+\nu-}$), even if one uses very smooth perturbations (ϵ_0, ϵ_1) of the data, the interactions of u_ν with perturbation in the nonlinearity

$$u(|u_t|^2 - |\nabla u|^2)$$

lead to terms of same regularity as u_ν. Modulations needed, *but only of the kind preserving the locus of the shock.*

Stability of KST blow up under general (non-equivariant) perturbations II

- **Theorem**(K.-Miao-Schlag '20) The KST finite time blow up solutions for critical co-rotational wave maps $u : \mathbf{R}^{2+1} \longrightarrow S^2$ are stable under sufficiently smooth and small *generic* perturbations, provided $\nu > 0$ is sufficiently small. The perturbed solutions are of the form

$$u(t,x) = \mathcal{R}_{h(t)}^{\alpha(t),\beta(t)} \mathcal{L}_{v(t)} \mathcal{S}_{c(t)} \big(Q(\lambda(t)r) + \epsilon(t,x) \big).$$

where $\mathcal{R}_{h(t)}^{\alpha(t),\beta(t)}$ represents a suitable combination of rotations on the target in terms of Euler angles, $\mathcal{L}_{v(t)}$ a suitable Lorentz transform, and $\mathcal{S}_{c(t)}$ a suitable scaling transformation.

Stability of KST blow up under general (non-equivariant) perturbations III

- Key aspects of this work : non-equivariant setting forces one to work in suitable frame for tangent bundle :

$$\begin{pmatrix} \cos\theta\cos U \\ \sin\theta\cos U \\ -\sin U \end{pmatrix}, \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$$

Stability of KST blow up under general (non-equivariant) perturbations III

- Key aspects of this work : non-equivariant setting forces one to work in suitable frame for tangent bundle :

$$\begin{pmatrix} \cos\theta\cos U \\ \sin\theta\cos U \\ -\sin U \end{pmatrix}, \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$$

- If we write $\phi_1 E_1 + \phi_2 E_2$ for the tangential part of perturbation, then. $\phi_1 \pm i\phi_2$ can be decomposed into Fourier series with respect to θ, resulting in

$$\epsilon_\pm(n) = \phi_1(n) \pm i\phi_2(n).$$

Stability of KST blow up under general (non-equivariant) perturbations III

- Key aspects of this work : non-equivariant setting forces one to work in suitable frame for tangent bundle :

$$\begin{pmatrix} \cos\theta\cos U \\ \sin\theta\cos U \\ -\sin U \end{pmatrix}, \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$$

- If we write $\phi_1 E_1 + \phi_2 E_2$ for the tangential part of perturbation, then. $\phi_1 \pm i\phi_2$ can be decomposed into Fourier series with respect to θ, resulting in

$$\epsilon_\pm(n) = \phi_1(n) \pm i\phi_2(n).$$

- Schrodinger operators

$$H_n^\pm = \partial_{RR} + \frac{1}{R}\partial_R - f_n(R) \pm g_n(R),\ f_n = \frac{n^2+1}{R^2} - \frac{8}{(1+R^2)^2}$$

Stability of KST blow up under general (non-equivariant) perturbations III

- We require an asymptotic (in n) spectral analysis for these operators, in particular asymptotics for the generalized eigenfunctions $\phi_n(R;\xi)$.

Stability of KST blow up under general (non-equivariant) perturbations III

- We require an asymptotic (in n) spectral analysis for these operators, in particular asymptotics for the generalized eigenfunctions $\phi_n(R;\xi)$.
- 'Semiclassical variable' $h = \frac{1}{|n|+1}$, $\alpha = h \cdot E$,

$$\phi_n(R;\xi) = h^{\frac{1}{3}}\alpha^{-\frac{1}{2}}q^{-\frac{1}{4}}(\tau)\mathrm{Ai}(h^{-\frac{2}{3}}\tau)(1 + ha_0(-\tau,\alpha,h)).$$

Stability of KST blow up under general (non-equivariant) perturbations III

- We require an asymptotic (in n) spectral analysis for these operators, in particular asymptotics for the generalized eigenfunctions $\phi_n(R;\xi)$.
- 'Semiclassical variable' $h = \frac{1}{|n|+1}$, $\alpha = h \cdot E$,

$$\phi_n(R;\xi) = h^{\frac{1}{3}}\alpha^{-\frac{1}{2}}q^{-\frac{1}{4}}(\tau)\mathrm{Ai}(h^{-\frac{2}{3}}\tau)(1 + ha_0(-\tau,\alpha,h)).$$

- For each mode n, one tries to mimic the estimates in the co-rotational case (as in K.-Miao).

Stability of KST blow up under general (non-equivariant) perturbations III

- We require an asymptotic (in n) spectral analysis for these operators, in particular asymptotics for the generalized eigenfunctions $\phi_n(R;\xi)$.
- 'Semiclassical variable' $h = \frac{1}{|n|+1}$, $\alpha = h \cdot E$,

$$\phi_n(R;\xi) = h^{\frac{1}{3}}\alpha^{-\frac{1}{2}}q^{-\frac{1}{4}}(\tau)\mathrm{Ai}(h^{-\frac{2}{3}}\tau)(1 + ha_0(-\tau,\alpha,h)).$$

- For each mode n, one tries to mimic the estimates in the co-rotational case (as in K.-Miao).
- The symmetries of the problem lead to certain algebraic instabilities which manifest in the Fourier modes $n = 0, \pm 1$. This is where the modulations are being used.

Outlook : classification in terms of radiation at blow up time ?

- Recent(2019) work by Jendrej-Lawrie-Rodriguez : write co-rotational WM blow up solution as

$$u(t,r) = Q(\lambda(t)r) + u_*(r) + g(t)$$

where $\lim_{t\to 0} g(t) = 0$. Then if

$$u_*(r) = qr^\nu + o(r^\nu),\ \nu > \frac{9}{2},$$

then $\lambda(t) \sim \frac{|\log t|}{t^{\nu+1}}$.

Outlook : classification in terms of radiation at blow up time ?

- Recent(2019) work by Jendrej-Lawrie-Rodriguez : write co-rotational WM blow up solution as

$$u(t,r) = Q(\lambda(t)r) + u_*(r) + g(t)$$

where $\lim_{t\to 0} g(t) = 0$. Then if

$$u_*(r) = qr^{\nu} + o(r^{\nu}),\ \nu > \frac{9}{2},$$

then $\lambda(t) \sim \frac{|\log t|}{t^{\nu+1}}$.

- KST solutions have similar radiation part (but also with $\nu > 0$ very small). Probably similar classification ?

Outlook : classification in terms of radiation at blow up time ?

- Recent(2019) work by Jendrej-Lawrie-Rodriguez : write co-rotational WM blow up solution as

$$u(t,r) = Q(\lambda(t)r) + u_*(r) + g(t)$$

where $\lim_{t\to 0} g(t) = 0$. Then if

$$u_*(r) = qr^{\nu} + o(r^{\nu}),\ \nu > \frac{9}{2},$$

then $\lambda(t) \sim \frac{|\log t|}{t^{\nu+1}}$.

- KST solutions have similar radiation part (but also with $\nu > 0$ very small). Probably similar classification ?
- How about more general radiation part asymptotics near $r = 0$. More exotic blow up rates ?

Outlook : multibubble solutions ; how much freedom ?

- Precise characterization of two bubble solutions for equivariant wave maps and under a minimal energy condition (threshold blow up) by Jendrej-Lawriew ('20) for $k \geq 2$ and in the co-rotational case $k = 1$ by Rodriguez('18). Rigid blow up rates.

Outlook : multibubble solutions ; how much freedom ?

- Precise characterization of two bubble solutions for equivariant wave maps and under a minimal energy condition (threshold blow up) by Jendrej-Lawriew ('20) for $k \geq 2$ and in the co-rotational case $k = 1$ by Rodriguez('18). Rigid blow up rates.
- In these scenarios only one bubble collapses while the other one converges to a 'limiting bubble'.

Outlook : multibubble solutions ; how much freedom ?

- Precise characterization of two bubble solutions for equivariant wave maps and under a minimal energy condition (threshold blow up) by Jendrej-Lawriew ('20) for $k \geq 2$ and in the co-rotational case $k = 1$ by Rodriguez('18). Rigid blow up rates.
- In these scenarios only one bubble collapses while the other one converges to a 'limiting bubble'.
- Can there be multi-bubble solutions where all bubbles collapse in finite or infinite time ? This will require more than the threshold energy. Is there a link between the topology one is working with and the possible collapsing rates ?