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A bulk gap in the presence of edge states for a Haldane pseudopotential

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Background: Haldane Pseudopotentials

Haldane pseudopotentials were originally introduced as Hamiltonian models for the fractional quantum Hall effect for $\nu = 1/(p+2)$ with $p \geq 0$ odd.

- Laughlin '83: Ansatz for many-body ground state wave function Ψ_p.
- Haldane, '83: Pseudopotential $W^p \geq 0$ obtained via projection onto lowest Landau level of repulsive, short-range, radially symmetric pair potential.

$$W^p = \sum_{i<j} P_{LLL} v_p(z_i, z_j') P_{LLL}, \quad v_p \propto \Delta^p \delta, \quad z = x + iy$$

  Tailored so $\Psi_p \in \ker W^p$.
- Haldane-Rezayi '85, Trugman-Kivelson '85, Lee-Papic-Thomale '17, ...: More generalized study of various pseudopotentials on different 2D geometries.
- Regnault-Jolicoeur '04, Cooper '08,...: Also model rapidly rotating Bose gases (p even).
- Lewin-Seiringer '09, Seiringer-Yngvason '20: Obtained as scaling limit.
- Johri-Papic-Schmitteckert-Bhatt-Haldane '12: Properties of pseudopotentials robust under change of geometry.
- Nachtergale, Warzel, Y. '21: Gap of truncated $p = 1$ fermionic pseudopotential in thin cylinder regime.

Properties and Conjectures:

$$\Lambda = \text{ physical space}, \quad N = \text{ number of particles}, \quad \nu = \frac{N}{|\Lambda|} \text{ filling factor}$$

1. Ground States: Zero energy states $\psi \in \mathcal{G}_\Lambda = \ker W_\Lambda^p$ satisfy $\nu \leq \nu(p) := \dfrac{1}{p+2}$.
   - E.g. Ψ_p has maximal filling $\nu(p)$.
2. Spectral Rigidity: For states with higher fillings $\nu > \nu(p)$:

$$E_0(\mathcal{H}_\Lambda^N) = \inf_{0 \neq \psi \in \mathcal{H}_\Lambda^N} \frac{\langle \psi | W_\Lambda^p \psi \rangle}{\|\psi\|^2} \propto |\Lambda| \times \text{ increasing function of } \nu$$

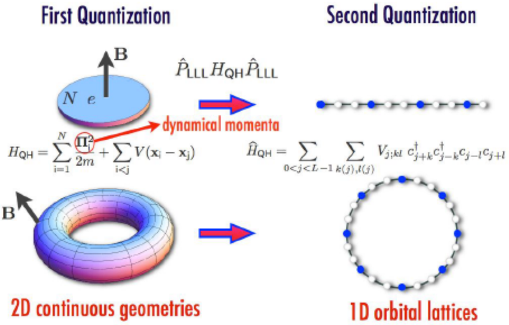
   - Determines Yrast line for Bose gases: Viefers-Hansson-Reimann '00, Regnault-Jolicoeur '04, Lewin-Seiringer '09,...
3. Spectral Gap Conjecture: Haldane '83, Haldane-Rezayi '85, ...

$$\gamma := \inf_\Lambda \operatorname{gap}(W_\Lambda^p) > 0 \quad \text{where} \quad \operatorname{gap}(W_\Lambda^p) = \inf_{0 \neq \psi \perp \mathcal{G}_\Lambda} \frac{\langle \psi | W_\Lambda^p \psi \rangle}{\|\psi\|^2}$$

   - The gap is responsible for the incompressibility of the FQH fluid: $E_0(\mathcal{H}_\Lambda^N) = 0$ for $\nu \leq \nu(p)$ and $E_0(\mathcal{H}_\Lambda^N) > \gamma$ for $\nu > \nu(p)$.

4 Anyonic Excitations with Fractional Charge and their topological stability: Hastings-Michalakis '15, Haah '16, Cha-Naaijkens-Nachtergaele '20,...

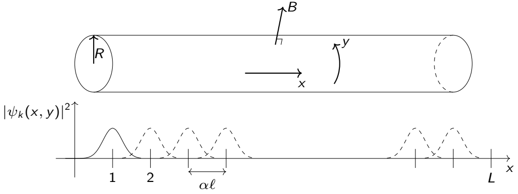
The 1/2-Haldane Pseudopotential



In second quantization, the pseudopotentials become one-dimensional (orbital) lattice models. Lee-Leinaas '04, Bergholtz-Karlhede, '05, Nakamura-Wang-Bergholtz '12,...

Today: We study the low-lying spectral properties of a truncated version of the lattice model for the 1/2-pseudopotential ($p = 0$) in the thin cylinder geometry.

The 1/2-Haldane Pseudopotential



magnetic length: ℓ

ratio: $\alpha = \ell/R$

Figure: The Landau orbitals. A magnetic flux $2\pi\beta$ along the cylinder axis shifts the orbitals by $\beta\alpha\ell$ (not shown).

Landau orbitals: Single particle Hilbert space is $\mathcal{H}_{LLL} = \mathrm{span}\{\psi_k | k \in \mathbb{Z}\}$ where

$$\psi_k(x, y) \propto \exp\left(ik\frac{\alpha y}{\ell}\right) \exp\left(-\frac{1}{2}\left[\frac{x}{\ell} - k\alpha\right]^2\right).$$

W^0: Projection of $v_0 \propto \delta$ onto bosonic Fock space $\mathcal{F} = \bigoplus_{N\geq 0} \mathrm{Sym}^N \mathcal{H}_{LLL}$:

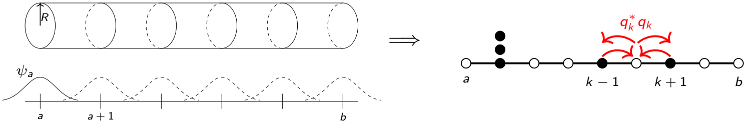
$$W^0 = \sum_{s\in\mathbb{Z}/2} B_s^* B_s, \quad B_s = \sum_{k\in s+\mathbb{Z}} e^{-\alpha^2 k^2} a_{s-k} a_{s+k}$$

where a_k is the bosonic annihilation operator for ψ_k.

Truncated model: We consider the model restricted to $|k| \leq 1$:

$$s \in \mathbb{Z} + \frac{1}{2} : \; B_s = 2e^{-\frac{\alpha^2}{4}} a_{s-\frac{1}{2}} a_{s+\frac{1}{2}}, \qquad s \in \mathbb{Z} : \; B_s = a_s^2 + 2e^{-\alpha^2} a_{s-1} a_{s+1}$$

The Finite-Volume Truncated 1/2-Model



Finite volume model: $\Lambda = [a, b] \subseteq \mathbb{Z}$ with open and periodic boundary conditions:

$$H_\Lambda = \sum_{k=a}^{b-1} n_k n_{k+1} + \kappa \sum_{k=a+1}^{b-1} q_k^* q_k , \qquad H_\Lambda^{\mathrm{per}} = \sum_{k=a}^{b} n_k n_{k+1} + \kappa \sum_{k=a}^{b} q_k^* q_k$$

$$n_k = a_k^* a_k , \quad q_k = a_k^2 - \lambda a_{k-1} a_{k+1} \text{ for some } \kappa > 0, \ \lambda \in \mathbb{C}$$

are self-adjoint operators with dense domain in the Fock space

$$\mathcal{H}_\Lambda = \overline{\mathrm{span}} \{ |\mu_a, \ldots, \mu_b\rangle : \mu_k \in \mathbb{N}_0 \} , \qquad \mu_k = \text{ number of particles occupying } \psi_k$$

Symmetries:

Particle number: $N_\Lambda = \sum_{k=a}^{b} n_k$, Center of mass: $M_\Lambda = \sum_{k=a}^{b} k n_k$

Physical regime: $\kappa = \dfrac{e^{\alpha^2/2}}{4}$ and $\lambda = -2e^{-\alpha^2}$ where $\alpha = \frac{\ell}{R}$.

Tao-Thouless limit: $\lambda \to 0$ as $R \to 0$.

Spectral Gap

For any finite interval $\min \mathrm{spec}(H_\Lambda^\#) = 0$ for $\# \in \{\text{obc, per}\}$ implying

$$\mathrm{gap}(H_\Lambda^\#) := \sup\{\delta > 0 : \mathrm{spec}(H_\Lambda) \cap (0, \delta) = \emptyset\}.$$

Simple Case $\lambda = 0$: $\mathrm{gap}(H_\Lambda^\#) = \min\{1, 2\kappa\}$.

Edge Modes for OBC: E.g. Lowest eigenvalue of $H_\Lambda \equiv H_\Lambda^{\mathrm{obc}}$ is $\frac{2\kappa|\lambda|^2}{\kappa+1} + \mathcal{O}(|\lambda|^4)$ in invariant subspace $\mathrm{span}\{|2010\ldots0\rangle, |1200\ldots0\rangle\}$

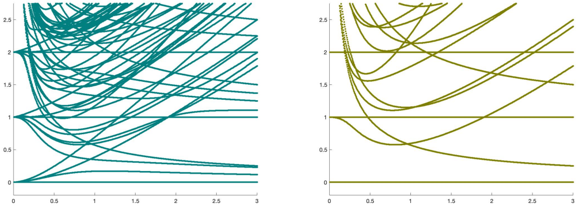


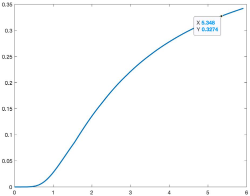
Figure: Plot of the spectrum for $H_{[1,9]}(\lambda)$ and $H_{[1,9]}^{\mathrm{per}}(\lambda)$, resp., for 1/3-truncated model in physical regime.

Main Result: Spectral Gap

Edge Modes for OBC: E.g. Lowest eigenvalue of H_Λ is $\frac{2\kappa|\lambda|^2}{\kappa+1} + \mathcal{O}(|\lambda|^4)$ in invariant subspace $\mathrm{span}\{|2010\ldots0\rangle, |1200\ldots0\rangle\}$

$$\gamma^{\mathrm{obc}} = \frac{1}{5}\min\left\{4\gamma^{\mathrm{per}}, \frac{2\kappa|\lambda|^2}{\kappa+1}\right\}$$

$$\gamma^{\mathrm{per}} = \frac{1}{4}\min\left\{1, \frac{2\kappa}{\kappa+1}, \frac{2\kappa}{1+\kappa|\lambda|^2}\right\}$$



Theorem: [Warzel, Y. '21] *For all* $\lambda \neq 0$ *with* $f(|\lambda|^2/2) < 1/3$ *(i.e.* $|\lambda| < 7.49\ldots$*)*

- **OBC gap:** $$\inf_{|\Lambda|\geq 10} \mathrm{gap}(H_\Lambda) \geq \min\left\{\gamma^{\mathrm{obc}}, \frac{2\kappa}{3}\left(1-\sqrt{3f(|\lambda|^2/2)}\right)^2\right\}$$
- **Bulk gap:** $$\liminf_{|\Lambda|\to\infty} \mathrm{gap}(H_\Lambda^{\mathrm{per}}) \geq \min\left\{\gamma^{\mathrm{per}}, \frac{\kappa}{3(1+|\lambda|^2)}\left(1-\sqrt{3f(|\lambda|^2/2)}\right)^2\right\}$$

Remarks:

- Bulk gap stays open despite edge states for OBC.
- Same strategy strengths the gap results in [Nachtergaele, Warzel, Y. '21]

Invariant Subspace Strategy

We identify a closed subspace $\mathcal{C}_\Lambda^\# \subseteq \mathrm{dom}(H_\Lambda^\#)$ for both $\# \in \{\mathrm{obc}, \mathrm{per}\}$ such that:

- Invariant under $H_\Lambda^\#$: $H_\Lambda^\# \mathcal{C}_\Lambda^\# \subseteq \mathcal{C}_\Lambda^\#$
- Contains ground state space: $\mathcal{G}_\Lambda^\# := \ker(H_\Lambda^\#) \subseteq \mathcal{C}_\Lambda^\#$

Since $\mathcal{H}_\Lambda^\# = \mathcal{C}_\Lambda^\# \oplus (\mathcal{C}_\Lambda^\#)^\perp$, the gap is given by

$$\mathrm{gap}(H_\Lambda^\#) = \min\left\{E_1(\mathcal{C}_\Lambda^\#),\ E_0\big((\mathcal{C}_\Lambda^\#)^\perp\big)\right\} \quad \text{where}$$

$$E_1(\mathcal{C}_\Lambda^\#) := \inf_{\psi \in \mathcal{C}_\Lambda^\# \cap (\mathcal{G}_\Lambda^\#)^\perp} \frac{\langle \psi | H_\Lambda^\# \psi \rangle}{\|\psi\|^2}, \qquad E_0\big((\mathcal{C}_\Lambda^\#)^\perp\big) := \inf_{\varphi \in (\mathcal{C}_\Lambda^\#)^\perp \cap \mathrm{dom}(H_\Lambda^\#)} \frac{\langle \varphi | H_\Lambda^\# \varphi \rangle}{\|\varphi\|^2}.$$

For Bulk Gap: Edge states of $H_\Lambda \equiv H_\Lambda^{\mathrm{obc}}$ are contained in $(\mathcal{C}_\Lambda^{\mathrm{obc}})^\perp \subseteq (\mathcal{C}_\Lambda^{\mathrm{per}})^\perp$

Identifying Invariant Subspaces

We first identify invariant subspaces for the model with open boundary conditions:

$$H_\Lambda = \sum_{k=a}^{b-1} n_k n_{k+1} + \kappa \sum_{k=a+1}^{b-1} q_k^* q_k, \qquad q_k = a_k^2 - \lambda a_{k-1} a_{k+1}$$

Observation 1: The occupation basis

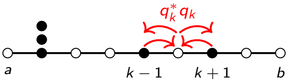
$$\{|\mu\rangle = |\mu_a \dots \mu_b\rangle : \mu_k \in \mathbb{N}_0 \ \forall\, k\} \subseteq \mathrm{dom}(H_\Lambda)$$

forms an orthonormal basis of eigenstates for the electrostatic terms:

$$\sum_{k=a}^{b-1} n_k n_{k+1} |\mu\rangle = \left(\sum_{k=a}^{b-1} \mu_k \mu_{k+1} \right) |\mu\rangle.$$

Observation 2: For any subspace $\mathcal{V}_\Lambda \subseteq \mathrm{dom}(H_\Lambda)$ spanned by occupation states

$$q_k^* q_k \mathcal{V}_\Lambda \subseteq \mathcal{V}_\Lambda \ \ \forall\, k \implies H_\Lambda \mathcal{V}_\Lambda \subseteq \mathcal{V}_\Lambda.$$



Example: Constructing Invariant Subspaces with Ground States

Since the interaction terms are all nonnegative:

$$\ker(H_\Lambda) = \bigcap_{k=a}^{b-1} \ker(n_k n_{k+1}) \cap \bigcap_{k=a+1}^{b-1} \ker(q_k).$$

Observations:

- $|\mu\rangle$ is a ground state of the electrostatic terms iff $\mu_k \mu_{k+1} = 0$ for all k.
- $q_k = a_k^2 - \lambda a_{k-1} a_{k+1}$ acts nontrivially on the sites $\{k-1, k, k+1\}$:

$$q_k \left(|101\rangle + \frac{\lambda}{\sqrt{2}} |020\rangle \right) = 0$$

Moreover, A simple calculation shows $q_k^* q_k \{|101\rangle, |020\rangle\} \subseteq \mathrm{span}\{|101\rangle, |020\rangle\}$.

Starting with the alternating string $|1010...\rangle$ can construct a set of occupation states by replacing '101' with '020' that span an invariant subspace of H_Λ:

1	0	1	0	1	0	1	0
0	2	0	0	1	0	1	0
1	0	0	2	0	0	1	0
1	0	1	0	0	2	0	0
0	2	0	0	0	2	0	0

$$\psi_\Lambda(R) = \sum_{T \leftrightarrow R} \left(\frac{\lambda}{\sqrt{2}} \right)^{d(T)} |\sigma(T)\rangle$$

[Jansen '12], [Nakamura, Wang, Bergholtz '12]

BVMD Tiling Spaces

We describe invariant subspaces of H_Λ using lattice tilings. Each subspace is generated by a root tiling R of **void**, **monomer**, and **boundary tiles**:

0	1 0	Left:	n 0 0	Right:	1	0 n

where $n \geq 2$. E.g. Root Tiling:

n 0 0	1 0	0	1 0	1 0	0	1 0	1

The Boundary-Void-Monomer-Dimer (BVMD) space generated by R is

$$\mathcal{C}_\Lambda(R) = \mathrm{span}\{|\sigma(T)\rangle : T \leftrightarrow R\}, \quad \sigma(T) \in \mathbb{N}_0^{|\Lambda|}$$

n 0 0	1 0	0	0 2 0 0		0	0 2 0	
n 0 0	1 0	0	1 0	1 0	0	0 2 0	
n 0 0	1 0	0	0 2 0 0		0	1 0	1

n 0 0	1 0	0	1 0	1 0	0	1 0	1

$\mathcal{C}_\Lambda(R)$ contains a unique ground state: $\psi_\Lambda(R) = \sum_{T \leftrightarrow R} \left(\frac{\lambda}{\sqrt{2}}\right)^{d(T)} |\sigma(T)\rangle.$

BVMD Tiling Spaces

Lemma: [Warzel, Y. '21] *For any root tilings $R \in \mathcal{R}_\Lambda$ on $\Lambda = [1, L]$ with $L \geq 3$,*

1. *$\mathcal{C}_\Lambda(R)$ is a finite dimensional invariant subspace of H_Λ.*
2. $\mathcal{G}_\Lambda \cap \mathcal{C}_\Lambda(R) = \mathrm{span}\{\psi_\Lambda(R)\}$.
3. *$\mathcal{C}_\Lambda(R) \perp \mathcal{C}_\Lambda(R')$ for all $R' \neq R$.*
4. *$H_\Lambda \restriction_{\mathcal{C}_\Lambda}$ is bounded where $\mathcal{C}_\Lambda = \bigoplus_R \mathcal{C}_\Lambda(R)$. Thus, $H_\Lambda \mathcal{C}_\Lambda \subseteq \mathcal{C}_\Lambda$.*

Theorem: [Warzel, Y. '21] *For all $\Lambda = [1, L]$ with $L \geq 5$,*

$$\mathcal{G}_\Lambda = \mathrm{span}\{\psi_\Lambda(R) : R \text{ a root tiling}\}.$$

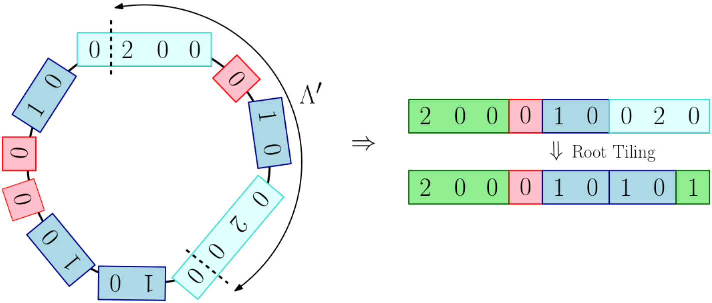
Thus, $\mathcal{G}_\Lambda \subseteq \mathcal{C}_\Lambda$ and $\dim(\mathcal{G}_\Lambda) = \infty$.

For PBC: Analogous construction of $\mathcal{C}_\Lambda^{\mathrm{per}}$ and $\mathcal{G}_\Lambda^{\mathrm{per}}$ using root tilings of the ring $\Lambda = [a, b]$ consisting of monomers and voids:

0   1 0

Properties: 1. $\dim \mathcal{G}_\Lambda^{\mathrm{per}} \propto \left(\frac{1+\sqrt{5}}{2}\right)^{|\Lambda|}$ 2. Maximum filling: $N_\Lambda(R)/|\Lambda| \leq 1/2$

Relation Between Ring and Interval Tilings



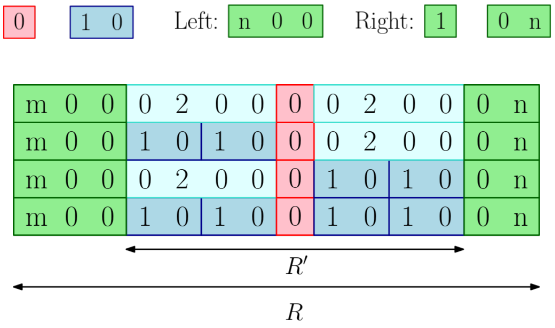
The bulk tiling space $\mathcal{C}_\Lambda^\infty \subseteq \mathcal{C}_\Lambda$ is the subspace spanned by tilings obtained as a truncation of a periodic tiling, i.e.

$$\mathcal{C}_\Lambda^\infty = \bigoplus_{R \text{ bulk root}} \mathcal{C}_\Lambda(R)$$

where a bulk root tiling R is any root tiling of Λ constructed from the tiles:

Relation Between $E_1(\mathcal{C}_\Lambda)$ and $E_1(\mathcal{C}_\Lambda^\infty)$: An Isospectral Reduction

Note that $\mathcal{C}_\Lambda^{\mathrm{obc}} \equiv \mathcal{C}_\Lambda = \bigoplus_R \mathcal{C}_\Lambda(R)$ is infinite dimensional.



$$H_\Lambda \mathcal{C}_\Lambda(R) = |m00\rangle \otimes H_{\Lambda'} \mathcal{C}_{\Lambda'}(R') \otimes |0n\rangle \implies E_1(\mathcal{C}_\Lambda(R)) = E_1(\mathcal{C}_{\Lambda'}(R'))$$

However, $\mathcal{C}_\Lambda^\infty$ is finite-dimensional and the above implies

$$E_1(\mathcal{C}_\Lambda) \geq \min_{\substack{|\Lambda'|=|\Lambda|-k: \\ k=0,2,3,5}} E_1(\mathcal{C}_{\Lambda'}^\infty).$$

Gap Estimating Strategy

Recap: For both $\# \in \{\mathrm{obc}, \mathrm{pbc}\}$, we have written $\mathcal{H}_\Lambda = \mathcal{C}_\Lambda^\# \oplus (\mathcal{C}_\Lambda^\#)^\perp$ where

$$\mathcal{G}_\Lambda^\# \subseteq \mathcal{C}_\Lambda^\# = \bigoplus_{R \in \mathcal{R}_\Lambda^\#} \mathcal{C}_\Lambda^\#(R), \qquad (\mathcal{C}_\Lambda^\#)^\perp = \overline{\mathrm{span}}\{|\mu\rangle = |\mu_a \ldots \mu_b\rangle : |\mu\rangle \notin \mathcal{C}_\Lambda^\#\}$$

are both invariant subspaces of $H_\Lambda^\#$. Moreover, the edge states of H_Λ are contained in $(\mathcal{C}_\Lambda^{\mathrm{obc}})^\perp \subseteq (\mathcal{C}_\Lambda^{\mathrm{per}})^\perp$. Thus,

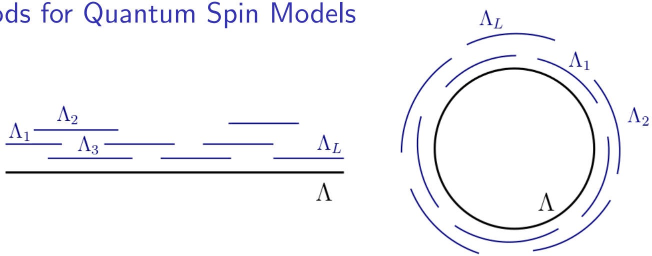
$$\mathrm{gap}(H_\Lambda^\#) = \min\left\{E_1(\mathcal{C}_\Lambda^\#),\ E_0\big((\mathcal{C}_\Lambda^\#)^\perp\big)\right\}$$

Methods: Lower bounds uniform in $|\Lambda|$ are obtained with the following approaches:

- For $E_1(\mathcal{C}_\Lambda^\#)$, apply gap techniques previously developed for quantum spin models and lattice fermions:
  - $E_1(\mathcal{C}_\Lambda^{\mathrm{obc}})$: martingale method on $\mathcal{C}_\Lambda^\infty$ [Nachtergaele, '96], [Nachtergaele, Sims, Y. '18]
  - $E_1(\mathcal{C}_\Lambda^{\mathrm{per}})$: finite size criterion [Knabe, '89] + OBC result for $E_1(\mathcal{C}_\Lambda^\infty)$
- For $E_0\big((\mathcal{C}_\Lambda^\#)^\perp\big)$, use electrostatic estimates to lower bound minimum energy.

  Important: PBC estimate does not require use of OBC estimate!

Gap Methods for Quantum Spin Models



- For $\dim(\mathcal{H}_\Lambda) < \infty$ and frustration-free model: $\mathcal{G}_{\Lambda'} \equiv \ker(H_{\Lambda'} \otimes \mathbb{1}_{\Lambda\setminus\Lambda'}) \ \ \forall \Lambda' \subseteq \Lambda$.
- Uses a covering of Λ by a sequence of smaller intervals with OBC.

$$\gamma = \inf_i \mathrm{gap}(H_{\Lambda_i}), \qquad \Gamma = \sup_i \|H_{\Lambda_i}\|, \qquad \Lambda_{n,k} = \bigcup_{i=k}^{n-k+1} \Lambda_i$$

- Martingale Method: If $\epsilon := \sup_n \|G_{\Lambda_{n+1}}(\mathbb{1} - G_{\Lambda_{n+1,1}})G_{\Lambda_{n,1}}\| < 1/\sqrt{\ell}$, then

$$\mathrm{gap}(H_\Lambda) \geq \frac{\gamma}{\ell}(1 - \epsilon\sqrt{\ell})^2.$$

- Finite Size Criterion: For any n such that $|\Lambda_{n,k}| < |\Lambda|$ for all k:

$$\mathrm{gap}(H_\Lambda^{\mathrm{per}}) \geq \frac{\gamma(n-1)}{\ell\Gamma n}\left[\inf_{1\leq k\leq L} \mathrm{gap}(H_{\Lambda_{n,k}}) - \frac{\Gamma}{n}\right]$$

Spectral Rigidity: Motivation for Bounding $E_0\big((\mathcal{C}_\Lambda^{\#})^\perp\big)$

Question: How does the lowest energy in the N-particle sector depend on $\nu := N/|\Lambda|$?

$$E_0(\mathcal{H}_\Lambda^N) := \inf_{\substack{\psi\neq 0:\\ N_\Lambda\psi = N\psi}} \frac{\langle\psi|H_\Lambda^{\mathrm{per}}\psi\rangle}{\|\psi\|^2}$$

Spectral Rigidity: By Cauchy-Schwarz

$$q_k^* q_k \geq (1-\delta) n_k(n_k-1) - |\lambda|^2 \frac{1-\delta}{\delta} n_{k-1}n_{k+1}, \quad \forall\, \delta \in (0,1),$$

which bounds the Hamiltonian from below by purely electrostatic operators

$$H_\Lambda^{\mathrm{per}} \geq \sum_{k\in\Lambda} n_k \left[n_{k+1} + \frac{\kappa}{2}(n_k - 1) - \kappa|\lambda|^2 n_{k+2}\right].$$

For fillings $\nu > 1$ and $\kappa|\lambda|^2 < 1$, which includes the thin cylinder regime, this yields

$$E_0(\mathcal{H}_\Lambda^{\nu|\Lambda|}) \geq \nu|\Lambda| \left[\nu(1+\kappa/2 - \kappa|\lambda|^2) - \kappa/2\right].$$

Idea for $E_0\big((\mathcal{C}_\Lambda^{\#})^\perp\big)$: Refine CS bound to treat individual non-BVMD occupation states.

Bounding $E_0((\mathcal{C}_\Lambda^{\#})^\perp)$

Partition set of non-BVMD tiling configurations $\mathcal{S}_\Lambda^{\#} = \{\mu : |\mu\rangle \notin \mathcal{C}_\Lambda^{\#}\}$ as

$$\mathcal{S}_\Lambda^{\#} = \mathcal{S}_E^{\#} \dot\cup \mathcal{S}_D^{\#}, \quad \mathcal{S}_E^{\#} = \left\{\mu \in \mathbb{N}_0^\Lambda : e_\Lambda^{\#}(\mu) > 0\right\}$$

where $e_\Lambda^{\#}(\mu)$ is the electrostatic energy:

$$e_\Lambda^{\mathrm{obc}}(\mu) = \sum_{k=a}^{b-1} \mu_k \mu_{k+1}, \quad e_\Lambda^{\mathrm{per}}(\mu) = e_\Lambda^{\mathrm{obc}}(\mu) + \mu_b \mu_a.$$

For any $\psi = \sum_{\mu \in \mathcal{S}_\Lambda^{\#}} \psi(\mu)|\mu\rangle \in (\mathcal{C}_\Lambda^{\#})^\perp \cap \mathrm{dom}(H_\Lambda^{\#})$, the expected energy is

$$\langle \psi | H_\Lambda^{\#} \psi \rangle = \sum_{\mu \in \mathcal{S}_E^{\#}} e_\Lambda^{\#}(\mu)|\psi(\mu)|^2 + \sum_{\nu \in \mathbb{N}_0^\Lambda} \sum_{k \in \Lambda^{\#}} |\langle \nu | q_k \psi \rangle|^2$$

where $\Lambda^{\mathrm{obc}} = [a+1, b-1]$ and $\Lambda^{\mathrm{per}} = [a, b]$.

Goal: Choose (ν_μ, k_μ) for each $\mu \in \mathcal{S}_D^{\#}$ and apply CS to show $\langle \psi | H_\Lambda^{\#} \psi \rangle \geq \gamma^{\#} \sum_{\mu \in \mathcal{S}_\Lambda^{\#}} |\psi(\mu)|^2$:

$$\gamma^{\mathrm{obc}} = \frac{1}{5} \min\left\{4\gamma^{\mathrm{per}}, \frac{2\kappa|\lambda|^2}{\kappa+1}\right\}, \qquad \gamma^{\mathrm{per}} = \frac{1}{4} \min\left\{1, \frac{2\kappa}{\kappa+1}, \frac{2\kappa}{1+\kappa|\lambda|^2}\right\}$$

Bounding $E_0\big((\mathcal{C}_\Lambda^{\rm obc})^\perp\big)$

Example: $\mu = (201000\ldots0) \in \mathcal{S}_D^{\rm obc}$ (an edge state configuration).

For OBC: Choosing $k_\mu = a+1$ there is a ν_μ so that

$$|\langle \nu_\mu | q_{k_\mu} \psi\rangle|^2 = |\sqrt{2}\psi(\eta) - \overline{\lambda}\sqrt{2}\psi(\mu)|^2 \geq -2\frac{1-\delta}{\delta}|\psi(\eta)|^2 + 2|\lambda|^2(1-\delta)|\psi(\mu)|^2$$

where $\eta = (1200\ldots0) \in S_E^{\rm obc}$.

Bounding $E_0\big((\mathcal{C}_\Lambda^{\mathrm{obc}})^\perp\big)$

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where $\eta = (1200\ldots0) \in S_E^{\mathrm{obc}}$. Picking $\delta = 2\kappa/(1+2\kappa)$ produces the estimate

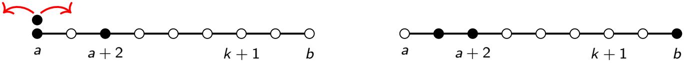
$$e_\Lambda^{\mathrm{obc}}(\eta)|\psi(\eta)|^2 + \kappa|\langle \nu_\mu | q_{k_\mu}\psi\rangle|^2 \geq |\psi(\eta)|^2 + \frac{2\kappa|\lambda|^2}{1+2\kappa}|\psi(\mu)|^2.$$

This reflects $\gamma^{\mathrm{obc}} = \mathcal{O}(|\lambda|^2)$ and goes to zero in Tao-Thouless limit!

General Strategy: Systematically chose a unique (ν_μ, k_μ) for each $\mu \in \mathcal{S}_D^{\mathrm{obc}}$ so that don't "eat" all electrostatic energy of any given $\eta \in \mathcal{S}_E^{\mathrm{obc}}$.

Bounding $E_0\big((\mathcal{C}_\Lambda^{\mathrm{per}})^\perp\big)$

Example: $\mu = (201000\ldots0) \in \mathcal{S}_D^{\mathrm{per}}$ (same configuration).



For PBC: For same μ, choosing $k_\mu = a$ with appropriate ν_μ produces:

$$|\langle \nu_\mu | q_{k_\mu} \psi \rangle|^2 = |\sqrt{2}\psi(\mu) - \lambda\psi(\eta)|^2$$

and applying similar strategy with a particular choice of $0 < \delta < 1$ yields:

$$e_\Lambda^{\mathrm{per}}(\eta)|\psi(\eta)|^2 + \kappa|\langle \nu_\mu | q_{k_\mu}\psi\rangle|^2 \geq \frac{1}{2}|\psi(\eta)|^2 + \frac{2\kappa}{1+2\kappa|\lambda|^2}|\psi(\mu)|^2$$

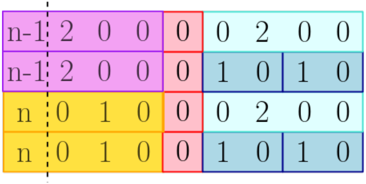
where $\eta = (0110\ldots01)$. Recall in physical regime $\kappa = \mathcal{O}(1)$ when $\lambda << 1$.

Edge Tiling Spaces

For open boundary conditions, every state with energy $\mathcal{O}(|\lambda|^2)$ belongs to an invariant subspace generated from a root tiling consisting of the BVMD-tiles

and at least one edge boundary tile: for $n \geq 2$

The edge tiling spaces require several new tiles and replacement rules. Nevertheless, these lattice tilings only differ from BVMD tilings at the edge.



Bulk Excitations

Lowest bulk excitations expected in quasi-hole/particle state. [Girvin, MacDonald, Platzman '85], [Yang, Hu, Papic, Haldane '12], [Wang, Nakamura '15]

We conjecture such states belong to invariant subspaces $D_{l,r}^{(m)}$ generated by roots of the form:

1	0	1	0	0	1	0	1	1	0	1	0

$\longleftarrow 2l \longrightarrow \longleftarrow 2m \longrightarrow \longleftarrow 2r \longrightarrow$

These spaces generate two new tiles and replacement rules:

0	1	2	0	0	1	0	1	0
0	2	0	1	0	1	0	1	0
1	0	1	1	0	1	0	1	0

1	0	0	2	1	0	0	1	0
1	0	1	0	2	0	0	1	0
1	0	1	1	0	1	0	1	0

Theorem: [Warzel, Y. '21] *For any l, $r \geq 3$, $\kappa > 1/2$ and $|\lambda|$ sufficiently small:*

$$\min_{m \in \mathbb{N}} E_0(D_{l,r}^{(m)}) = 1 - \frac{2\kappa}{2\kappa - 1} + \mathcal{O}(|\lambda|^4).$$

Can also identify a slew of many-body scars of mid- and high- energy:

1	0	1	0	0	0	1	1	0	0	1	0	1	0

Conclusion:

Summary:

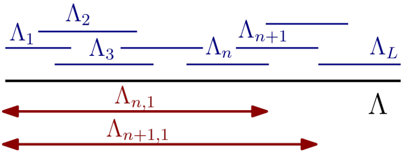
1. Proved conjectured low-lying energy properties of truncated pseudopotentials in a cylinder geometry facilitated by invariant subspaces described in terms of tilings.
2. A bulk gap strategy: approach valid for other models where edge states and ground states can be separated into different invariant subspaces.
   - E.g. our approach improves bulk gap result for 1/3-truncated model studied in [Nachtergaele, Warzel, Y. '21]

Interesting Questions and Future Directions:

1. Additional algebraic structure beyond symmetries?
2. Better control of first and second excited states. Low complexity?
3. Longer range truncations? Stability of the gap?
4. The untruncated Haldane model.

Thank you for your attention!

Bounding $E_1(\mathcal{C}_\Lambda^\infty)$



We apply the martingale method to the Hilbert space $\mathcal{C}_\Lambda^\infty$ and Hamiltonians

$$H_{\Lambda'}^\infty := (H_{\Lambda'} \otimes \mathbb{1}_{\Lambda\setminus\Lambda'}) \restriction_{\mathcal{C}_\Lambda^\infty} \quad \forall \Lambda' \subseteq \Lambda$$

where the overlapping intervals are chosen so that $\ell = 3$ and either $|\Lambda_i| = 5, 6$, for all i.

Calculations using isospectrality and orthogonality of the BVMD states/spaces gives

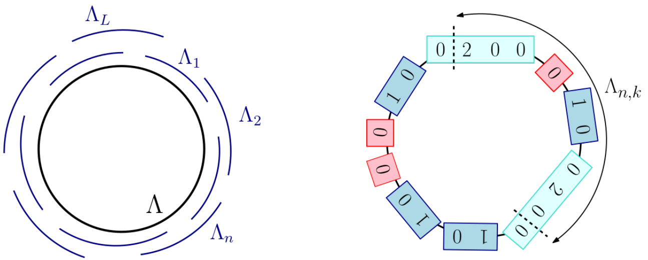
$$\gamma = \mathrm{gap}(H_{\Lambda_i}^\infty) = 2\kappa$$

$$\epsilon = \sup_n \| G_{\Lambda_{n+1}}^\infty (\mathbb{1} - G_{\Lambda_{n+1,1}}^\infty) G_{\Lambda_{n,1}}^\infty \| \leq \sqrt{f(|\lambda|^2/2)}$$

where $\mathcal{G}_{\Lambda'}^\infty = \ker(H_{\Lambda'}^\infty) \subseteq \mathcal{C}_\Lambda^\infty$. This produces the final estimate:

$$E_1(\mathcal{C}_\Lambda^\infty) \geq \frac{2\kappa}{3}\left(1 - \sqrt{3f(|\lambda|^2/2)}\right)^2.$$

Bounding $E_1(\mathcal{C}_\Lambda^{\mathrm{per}})$



We apply Knabe's finite size criteria to the Hilbert space $\mathcal{C}_\Lambda^{\mathrm{per}}$ and Hamiltonians

$$H_\Lambda^{\mathrm{per}} \restriction_{\mathcal{C}_\Lambda^{\mathrm{per}}} \quad \text{and} \quad H_{\Lambda_{n,k}} \restriction_{\mathcal{C}_\Lambda^{\mathrm{per}}}$$

where $\Lambda_{n,k} = \bigcup_{i=k}^{n+1-k} \Lambda_i$ chosen so $|\Lambda_i| = 5, 6$. Another isospectral argument shows

$$\|H_{\Lambda_i} \restriction_{\mathcal{C}_\Lambda^{\mathrm{per}}}\| = 2\kappa(1+|\lambda|^2) \quad \text{and} \quad \mathrm{gap}(H_{\Lambda_{n,k}} \restriction_{\mathcal{C}_\Lambda^{\mathrm{per}}}) = E_1(\mathcal{C}_\Lambda^\infty)$$

This yields:

$$E_1(\mathcal{C}_\Lambda^{\mathrm{per}}) \geq \frac{n-1}{2n(1+|\lambda|^2)} \left(E_1(\mathcal{C}_{\Lambda_{n,k}}^\infty) - \frac{2\kappa(1+|\lambda|^2)}{n} \right)$$