Harmonic analysis and gamma functions

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Basic Functions, Orbital Integrals, and Beyond Endoscopy
in honor of Prof. Casselman's 80th birthday
November 15, 2021

Riemann zeta function

- (Euler) For $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}, \quad \begin{cases} \text{ab. cov.} \\ = \prod_p \zeta_p(s) \end{cases}$$

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$$\xi(s) = \pi^{-s/2} \cdot \Gamma(s/2) \cdot \zeta(s) = \int_0^\infty \left(\frac{\theta(it) - 1}{2} \right) \cdot t^{s/2} \cdot \frac{\mathrm{d}t}{t}$$

with $\theta(\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau}$ (Jacobi's theta)

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- Dirichlet & Hecke L-functions.

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- Ingredients:

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  $$\mathcal{Z}(s,f,\chi) = \int_{\mathbb{A}^\times} f(x)\chi(x)|x|^s \,\mathrm{d}^*x, \quad f \in \mathcal{S}(\mathbb{A})$$

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  - (Fourier transform) $\mathcal{F}_\psi = \bigotimes_{\mathfrak{p}} \mathcal{F}_{\psi,\mathfrak{p}} : \mathcal{S}(\mathbb{A}) \simeq \mathcal{S}(\mathbb{A})$;

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- $$\mathcal{Z}(1-s,\mathcal{F}_{\psi_{\mathfrak{p}}}(\cdot),\chi_{\mathfrak{p}}^{-1}) = \gamma(s,\chi_{\mathfrak{p}},\psi_{\mathfrak{p}}) \cdot \mathcal{Z}(s,\cdot,\chi_{\mathfrak{p}});$$

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- Set
  $$\mathbb{L}_{\mathfrak{p}} = \begin{cases} \mathbb{1}_{\mathfrak{o}_{\mathfrak{p}}} & \mathfrak{p} \text{ non-Archi.} \\ \text{Gaussian} & \mathfrak{p} \text{ Archi.} \end{cases}$$
  Then $\mathcal{F}_{\psi_{\mathfrak{p}}}(\mathbb{L}_{\mathfrak{p}}) = \mathbb{L}_{\mathfrak{p}}$ & $\mathcal{Z}(s,\mathbb{L}_{\rho},\chi_{\mathfrak{p}}) = L(s,\chi_{\mathfrak{p}})$ for $\chi_{\mathfrak{p}}$ unramified;

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- Global Poisson summation for $\mathcal{F}_{\psi}\Rightarrow$ mero. cont. & fun. eq. $\mathcal{Z}(s,\cdot,\chi)\Rightarrow L(s,\chi)$;

Gelfand-Graev-Piatetski-Shapiro

- Observation: As distr. on $k_{\mathfrak{p}}$,

$$\gamma(s, \chi_{\mathfrak{p}}, \psi_{\mathfrak{p}}) \cdot \mathcal{F}_{\psi_{\mathfrak{p}}}(\chi_{\mathfrak{p}}|\cdot|^{s-1}) = \chi_{\mathfrak{p}}^{-1}|\cdot|^{-s};$$

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- $$\psi \longleftrightarrow \gamma(s, \chi_\mathfrak{p}, \psi_\mathfrak{p})$$

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- (Langlands) $L^S(s, \pi, \rho)$ ab. cov. for $\mathrm{Re}(s)$ large;

Functoriality conjecture

Conjecture (Langlands)

$L(s, \pi, \rho)$ has a mero. cont. to $s \in \mathbb{C}$ & fun. eq.

$$L(1-s, \pi^\vee, \rho) = \varepsilon(s, \pi, \rho) \cdot L(s, \pi, \rho)$$

w/ $\varepsilon(s, \pi, \rho)$ nonzero entire in $s \in \mathbb{C}$.

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- Methods: Tate, Godement-Jacquet; Rankin-Selberg; Langlands-Shahidi; Trace formula;

A question

Question

Understand the analytical properties of $L(s, \pi, \rho)$ and its local factors $L(s, \pi_{\mathfrak{p}}, \rho)$ through

$$\text{harmonic analysis} \begin{cases} \text{Schwartz space} \\ \text{Fourier transform} \\ \text{Poisson summation} \end{cases}$$

on G (& related spherical varieties);

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- (Sakellaridis) Generalize to affine spherical varieties (?);

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Conjectural ingredients

- Schwartz space $\mathcal{C}_c^\infty(G(k_\mathfrak{p})) \subset \mathcal{S}_\rho(G(k_\mathfrak{p})) \subset \mathcal{C}^\infty(G(k_\mathfrak{p}))$;

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- For $f \in \mathcal{S}_\rho(G(k_\mathfrak{p}))$, and $\varphi_{\pi_\mathfrak{p}} \in \mathcal{C}(\pi_\mathfrak{p})$, set

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$$n_\rho = \langle \rho_B, \lambda_\rho \rangle;$$

- In general different n_ρ differ by unramified shift;

Expectation: Schwartz space

- $\mathcal{Z}(s, f, \varphi_{\pi_\mathfrak{p}})$ is ab. cov. for $\mathrm{Re}(s)$ large, with a mero. cont. to $s \in \mathbb{C}$ and is a hol. multiple of $L(s, \pi_\mathfrak{p}, \rho)$;

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- For $\mathfrak{p}$ non-Archimedean, $\{\mathcal{Z}(s, f, \varphi_{\pi_\mathfrak{p}}) \mid f \in \mathcal{S}(G(k_\mathfrak{p})), \varphi_\mathfrak{p} \in \mathcal{C}(\pi_\mathfrak{p})\} = L(s, \pi, \rho) \cdot \mathbb{C}[q^s, q^{-s}]$;

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- There exists $\mathbb{L}_{\rho,\mathfrak{p}} \in \mathcal{S}(G(k_\mathfrak{p}))^{K_\mathfrak{p} \times K_\mathfrak{p}}$, such that $\mathcal{F}_{\rho,\psi_\mathfrak{p}}(\mathbb{L}_{\rho,\mathfrak{p}}) = \mathbb{L}_{\rho,\mathfrak{p}}$ and $\mathcal{Z}(s, \mathbb{L}_{\rho,\mathfrak{p}}, \varphi_\mathfrak{p}) = L(s, \pi_\mathfrak{p}, \rho)$ for $\pi_\mathfrak{p}$ unramified and $\varphi_\mathfrak{p}$ zonal sperhical (Casselman-Shalika formula);

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- $\mathbb{L}_{\rho,\mathfrak{p}} = \mathrm{char}(\mathrm{M}_n(\mathfrak{o}_\mathfrak{p}))$;

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- $$\mathcal{Z}(1-s, \mathcal{F}_{\psi_{\mathfrak{p}}}(f), \varphi_{\pi_{\mathfrak{p}}}^{\vee}) = \gamma(s, \pi_{\mathfrak{p}}, \rho, \psi_{\mathfrak{p}}) \cdot \mathcal{Z}(s, f, \varphi_{\mathfrak{p}});$$

Expectation: Fourier transform

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- $\mathcal{F}_{\rho,\psi_{\mathfrak{p}}} \circ \mathcal{F}_{\rho,\psi_{\mathfrak{p}}^{-1}} = \mathrm{Id};$
- For $\rho = \mathrm{std}$ of GL_n, $\Phi_{\rho,\psi_{\mathfrak{p}}}(\cdot) = \psi(\mathrm{tr}(\cdot)) \cdot |\det(\cdot)|^{n}$;

Preliminary analysis for local unramified

Proposition (L.)

- For $\mathfrak{p}$ non-Archimedean, we have the equalities

$$\mathcal{S}_\rho(G(k_\mathfrak{p}))^{K_\mathfrak{p}\times K_\mathfrak{p}} = \mathbb{L}_{\rho,\mathfrak{p}} * \mathcal{C}_c^\infty(G(k_\mathfrak{p}))^{K_\mathfrak{p}\times K_\mathfrak{p}}$$
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- For $\mathfrak{p}$ Archimedean, take $\mathbb{L}_{\rho,\mathfrak{p}}$ as the inverse Harish-Chandra transform of $L(s,\pi_\mathfrak{p},\rho)$. Then for $\mathrm{Re}(s)$ large,

$$\mathbb{L}_{\rho,\mathfrak{p}}\cdot|\sigma(\cdot)|^s \quad \text{and } \Phi^{K_\mathfrak{p}}_{\rho,\psi_\mathfrak{p}}\cdot|\sigma(\cdot)|^s$$

  can be plugged into the Arthur-Selberg trace formula.

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- (Braverman-Kazhdan) Algebraic integration and the datum on tori;
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- Finite field analogue has been resolved by Cheng-Ngô for $G = \mathrm{GL}_n$, T.-H. Chen for $\mathcal{D}$-module setting and almost all finite fields, Laumon-Letellier over any finite fields;

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  - The space of regular functions on the Bernstein variety

$$\Omega(G(F)) = \bigsqcup_{(M,\sigma)} X_{M,\sigma},$$

with $X_{M,\sigma} = \{[M, \chi \cdot \sigma]_G \mid \chi \in \Psi(M)\}$;

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- (Bernstein) $\Phi_{\mathrm{std}}|_{\mathrm{SL}_n} \in \mathfrak{Z}(\mathrm{SL}_n(F))$

π-Hankel transform

- Motivated from the observation of Bernstein, (Jiang-L.) construct π-Hankel transform over any local field F of characteristic zero

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\mathcal{S}(\mathrm{M}_n(F)) \otimes \mathcal{C}(\pi) & \xrightarrow{\mathcal{F}_{\psi}, (\cdot)^{\vee}} & \mathcal{S}(\mathrm{M}_n(F)) \otimes \mathcal{C}(\pi^{\vee}) \\
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  for $\{\mathfrak{c}_\ell\}$ any delta sequence to Id_n;
- Independent of $\{\mathfrak{c}_\ell\}$ and $\varphi_{\widetilde{\pi}}(\mathrm{I}_n) = 1$;
- More general: Kirillov model: generic representations of π can be realized on the same variety P_n/U_n with P_n mirabolic, but different Schwartz and Fourier captured by π;

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- (Piatetski-Shapiro, Rallis) $\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2n} \curvearrowright \mathrm{Ind}_P^{\mathrm{Sp}_{4n}}(\chi_s)$ with analytical properties of zeta integrals captured by intertwining operators;

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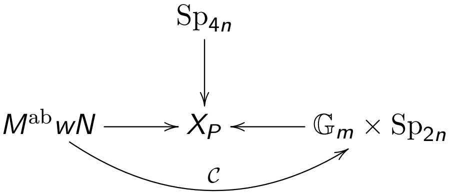
The following diagram illustrates the transition between the work of Piatetski-Shapiro and Rallis to Jiang-L.-Zhang,



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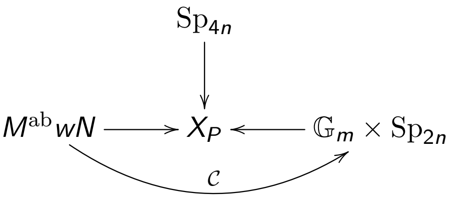

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- This formulation has been further studied by Getz-Hsu-Leslie for G split, simple and simply connected.

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- For $f \in \mathcal{C}_c^\infty(X_P(F))$, set

$$\mathcal{F}_{X,\psi}(f)(g) = \int_{F^\times}^{\mathrm{reg}} \eta_{\mathrm{pvs},\psi}(x)|x|^{-\frac{2n+1}{2}} \int_{N(F)} f(wn\mathfrak{s}(x)g)\,\mathrm{d}n\,\mathrm{d}x$$

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- $$\begin{aligned}&\int_{F^\times}^{\mathrm{reg}} \eta_{\mathrm{pvs},\psi}(x)\chi_s^{-1}(x)\,\mathrm{d}^\times x\\ =&\gamma(s-\frac{2n-1}{2},\chi,\psi)\cdot\prod_{i=0}^{n-1}\gamma(2s-2n+2i,\chi^2,\psi).\end{aligned}$$

Jiang-L.-Zhang: Schwartz space

Proposition (JLZ)

- Set

$$\mathcal{S}(X_P(F)) := \mathcal{C}_c^\infty(X_P(F)) + \mathcal{F}_{X,\psi}(\mathcal{C}_c^\infty(X_P(F));$$

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- A function $f \in \mathcal{C}^\infty(X_P(F))$ belongs to $\mathcal{S}(X_P(F))$ if and only if f is right $K_{\mathrm{Sp}_{4n}}$-finite, and as a function in $a \in F^\times$,

$$|a|^{2n+1} \cdot f(\mathfrak{s}_a^{-1} k)$$

  belongs to $\mathcal{S}^-_{\mathrm{pvs}}(F^\times)$ for any fixed $k \in K_{\mathrm{Sp}_{4n}}$;

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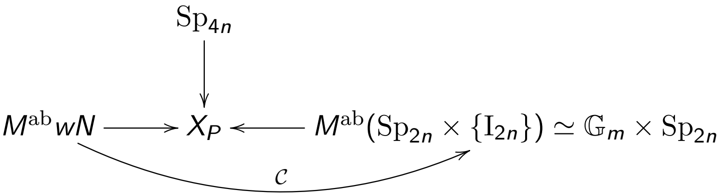
- In particular, the reductive monoid $\mathcal{M}_\rho$ attached to (G,ρ) in this situation is exactly given by $\overline{X}_P^{\mathrm{aff}}$;

Jiang-L.-Zhang: Fourier operator



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$$\begin{array}{ccc} & \mathrm{Sp}_{4n} & \\ & \downarrow & \\ M^{\mathrm{ab}}wN & \longrightarrow X_P \longleftarrow & M^{\mathrm{ab}}(\mathrm{Sp}_{2n} \times \{\mathrm{I}_{2n}\}) \simeq \mathbb{G}_m \times \mathrm{Sp}_{2n} \end{array}$$

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- $\mathcal{C}$ is given by the Cayley transform;
- $$\mathrm{j}_{\mathcal{C}^{-1}}(h) = \frac{1}{\zeta_F(2i)} \cdot |\det(h - \mathrm{I}_{2n})|^{-(2n+1)};$$

Jiang-L.-Zhang: Fourier operator

- For $f \in \mathcal{S}(X_P(F))$, set

$$\phi_f(a, h) := f(\mathfrak{s}(a)^{-1} \cdot (h, \mathrm{I}_{2n}))|a|^{\frac{2n+1}{2}}$$

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$$\mathcal{F}_{\rho,\psi}(\phi_f)(a,h) := \int_{F^\times}^{\mathrm{reg}} \int_{\mathrm{Sp}_{2n}(F)} \Phi_{\rho,\psi}(ax, gh) \cdot \phi_f(x,g)\, \mathrm{d}g\, \mathrm{d}x.$$

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L.-Ngô (in progress)

- For $G = \mathrm{GL}_2$, $\rho : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$,

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- (L.) Modulo convergence, true for $\rho = \mathrm{Sym}^2$ (= JLZ); Also for $\rho = \mathrm{std}$ of $G = \mathbb{G}_m \times \mathrm{SO}_4$;

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where

$$\mathrm{D}(\alpha_i, a_i)$$

is the symmetric polynomial attached to variables $(t_i)_{1\le i\le n}$ with

$$\mathrm{D}(\alpha_i, \mathrm{tr}\wedge^i t) = \sum_{I\in\mathcal{I}_{n-2}} \Big(\sum_{j=1}^{n-1} \alpha_i \cdot \mathrm{tr}\wedge^{i-2} t_I\Big).$$

Here $\mathcal{I}_{n-2} = \{(i_1, \dots, i_{n-2}) \mid 1 \le i_1 < \dots < i_{n-2} \le n\}$.

Thank You and happy birthday Prof. Casselman!