THE ALGEBRAIC AND TRANSCENDENTAL PARTS OF THE SPECTRA OF ARITHMETIC MANIFOLDS

PETER SARNAK

BIRS NOV 2021

W. CASSELMAN'S 80TH BIRTHDAY (NOV 27).

CASSELMAN'S MANY INFLUENTIAL WORKS INCLUDE:

- CASSELMAN-SHALIKA; WHITAKER FUNCTIONS FOR UNRAMIFIED PRINCIPAL SERIES FOR p-ADIC GROUPS.
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CELEBRATED RESULTS

- "GL_n" IN ALGEBRAIC NUMBER FIELDS L-FUNCTIONS GALOIS PROPERTIES DURHAM (1975)

CONTAINS A CLEAR EXPLANATION OF MAASS FORMS OF EIGENVALUE 1/4.

- "A CONJECTURE ABOUT THE ANALYTIC BEHAVIOR OF EISENSENSTEIN SERIES" PURE APP. QUAT (2005)

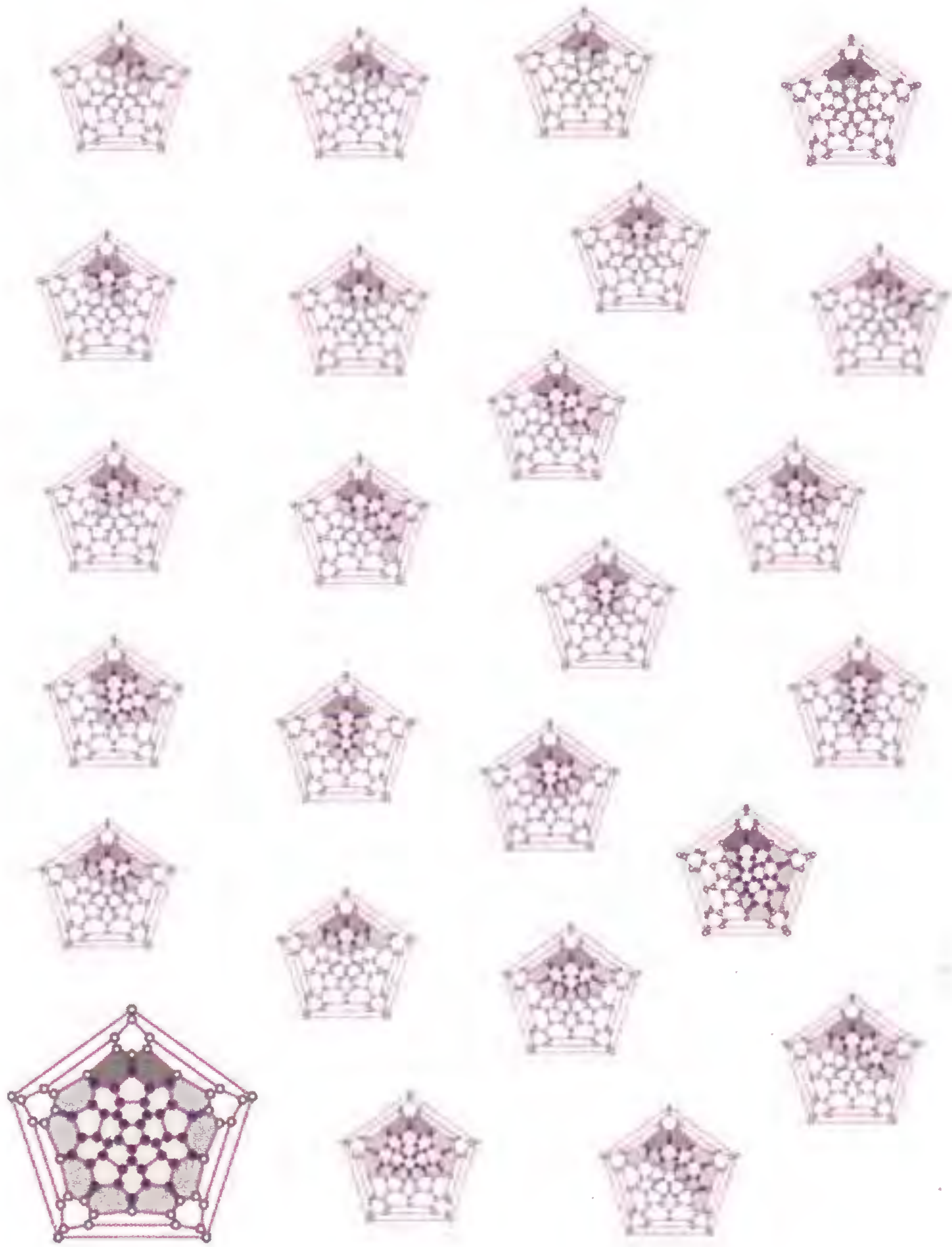
LIKE AN OLDER COUSIN TO ME, FOR BILL THE ANALYTIC CONTINUATION OF EISENSTEIN SERIES IS NOT A BLACK BOX; ANALYSIS OF THE METHOD, OF POLES, GROWTH RATES, ARE FUNDAMENTAL!

ISSN 0002-9920

Notices

of the American Mathematical Society

August 2004 Volume 51, Number 7



Calculating an Expansion Constant *(see page 815)*

Reference and Book List

Sync: The Emerging Science of Spontaneous Order, by Steven Strogatz. Hyperion, February 2003. ISBN 0-786-86844-9. (Reviewed March 2004.)

Travels in Four Dimensions: The Enigmas of Space and Time, by Robin Le Poidevin. Oxford University Press, February 2003. ISBN 0-19-875254-7.

* *Turing (A Novel about Computation)*, by Christos H. Papadimitriou. MIT Press, November 2003. ISBN 0-262-16218-0.

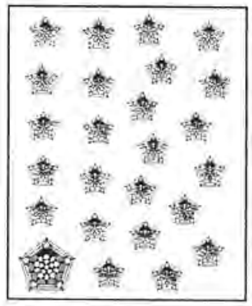
What Is Thought?, by Eric B. Baum. MIT Press, January 2004. ISBN 0-262-02548-5.

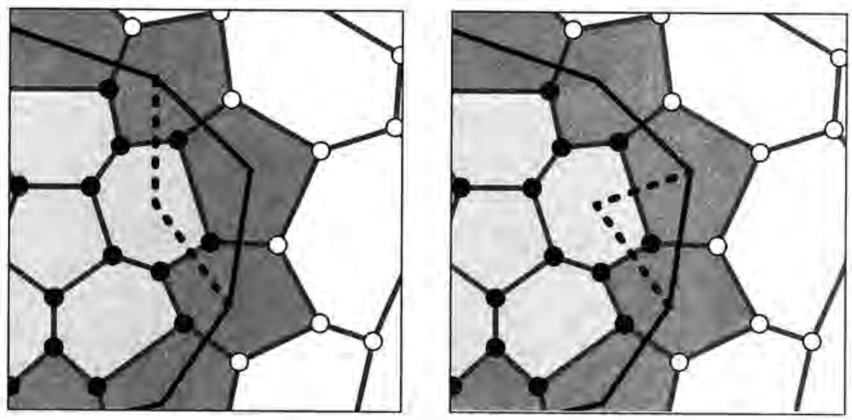
What the Numbers Say: A Field Guide to Mastering Our Numerical World, by Derrick Niederman and David Boyum. Broadway Books, April 2003. ISBN 0-767-90998-4.

When Least Is Best: How Mathematicians Discovered Many Clever Ways to Make Things As Small (or As Large) As Possible, by Paul J. Nahin. Princeton University Press, November 2003. ISBN 0-691-07078-4.

About the Cover

In preparing diagrams for Peter Sarnak's article in this issue on Ramanujan graphs, we decided that it would be an interesting exercise to verify that its expansion constant h is $1/4$. I recall that if the graph has N nodes, then this constant is the minimum value of $|\partial X|/|X|$, where X varies over the subsets of nodes of size at most $N/2$. Thus a priori one might expect to have to look at close to 2^{80} subsets of nodes, and, indeed, it has been shown by M. Blum et al. (*Inform. Process. Lett.* **13** (1981)) that this is a very difficult problem. For the graph at hand, however, Sarnak was able to verify by hand that $h = 1/4$, and it was also possible to verify the calculation with a computer program that might work as well for more general 3-regular graphs. The basic idea of the program is to look at the possible *cut sets* separating X from its complement. There are two key observations that the program is based on. The first is that one need only look at connected subsets X, and in fact only at cut sets that are Jordan curves. The second observation is that in certain circumstances one need only look at cut sets that satisfy a kind of convexity condition at each vertex. The exact conditions ought to be clear from the accompanying diagrams, where the dashed lines cannot be the cut sets for a candidate X, since adjusting them in a simple way increases $|X|$ without decreasing $|\partial X|$. (The nodes in X are dark.)





It is straightforward and entirely practical to make up an algorithm that constructs all admissible cut sets. If $|X|$ for all of these is not greater than half the number of nodes, the convexity argument above shows that h can be calculated by perusing the list. Because of the symmetries in the graph at hand, it is necessary to consider only two types of cut sets, and the cover illustration is, in effect, the program output for one of these types. It shows all convex cut sets passing through the top two gray faces (up to mirror symmetry). The minimum value $1/4$ is achieved in the large diagram at lower left, where $|X|$ is also the maximum value of 40. This graph is a Ramanujan graph. A result of Lipton and Tarjan (*SIAM J. Appl. Math.* **36** (1979)) implies that there are at most finitely many planar Ramanujan graphs. The largest ones known are 84:20, and 84:23 in the *Atlas of Fullerenes* by P. Fowler and D. Manolopoulos. I'd like to thank A. Gamburd for calling my attention to the graph used here, which he found in a paper by P. Frankin on the four-color problem, and also for telling me about Fullerenes.

—Bill Casselman
Covers/Graphics Editor
(notices-covers@ams.org)

G SEMISIMPLE REAL

Γ AN ARITHMETIC (EVEN CONGRUENCE) SUBGROUP.

K A SUBGROUP STABILIZING A CARTAN INVOLUTION

$G/K = S$ IS A GLOBALLY SYMMETRIC SPACE.

$$X = \Gamma \backslash S = \Gamma \backslash G / K$$

SPECTRUM OF $L^2(X)$; SPECTRUM OF THE RING OF INVARIANT OPERATORS ON X.

BETTER STILL $Y = \Gamma \backslash G$

SPECTRUM OF $L^2(Y)$ UNDER THE REGULAR REPRESENTATION OF G.

ASSUME THAT $\Gamma \backslash G$ IS COMPACT SO THAT THE SPECTRUM IS DISCRETE.

• OUR INTEREST IS IN THE ALGEBRAIC AND TRANSCENDENTAL PARTS.

SIMPLEST EXAMPLE:

$X = \mathbb{R}/\mathbb{Z}$ IS A CIRCLE OF LENGTH ℓ.

$$\Delta = \frac{d^2}{dx^2}$$

$$\text{SPEC}\left(\pm\sqrt{\Delta}\right) = \left\{\frac{2\pi}{\ell} m\right\}_{m\in\mathbb{Z}} \text{ ; MULT } 2$$

SO IF $\ell \in \overline{\mathbb{Q}}$ THEN SPEC(X) CONSISTS OF 0 AND THE REST ARE TRANSCENDENTAL, BUT

$$\text{TRANS DEG}_{\mathbb{Q}}(\text{SPEC}(X)) = 1.$$

THE SPECTRUM IS AN ARITHMETIC PROGRESSION.

TRACE FORMULA (POISSON-SUMMATION)

NORMALIZE FOURIER TRANSFORM

$$\int f(x)\, e^{-2\pi i \xi x}\, dx$$

$$\widehat{\sum_{m\in \text{spec}(X)} \delta_m} = \sum_{m\in \text{SPEC}(X)} \delta_m .$$

EG 2: $X = S^2$ ROUND 2-SPHERE

THEN BY USING SPHERICAL HARMONICS

$\text{SPEC}(\Delta_X)$ IS AFTER NORMALIZATION

$\lambda_j = j(j+1),\ j \geq 0$ WITH MULT $2j+1$.

SO IS ALGEBRAIC.

· MORE GENERALLY AND REMARKABLY IF S IS ANY COMPACT GLOBALLY SYMMETRIC SPACE THE SPECTRUM CAN BE COMPUTED AND IS ALGEBRAIC,

IN FACT THE FULL SPECTRUM OF $L^2(G)$ IS COMPUTED BY WEYL (WEYL'S CHARACTER FORMULA).

- IN THE CASE THAT G AND S ARE NOT COMPACT, $L^2(G)$ AND $L^2(S)$ HAVE VERY LITTLE DISCRETE SPECTRA, IF AT ALL.

$\underline{L^2(G):}$ HARISH-CHANDRA'S DISCRETE SERIES

HE DESCRIBES EXPLICITLY AND ALGEBRAICALLY (IN OUR SENSE) ~~[illegible]~~ ALL OF THE DISCRETE SPECTRUM (IFF RANK G = RANK K) THROUGH HIS CHARACTER THEORY AND ASYMPTOTICS OF MATRIX COEFFICIENTS.

- ON MANY OCCASIONS LANGLANDS HAS COMMENTED ON HOW INSTRUMENTAL THE ABOVE WAS TO HIS OWN WORK.

NOTE: SINCE WE ARE INTERESTED IN ADELE GROUPS $G(\mathbb{A})$ AND NOT JUST THE ARCHIMEDIAN SPECTRUM; IF SAY G IS DEFINED OVER $\mathbb{Q}$ AND $G(\mathbb{R})$ IS COMPACT, THEN NOT ONLY IS $\mathrm{SPEC}(G(\mathbb{R}))$ ALGEBRAIC BUT ALSO THE SPECTRA OF ALL THE HECKE OPERATORS.

(THESE ACT ON FINITE DIMENSIONAL $G(\mathbb{R})$ SPACES BY CONVOLUTION BY ALGEBRAIC MATRICES, OR BY THE TRACE FORMULA..)

- SO THE BASIC QUESTION FOR US FOR OUR LOOSE NOTION OF ALGEBRAIC SPECTRUM, IS WHEN $G = G(\mathbb{R})$ IS NON-COMPACT.

- THE TRANSCENDENCE IS TIED TO THE ARCHIMEDIAN PLACES!

A "TOY PROBLEM" (JOINT WITH P. KURASOV)

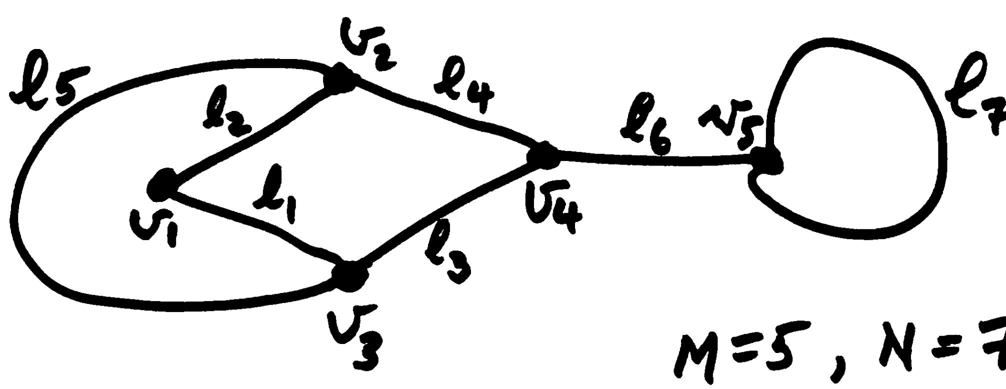
METRIC GRAPHS:

THE ONLY COMPACT SMOOTH ONE-DIMENSIONAL RIEMANNIAN MANIFOLD IS A CIRCLE OF LENGTH ℓ.

WHAT IF WE ALLOW A FINITE NUMBER OF SINGULARITIES, THAT IS HOMOGENEOUS INTERVALS OF LENGTHS $\ell_1, \dots, \ell_N$ GLUED AT VERTICES $v_1, \dots, v_M$ (SINGULARITIES)

$\Rightarrow$ CONNECTED METRIC GRAPH X

X:



$M=5$, $N=7$

TOPOLOGICALLY X IS ALREADY NON-TRIVIAL:

$\pi_1(X) = \Gamma$ IS A FREE GROUP OF RANK $N-M+1$.

THE LAPLACIAN Δ IS $\frac{d^2}{dx^2}$ ON EACH EDGE AND WE RESOLVE THE SINGULARITY AT THE VERTICES WITH NEUMANN BDRY CONDITIONS

$$\phi : X \to \mathbb{C}$$

- ϕ IS CONTINUOUS AT EACH v
- $\sum_e \partial_e \phi(v) = 0$ AT EACH v,

THE SUM IS OVER ALL DIRECTED e TERMINATING AT v
∂_e IS DERIVATIVE ALONG e.

$$\Delta \phi + k^2 \phi = 0$$

SELF-ADJOINT

$$\text{SPEC}(X) = \{ k's \}$$

DISCRETE SUBSET OF $\mathbb{R}$.

WHAT CAN BE SAID ABOUT THE ARITHMETIC STRUCTURE OF SPEC(X), ADDITIVE, TRANSCENDENTAL ?

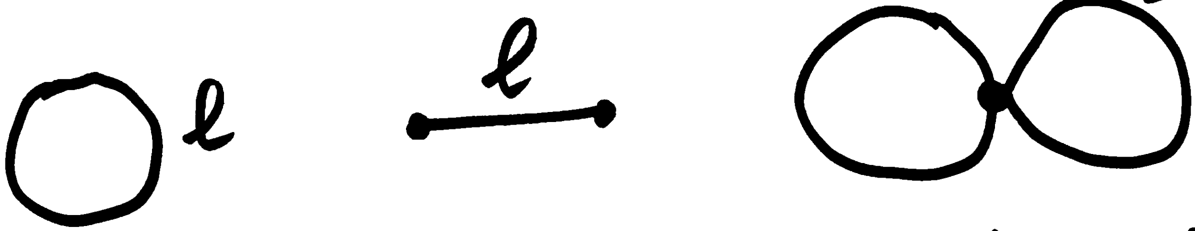
THESE ARE SPECIAL CASES OF QUANTUM GRAPHS, INTRODUCED BY CHEMISTS AND STUDIED BY PHYSICISTS, .., SMILANSKY, ...

- EACH LOOP OF LENGTH ℓ IN X GIVES A FULL ARITHMETIC PROGRESSION

$$\left\{ \frac{2\pi m}{\ell} \right\}_{m \in \mathbb{Z}} \quad \text{IN} \quad \mathrm{SPEC}(X)$$

$$\left(\quad \phi_m(x) = \sin\left(\frac{2\pi x m}{\ell}\right), \text{ IF } x \in \text{LOOP}, 0 \text{ OTHERWISE} \right)$$

- THE THREE X'S :



ARE SPECIAL, IN THAT SPEC(X) IS AN ARITHMETIC PROGRESSION IN THE FIRST TWO AND A UNION OF 3-ARITHMETIC PROG IN THE THIRD.

IN WHAT FOLLOWS WE AVOID THESE THREE METRIC GRAPHS.

~~LET $N(X)$ BE THE COMPLEMENT IN SPEC(X) OF THE ARITH. PROG.~~

· LET $N(X)$ BE THE COMPLEMENT IN $\underline{L^{11}}$ SPEC(X) OF THE ARITHMETIC PROGRESSIONS COMING FROM THE LOOPS.

<u>THEOREM</u> (KURASOV-S 2021)

ASSUME THAT $\ell_1, \dots, \ell_N$ ARE LINEARLY INDEPENDENT OVER $\mathbb{Q}$ THEN

(i) $\dim_{\mathbb{Q}} \operatorname{span}_{\mathbb{Q}} (N(X)) = \infty$

(ii) THERE IS $C = C(X)$ S.T. FOR ANY ARITHMETIC PROGRESSION $\underline{P} \subset \mathbb{R}$

$$|\underline{P} \cap N(X)| \leq C .$$

<u>SCHANUEL'S CONJ:</u>

IF $z_1, z_2, \dots, z_N$ ARE LINEARLY INDEPENDENT OVER $\mathbb{Q}$ THEN

$$\operatorname{TRANS\,DEG}_{\mathbb{Q}} (z_1, \dots, z_N, e^{z_1}, \dots e^{z_n}) \geq n .$$

<u>COR TO THEOREM:</u> IF $\ell_1, \ell_2, \dots, \ell_N$ ARE LINEARLY INDEPENDENT OVER $\mathbb{Q}$ AND ARE ALGEBRAIC AND ASSUMING SCHANUEL

$$\operatorname{TRANS\,DEG}_{\mathbb{Q}} (N(X)) = \infty .$$

TRACE FORMULA FOR X (ROTH, KOTTOS/SMILANSKY / KURASOV / ..)

X A METRIC GRAPH

$$\widehat{\sum_{k \in \mathrm{SPEC}(X)} \delta_k} = \frac{2(\ell_1 + \dots + \ell_N)}{\pi}\delta_0 + \frac{1}{\pi}\sum_{p \in \underline{P}} \ell(\mathrm{prim}\, p)\left[S_v(p)\,\delta_{\ell(p)} + \overline{S_v(p)}\,\delta_{-\ell(p)}\right].$$

WHERE:

- $\underline{P}$ IS THE SET OF ORIENTED PATHS IN X UP TO CYCLIC EQUIVALENCE (BACKTRACKING ALLOWED).
- $\mathrm{prim}(p)$ THE PRIMITIVE PART GOING AROUND p
- $\ell(p)$ THE LENGTH OF THE PATH
- $S_v(p)$ IS THE PRODUCT OF THE 'SCATTERING' COEFF AT THE VERTICES ENCOUNTERED ON TRAVERSING p

$$s = \begin{cases} -1 + \dfrac{2}{\deg(v)} \\[2ex] \dfrac{2}{\deg v} \\[2ex] 0 \quad \text{OTHERS} \end{cases}$$

$\mu = \sum_{k \in \mathrm{SPEC}(X)} \delta_k$ IS A POSITIVE MEASURE SUPPORTED ON A DISCRETE SET AND $\hat{\mu}$ IS SUPPORTED IN THE DISCRETE SET $\{m_1 \ell_1 + \cdots + m_N \ell_N : m_j \geq 0 \text{ in } \mathbb{Z}\}$.

"CRYSTALLINE MEASURE"

. μ PROVIDES EXOTIC SUCH MEASURES WHICH WERE SOUGHT AFTER — GENERALIZED POISSON SUMMATION WHICH ARE FAR FROM ARITHMETIC PROGRESSIONS.

THE PROOF OF THE THEOREM AND ITS COROLLARY MAKES USE OF THE FULL QUANTITATIVE VERSIONS OF "LANG'S G_m CONJECTURES" PROVED BY M. LAURENT, EVERTSE-SCHLICKEWEI-SCHMIDT ; ALL BASED ON SCHMIDT'S SUBSPACE THEOREM.

. THE CONNECTION TO SPEC(X) IS VIA AN ENTIRE QUASI-PERIODIC FUNCTION AND ITS ZEROS. (NOT EXPLICIT !)

FOR OUR TOY PROBLEM OF METRIC GRAPHS, THERE SOME 'OBVIOUS' POINTS IN THE SPECTRUM COMING FROM LOOPS GIVING ARITHMETIC PROGRESSIONS, THESE HAVE A BOUNDED TRANSCENDENCE DEGREE. THE REST OF THE SPECTRUM IS HIGHLY TRANSCENDENTAL WHEN $\ell_1, \dots, \ell_N$ ARE ALGEBRAIC (THE "ARITHMETIC X's").

WE EXPECT THAT A SIMILAR PICTURE HOLDS FOR $\Gamma\backslash G$ (G NON-COMPACT) THOUGH THE "OBVIOUS SPECTRUM" IS MUCH LESS OBVIOUS, AND VERY LITTLE CAN BE PROVED.

ABELIAN G , GL(1) :

CHARACTERS OF ABELIAN GROUPS CAN BE DESCRIBED EXPLICITLY SO THAT THESE ALGEBRAIC / TRANSCENDENCE QUESTIONS CAN BE ADDRESSED.

FOR GL_1 / K , K A NUMBER FIELD
HECKE DISTINGUISHED TWO TYPES:

- FINITE IMAGE
- INFINITE IMAGE "GROSSEN".

WEIL DEFINED AN EXTENSION W_K OF $\mathrm{GAL}(\bar{K}/K)$ WHOSE ONE DIMENSIONAL REPRESENTATIONS CORRESPOND TO ALL OF THE HECKE CHARACTERS.

HE DISTINGUISHES TWO TYPES OF GROSSEN-CHAR:

- TYPE A_0: WHOSE COEFF LIE IN A FIXED NUMBER FIELD
- NOT TYPE A_0 WHICH HE EXPECTS ARE TRANSCENDENTAL.

EXAMPLES: $K = \mathbb{Q}(\sqrt{D})$, $D > 0$ CLASS NUMBER ONE

$$\alpha \xrightarrow{\lambda_m} \left(\frac{\alpha}{\alpha'} \right)^{m i \pi / \log \varepsilon}$$

α' CONJ OF α, $m \in \mathbb{Z}$, ε FUND UNIT IN $\mathcal{O}(K)$.

CLOSELY RELATED IS

$$G = SO_f \quad , \quad f(x_1, x_2) = x_1^2 - D x_2^2$$

$G(\mathbb{R})$ NON-COMPACT!

ONE CAN SHOW (WALDSCHMIDT) THAT TAKING DIFFERENT PRIMES α YIELD TRANSCENDENTAL VALUES FOR $\lambda_m(\alpha)$, USING THE SIX EXPONENTIALS THEOREM

z_1, z_2, z_3 LINEARLY INDEP OVER $\mathbb{Q}$

w_1, w_2 LINEARLY INDEP OVER $\mathbb{Q}$

THEN AT LEAST ONE OF

$e^{z_i w_j}$ IS TRANSCENDENTAL.

$L^2(\Gamma \backslash G)$, $G(\mathbb{R})$ NON-COMPACT SEMISIMPLE.

CLOZEL AND MORE RECENTLY BUZZARD AND GEE HAVE FORMULATED ANALOGUES OF TYPE A_0 FOR GENERAL G OVER A NUMBER FIELD K.

- AN AUTOMORPHIC FORM IS ALGEBRAIC OF TYPE A_0 IF ITS SATAKE PARAMETERS AND L-PARAMETERS ARE IN A FIXED ALGEBRAIC EXT^{N} (THERE ARE SUBTLETIES WITH TWISTS ...)

WE EXAMINE THE SIMPLEST CASE $GL_2/\mathbb{Q}$ WHICH ALREADY CONTAINS MANY OF THE KEY ISSUES.

- THE DISCRETE SERIES OF $SL_2(\mathbb{R})$ CORRESPOND TO HOLOMORPHIC FORMS FOR WHICH EVERYTHING IS ALGEBRAIC.

MAASS FORMS

WT 0 (ALSO WT 1)

$$\Delta\phi + \left(\tfrac{1}{4} + k^2\right)\phi = 0$$

$$\phi(\gamma z) = \phi(z) \quad , \quad \gamma \in \Gamma \leq SL_2(\mathbb{Z})$$

A CONGRUENCE SUBGROUP.

$$\phi \in L^2_{cusp}(\Gamma \backslash \mathbb{H}) \quad , \text{ HECKE-EIGENFORM.}$$

WHAT IS THE ADDITIVE / ALGEBRAIC STRUCTURE OF THIS DISCRETE k-SPECTRUM IN $\mathbb{R}$?

$k = 0$; $\lambda = 1/4$: PRESUMABLY THESE CORRESPOND TO THE EVEN TWO DIMENSIONAL IRREDUCIBLE FINITE ARTIN GALOIS REPREPRENTATIONS ρ (AS EXPLAINED SO SUCCINCTLY BY CASSELMAN).

- MAASS ; LANGLANDS - TUNNEL GIVE ONE DIRECTION IF ρ IS SOLVABLE.

- NO ICOSAHEDRAL CASE IS KNOWN.

FOR $k \neq 0$; THE ONLY EXPLICIT EXAMPLES KNOWN ARE (INDIVIDUALLY) TRANSCENDENTAL AND LIE IN ARITHMETIC PROGRESSION

$$k_m = \frac{\pi m}{\log \varepsilon_D}, m \in \mathbb{Z} \quad ; \quad \Gamma = \Gamma(4D)$$

DUE TO MAASS VIA THETA LIFT FROM SO_f, $f = x^2 - Dy^2$ OR USING HECKE NON-TYPE A_0 GROSSEN CHARAKTERE.

(Y. PETRIDIS)

- FOR SUITABLE SUCH Γ ONE CAN CONSTRUCT A LARGE NUMBER OF SUCH ARITHMETIC PROGRESSIONS IN SPEC$(\Gamma \backslash \mathbb{H})$ AND THERE CLOSED GEODESICS ON X THAT CORRESPOND VIA POISSON SUM TO THESE — HOWEVER THE RELATION HERE IS NOT OBVIOUS LIKE THAT FOR LOOPS ON METRIC GRAPHS.

CONJECTURE:

$$\dim_{\mathbb{Q}} \operatorname{SPAN}\left[\operatorname{SPEC}\left(L^2_{cusp}(X_\Gamma)\right)\right] = \infty$$

$$\operatorname{TRANSDEG}_{\mathbb{Q}}\left[\operatorname{SPEC}\left(L^2_{cusp}(X_\Gamma)\right)\right] = \infty .$$

VERY LITTLE IS KNOWN IN THIS DIRECTION

THEOREM (S ; F. BRUMLEY)

IF π IS AN AUTOMORPHIC CUSP FORM ON $GL_2/\mathbb{Q}$ WHICH IS A MAASS FORM AT ∞ AND WHOSE COEFF ARE INTEGERS IN A QUADRATIC NUMBER FIELD K, $K \neq \mathbb{Q}(\sqrt{5})$, THEN $\lambda = 1/4$ AND π CORRESPONDS TO A SOLVABLE TWO DIMENSIONAL EVEN ARTIN GALOIS REPRESENTATION.

THE RELEVANCE OF THE TRANSCENDENTAL SPECTRUM

• FOR NUMBER THEORY WHY WORRY ABOUT THESE ELUSIVE TRANSCENDENTAL OBJECTS – WHY NOT STICK TO DISCRETE SERIES AND COHOMOLOGICAL FORMS?

(1) EVEN IF ONE'S INTEREST IS ONLY IN GALOIS REPRESENTATIONS, HALF OF THE FINITE 2-DIMENSIONAL SUCH REPRESTATIONS ARE EVEN, AND THESE SHOULD CORRESPOND TO:

MAASS CUSP FORMS WITH $\lambda = \frac{1}{4}$; ("ALGEBRAIC").

(2) THE PRIMARY TOOL IN PROVING VARIOUS INSTANCES OF LANGLANDS PRINCIPLE OF FUNCTORIALITY,

IS THE TRACE FORMULA.

IT INVOLVES COMPARISONS OF ORBITAL INTEGRALS ASSOCIATED WITH CONJUGACY CLASSES IN Γ, AND IT CANNOT SINGLE OUT THE ALGEBRAIC FROM THE TRANSCENDENTAL.

(3) ANALYTIC DIOPHANTINE APPLICATIONS OF AUTOMORPHIC FORMS MAKE USE OF THE FULL SPECTRUM OF $L^2(\Gamma\backslash G)$, WITH THE TRANSCENDENTAL PART OFTEN BEING PRIMARY.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{d(n)}{n^s} = \zeta(s)^2$; $d(n) = \#$ OF DIVISORS OF n.

(SELBERG) FOR $a \neq 0$ $\displaystyle\sum_{n=1}^{\infty} \frac{d(n)d(n+a)}{n^s}$, HAS A MEROMORPHIC CONTINUATION TO $\mathbb{C}$ WITH POLES ON $\mathrm{Re}(s) = 1/2$ AT $1/2 + ik$, $k \in \mathrm{spec}\left(L^2_{cusp}(SL_2(\mathbb{Z})\backslash\mathbb{H})\right)$.

(b) SUCH DIOPHANTINE PROBLEMS AS HILBERT'S 11-TH PROBLEM OF REPRESENTATIONS OF INTEGERS BY INTEGRAL QUADRATIC FORMS, ARE RESOLVED USING MAASS FORMS AS A CENTRAL TOOL.

⋮

(4) IN THE FUNCTION FIELD ($\mathbb{Q}$ REPLACED BY $\mathbb{F}_q(t)$) THERE IS NO ARCHIMEDIAN PLACE AND CORRESPONDINGLY NO TRANSCENDENTAL PART. THE BIG CONJECTURES; RIEMANN AND RAMANUJAN ARE KNOWN IN THAT CASE.

IT APPEARS THAT THE TRANSCENDENTAL PART STANDS AS A BLOCKADE.

VERTICAL AND HORIZONAL

ONE CAN COMBINE THESE

LET $\overline{\Gamma}$ BE THE DIVISION GROUP OF Γ

$$\overline{\Gamma} = \{ z \in T : z^{\ell} \in \Gamma \text{ FOR SOME } \ell \geq 1 \}$$

$$\overline{1} = \mathrm{tor}(T).$$

THE ULTIMATE VERSION WHICH IS ALSO UNIFORM OVER DEFINING FIELDS AND QUANTITATIVE IN THE RANK r OF Γ IS DUE TO

EVERTSE / SCHLICKEWEI / SCHMIDT

<u>THEOREM</u>: $V \subset (\mathbb{C}^*)^N$, Γ A FINITELY GENERATED SUBGROUP OF RANK r, THERE ARE $T_1, T_2, \ldots T_\nu$ TRANSLATES OF SUBTORI CONTAINED IN V SUCH THAT

$$\overline{\Gamma} \cap V = \overline{\Gamma} \cap (T_1 \cup T_2 \cup \ldots T_\nu)$$

AND $\quad \nu \leq (c(V))^r.$

<u>REMARK</u>: THE CONSTANT $c(V)$ CAN BE GIVEN EXPLICITLY, HOWEVER THE ACTUAL SAY ZERO DIMENSIONAL T_j's CANNOT IN GENERAL BE DETERMINED BY THIS PROOF.

THE PROOF INVOLVES SPECIALIZATION ARGUMENT & REDUCING TO $\Gamma \subset T(\overline{\mathbb{Q}})$ AND ABSOLUTE HEIGHT VERSIONS OF THE SCHMIDT SUBSPACE THEOREM AS WELL AS A STUDY OF POINTS OF SMALL HEIGHT.

A SPECIAL ROLE IS PLAYED BY

$$V: \quad a_1 x_1 + a_2 x_2 + \cdots + a_N x_N = 1,$$

IN $(\mathbb{C}^*)^N$.