# Invariants for families of gapped states

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work in progress with A. Kapustin

# Outline

- Motivation
- Formalism: the complex of currents
- Application 1: Hall invariants
- Application 2: Berry invariants

# Motivation

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- States are pure states with some extra locality properties (e.g. "gapped states").
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In general it's very difficult to describe (or even define!) $\mathcal{X}$. But we can at least hope to construct invariants of families $\mathcal{M} \to \mathcal{X}$.

# Example: Berry classes for 0d system

If we don't have any locality, the system is effectively zero-dimensional. In the limit when the number of degrees of freedom goes to infinity the space has the homotopy type of $\mathbb{CP}^\infty$. Only $\pi_2(\mathbb{CP}^\infty) = \mathbb{Z}$ is non-trivial.

For a smooth family $\mathcal{M}$ of states on a Hilbert space $\mathcal{H}$ with the corresponding rank-1 projector $P$, we can define a canonical line bundle $\mathcal{L}$ over $\mathcal{M}$ with the canonical curvature

$$F = \mathrm{Tr}(PdPdP). \tag{1}$$

$\frac{1}{2\pi i}[F] \in H^2(\mathcal{M}, \mathbb{Z})$ is known as Berry class. It gives an obstruction to the triviality of the family $\mathcal{M}$ (e.g. spin 1/2 in a magnetic field).

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Question: is there a meaningful generalization for states of (interacting) many-body systems in the thermodynamic limit?

# Uniformly local Hamiltonians

One way to define a local Hamiltonian is by a formal sum:

$$\mathsf{H} = \sum_{j \in \Lambda} \mathsf{h}_j \tag{2}$$

where $\mathsf{h}_j$ is a traceless uniformly bounded $\|\mathsf{h}_j\| \le C$ observable local on a ball of radius $R$ with the center at $j$. It defines an unbounded derivations on the algebra

$$\mathsf{H}(\mathcal{A}) = [\mathsf{H}, \mathcal{A}] = \sum_{j \in \Lambda} [\mathsf{h}_j, \mathcal{A}]. \tag{3}$$

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Such description is ambiguous. E.g.

$$\mathsf{H} = ... + (h_{-1} + \mathcal{A}) + (h_0 - \mathcal{A}) + h_1 + h_2 + ... \tag{4}$$

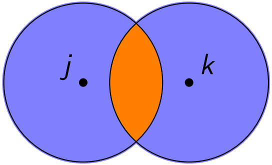
defines the same Hamiltonian.

# Uniformly local Hamiltonians: the complex of currents

Let $\mathfrak{d}_l$ be the Lie algebra of traceless local observables, and let $C_n(\mathfrak{d}_l)$ be a space of antisymmetric functions

$$\mathsf{f} : \Lambda^{n+1} \to \mathfrak{d}_l$$

such that $\mathsf{f}_{j_0 \ldots j_n}$ is a local uniformly bounded observable on a ball of radius $R$ with the center at $j_a$ for any $a \in \{0, ..., n\}$. We call them *n*-currents or *n*-chains.



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$$(\partial \mathsf{j})_k := \sum_{l \in \Lambda} \mathsf{j}_{kl} = -i[\mathsf{H}, \mathsf{q}_k] = -\dot{\mathsf{q}}_k$$

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- Similarly, $\mathrm{j}^E_{kl} = i[\mathrm{h}_k, \mathrm{h}_l]$ defines the energy current.

# Uniformly local Hamiltonians: the complex of currents

The map $\partial : C_{n+1}(\mathfrak{d}_l) \to C_n(\mathfrak{d}_l)$

$$(\partial \mathsf{f})_{j_0 \ldots j_n} = \sum_{j_{n+1} \in \Lambda} \mathsf{f}_{j_0 \ldots j_{n+1}}$$

defines a chain-complex

$$\ldots \xrightarrow{\partial_3} C_2(\mathfrak{d}_l) \xrightarrow{\partial_2} C_1(\mathfrak{d}_l) \xrightarrow{\partial_1} C_0(\mathfrak{d}_l) \to 0 \tag{5}$$

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**Lemma:** The homology of this complex is trivial except for $H_0(\mathfrak{d}_l)$. Moreover, there is an explicit contracting homotopy, i.e. a map $h_n : C_n(\mathfrak{d}_l) \to C_{n+1}(\mathfrak{d}_l)$ such that $\partial_{n+1} \circ h_n + h_{n-1} \circ \partial_n = \mathrm{Id}$.

# Uniformly local Hamiltonians: the complex of currents

We can define the augmented complex

$$\ldots \xrightarrow{\partial_3} C_2(\mathfrak{d}_I) \xrightarrow{\partial_2} C_1(\mathfrak{d}_I) \xrightarrow{\partial_1} C_0(\mathfrak{d}_I) \xrightarrow{\partial_0} \mathfrak{D}_I \to 0 \tag{6}$$

with $\mathfrak{D}_I := H_0(\mathfrak{d}_I)$.

The space of uniformly local Hamiltonians is identified with $\mathfrak{D}_I$.

# Uniformly local Hamiltonians: the complex of currents

| **Lattice $\Lambda \subset \mathbb{R}^d$** | **Fields on $\mathbb{R}^d$** |
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| 0-chain | $d$-form (density) |
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| $n$-chain | $(d-n)$-form (higher current) |
| $\partial$ | de Rham $d$ |
| | integration |
| | wedge product |

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Contraction with regions $A_0, \ldots, A_n$:

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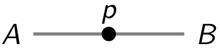
In $d = 1$ the contraction a 1-current $\mathrm{j}$ with two complementing half-lines $A, B$ defines a local observable $\mathrm{j}_{AB}$.



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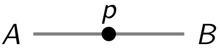
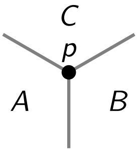



In $d = 2$ the contraction of a 2-current $\mathrm{m}$ with regions $A, B, C$ defines a local observable $\mathrm{m}_{ABC}$



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# Uniformly local Hamiltonians: the complex of currents

There is a canonical degree 1 graded-skew-symmetric bracket $\{\cdot,\cdot\} : C_n(\mathfrak{d}_l) \times C_m(\mathfrak{d}_l) \to C_{n+m+1}(\mathfrak{d}_l)$ defined by

$$\{\mathsf{f},\mathsf{g}\}_{j_0\ldots j_{|\mathsf{f}|+|\mathsf{g}|+1}} := \frac{1}{|\mathsf{f}|!|\mathsf{g}|!}[\mathsf{f}_{j_0\ldots j_{|\mathsf{f}|}}, \mathsf{g}_{j_{|\mathsf{f}|+1}\ldots j_{|\mathsf{f}|+|\mathsf{g}|+1}}] + (\text{s. perms}). \quad (8)$$

that satisfies graded Leibniz rule and graded Jacobi identity.

$$\partial\{\mathsf{f},\mathsf{g}\} = \{\partial\mathsf{f},\mathsf{g}\} + (-1)^{|\mathsf{f}|+1}\{\mathsf{f},\partial\mathsf{g}\}, \quad (9)$$

$$(-1)^{(|\mathsf{f}|+1)(|\mathsf{h}|+1)}\{\mathsf{f},\{\mathsf{g},\mathsf{h}\}\} + (\text{c. perms}) = 0. \quad (10)$$

It defines a (1-shifted) DG Lie algebra structure on $C_\bullet(\mathfrak{d}_l)$ and induces a Lie algebra structure on $\mathfrak{D}_l$.

Examples: $\mathsf{j} = \{\mathsf{h},\mathsf{q}\}$, $\mathsf{j}^E = \{\mathsf{h},\mathsf{h}\}$, ...

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# Uniformly almost local Hamiltonians

For practical application the assumption of uniform locality is too strong. Even for quasi-adiabatic evolution of finite-ranged gapped Hamiltonians we need subexponential decay. [Hastings 04; Osborne 06; Nachtergale et. al. 11; Ogata, Moon 19]

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We say that $\mathcal{A} \in \mathfrak{d}_l$ is $b(r)$-localized at $j$, if

$$\inf_{\mathcal{B} \in \mathscr{A}_{B_j(r)}} \|\mathcal{A} - \mathcal{B}\| \leq b(r). \tag{11}$$

We define uniformly almost local (UAL) chain complex by requiring that all components $\mathsf{f}_{j_0 \ldots j_n}$ of $n$-chain $\mathsf{f}$ are $b$-localized at each $j_a$ for some $b(r) \in \mathcal{O}(r^{-\infty})$.

# Uniformly almost local Hamiltonians

**Lemma:** the structures described above hold for UAL chain complex

$$\dots \xrightarrow{\partial_2} C_1(\mathfrak{d}_{al}) \xrightarrow{\partial_1} C_0(\mathfrak{d}_{al}) \to \mathfrak{D}_{al} \to 0 \qquad (12)$$

The corresponding Lie algebras $\mathfrak{d}_{al}$ and $\mathfrak{D}_{al} = H_0(\mathfrak{d}_{al})$ have the structure of Fréchet-Lie algebra.

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Remark: perhaps there is even the corresponding Fréchet-Lie group of automorphisms generated by such derivations.

# Pseudo-gapped states

Let $\mathfrak{d}_{al}{}^{\psi}$, $\mathfrak{D}_{al}^{\psi}$ be Lie-subalgebras, which do not excite a pure state $\psi$:

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**Lemma:** For a ground state $\psi$ of a gapped Hamiltonian with exponentially decaying interaction the complex

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is exact, and the there is UAL contracting homotopy $h^{\psi}_n : C_n(\mathfrak{d}^{\psi}_{al}) \to C_{n+1}(\mathfrak{d}^{\psi}_{al})$. In particular, for any closed $\mathsf{f} \in C_n(\mathfrak{d}^{\psi}_{al})$ we can construct $\mathsf{g} = h_n(\mathsf{f})$ such that $\mathsf{f} = \partial \mathsf{g}$.

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From now on we consider the class of states for which $h^{\psi}_n$ exists without any reference to the Hamiltonian (even if it exists). We call them "pseudo-gapped". (similar condition in [Bachmann, Bols, et al. 18])

# Pseudo-gapped states

Important subclass: invertible states. [Kitaev]

We call a state $\psi$ on a system $\Lambda$ invertible, if there is another system $\Lambda'$ with a state $\psi'$, such that $\psi \otimes \psi'$ on $\Lambda \cup \Lambda'$ is in the trivial phase, i.e. it can be disentangled by a local Hamiltonian evolution.

In the presence of symmetry we may consider $G$-invariant version of invertibility.

Modest goal: describe all invertible states (Kitaev's conjecture).

# Application 1: Hall invariants

Let $\psi$ be a pseudo-gapped state invariant under an on-site $U(1)$ symmetry, and let $Q \in \mathfrak{D}^{\psi}_{al}$ be the corresponding generator of the symmetry. We may consider $U(1)$-invariant part of $C_{\bullet}(\mathfrak{d}^{\psi}_{al})$.

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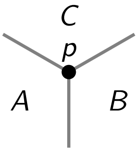
Claim: for $d = 2$ the contraction $\sigma = 4\pi i \langle q^{(2)}_{ABC} \rangle$ is an invariant of the phase. For gapped states $\sigma/2\pi$ coincides with the Hall conductance.



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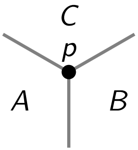


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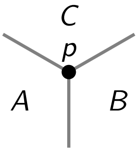


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- For invertible states one can show that $\sigma \in \mathbb{Z}$ for fermions and $\sigma \in 2\mathbb{Z}$ for bosons [Kapustin, NS 20] using methods similar to the proof of the quantization of Hall conductance on a torus [Hastings, Michalakis 13; Bachmann et al. 18].

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$$\ldots$$

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Claim: for even $d$ and a conical intersection $A_0, A_1, \dots, A_d$ the contraction

$$\langle q^{(d)}_{A_0 \dots A_d} \rangle \tag{16}$$

is an invariant of the phase.

These invariants are supposed to correspond to non-linear response described by the effective action $\int A dA \dots dA$.

# Application 1: Hall invariants

In the same way one can define invariants for non-abelian Lie group $G$ with Lie algebra $\mathfrak{g}$ taking values in invariant multi-linear forms on $\mathfrak{g}$. For example, for $d = 2$ with charges $\mathrm{Q}^a = \sum_j \mathrm{q}_j^a$ we can construct 2-current $\mathrm{m}^{ab}$ satisfying

$$\frac{1}{2}\{\mathrm{q}^a, \mathrm{q}^b\} = -\partial \mathrm{m}^{ab}$$

which defines an invariant quadratic form $\langle \mathrm{m}^{ab}_{ABC} \rangle$ on $\mathfrak{g}$.

# Application 2: Berry classes

Let $\mathcal{M}$ be a smooth manifold equipped with $\mathsf{G} \in \Omega^1(\mathcal{M}, \mathfrak{D}_{al})$, and let $\psi$ be a family of pseudo-gapped states. We say that it defines a smooth family of states if for any two points $\lambda_1, \lambda_2 \in \mathcal{M}$ and for any smooth path $p : [0,1] \to \mathcal{M}$ between $\lambda_1$ and $\lambda_2$ the state $\psi_{\lambda_2}$ can be obtained from $\psi_{\lambda_1}$ using $p^*\mathsf{G}$.

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Let $(\mathcal{M}, \mathsf{G}, \psi)$ be a smooth family of pseudo-gapped states. Then the bi-complex

$$\ldots \xrightarrow{\partial} \Omega^\bullet(\mathcal{M}, C_1(\mathfrak{d}^{\psi}_{al})) \xrightarrow{\partial} \Omega^\bullet(\mathcal{M}, C_0(\mathfrak{d}^{\psi}_{al})) \xrightarrow{\partial} \Omega^\bullet(\mathcal{M}, \mathfrak{D}^{\psi}_{al}) \xrightarrow{\partial} 0, \tag{17}$$

is exact with respect to $\partial$.

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satisfying

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that is $\mathsf{F} \in \Omega^2(\mathcal{M}, \mathfrak{D}^{\psi}_{al})$. The descent equation is given by

$$D\mathsf{f}^{\bullet} + \frac{1}{2}\{\mathsf{f}^{\bullet}, \mathsf{f}^{\bullet}\} = \mathsf{F} - \partial\mathsf{f}^{\bullet} \tag{20}$$

where $f^{\bullet} = f^{(0)} + f^{(1)} + f^{(2)} + \ldots$ with $f^{(n)} \in \Omega^{n+2}(\mathcal{M}, C_n(\mathfrak{d}^{\psi}_{al}))$.

# Application 2: Berry classes

Let

$$\mathsf{F} := d\mathsf{G} + \frac{1}{2}\{\mathsf{G}, \mathsf{G}\} \tag{18}$$

satisfying

$$D\mathsf{F} := d\mathsf{F} + \{\mathsf{G}, \mathsf{F}\} = 0, \quad \langle [\mathsf{F}, \mathcal{A}] \rangle_\psi = 0 \tag{19}$$

that is $\mathsf{F} \in \Omega^2(\mathcal{M}, \mathfrak{D}^\psi_{al})$. The descent equation is given by

$$D\mathsf{f}^\bullet + \frac{1}{2}\{\mathsf{f}^\bullet, \mathsf{f}^\bullet\} = \mathsf{F} - \partial \mathsf{f}^\bullet \tag{20}$$

where $f^\bullet = f^{(0)} + f^{(1)} + f^{(2)} + \dots$ with $f^{(n)} \in \Omega^{n+2}(\mathcal{M}, C_n(\mathfrak{d}^\psi_{al}))$.

Claim: $[\langle f^{(d)}_{A_0 \dots A_d} \rangle] \in H^{d+2}(\mathcal{M}, i\mathbb{R})$ is an invariant of a family $\mathcal{M}$. We call it higher Berry class. [Kitaev (unpublished); Kapustin, Spodyneiko 19]

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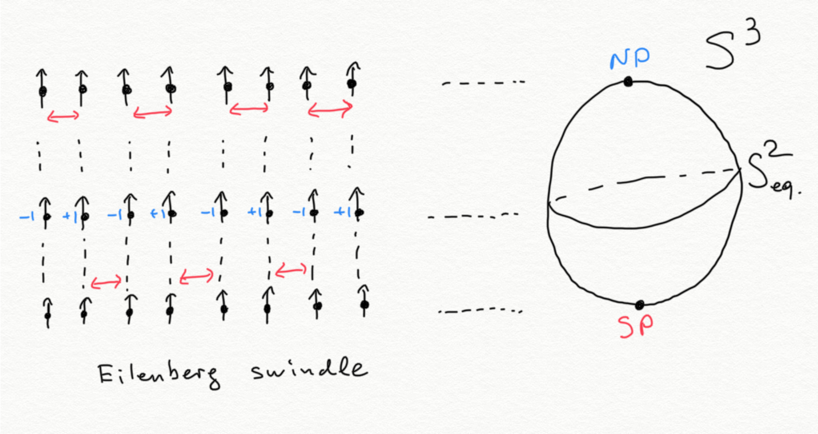
Example: $\mathcal{M} = S^3$ family of 1d states with non-trivial $[\langle f^{(1)}_{AB} \rangle] \in H^3(\mathcal{M}, i\mathbb{R})$.



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- A non-trivial family of 1d states may appear on the boundary of 2d system if some symmetry preserves the state in the bulk (WZW invariants).
- Not known if there is any quantization for $d > 1$.

# Final remarks

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- One can also consider a unifying equation

$$D\mathrm{b}^\bullet + \frac{1}{2}\{\mathrm{b}^\bullet, \mathrm{b}^\bullet\} = \left(\mathrm{F} + \sum_a \mathrm{Q}^a t^a\right) - \partial \mathrm{b}^\bullet \qquad (21)$$

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  that in addition contains Thouless invariants.
- It seems like all invariants for Lie group symmetry $G$ originating from Berry curvature can be obtained in this way.
- For discrete $G$ some invariants can be defined for invertible states. One has to use multiplicative version of pseudo-gap condition which is technically more challenging.

Thank you for your attention!