Equivalent definitions of Arthur packets for real quasisplit unitary groups

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Joint work with N. Arancibia

Unitary groups

- $G = \mathrm{GL}_N$, a complex group
- $G(\mathbb{R}) = \mathrm{U}(N/2, N/2)$ for even N
- $G(\mathbb{R}) = \mathrm{U}((N-1)/2, (N+1)/2)$ for odd N
- ${}^L G = {}^\vee\mathrm{GL}_N \rtimes \langle {}^\vee\delta_0 \rangle$

$$ {}^\vee\delta_0(g) = J\,(g^{-1})^\intercal J^{-1}, \quad g \in {}^\vee\mathrm{GL}_N $$

$$ J = \begin{bmatrix} 0 & & & 1 \\ & & -1 & \\ & \iddots & & \\ (-1)^{N-1} & & & 0 \end{bmatrix} $$

A-parameters

Fix

$$\psi_G : W_{\mathbb{R}} \times \mathrm{SL}_2 \to {}^L G$$

- $1 \to \mathbb{C}^\times \to W_{\mathbb{R}} \to \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to 1$ is the real Weil group
- $\psi_G | W_{\mathbb{R}}$ is a tempered L-parameter
- $\psi_G | \mathrm{SL}_2$ is a finite-dimensional representation

A-packets

$$
\begin{array}{ccccc}
 & & \psi_G & & \\
 & \swarrow & & \searrow & \\
\eta_{\psi_G}^{\mathrm{Mok}} & & \stackrel{?}{=} & & \eta_{\psi_G}^{\mathrm{ABV}} \\
\downarrow & & & & \downarrow \\
\Pi_{\psi_G}^{\mathrm{Mok}} & & \stackrel{?}{=} & & \Pi_{\psi_G}^{\mathrm{ABV}}
\end{array}
$$

- Each type of packet Π_{ψ_G} is defined here as the irreducible representations occurring in a stable virtual character η_{ψ_G}.
- [Mok 2015] follows [Arthur 2013] in using twisted endoscopy.
- [Adams, Barbasch, Vogan 1992] gives intrinsic definition using sheaf theory. It works for any real reductive group.
- The equality of packets is ongoing joint work with Arancibia. We follow [Adams, Arancibia, M.]

Strategy of Mok/Arthur

- Express $G(\mathbb{R})$ as a twisted endoscopic group of a nice group $H(\mathbb{R})$.
- Use ψ_G to define an A-parameter ψ for H.
- Define $\Pi^{\mathrm{Mok}}_{\psi_G}$ as the preimage of Π_ψ under endoscopic transfer.

The nice group H for the unitary group G satisfies

$$H(\mathbb{R}) = \mathrm{GL}_N(\mathbb{C})$$

Twisted endoscopic data I

- $\mathrm{GL}_N(\mathbb{C})$ is the real form of

$$\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N = \mathrm{GL}_N \times \mathrm{GL}_N$$

- $\mathrm{GL}_N(\mathbb{C}) = (\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N)(\mathbb{R})$
- ${}^L\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N = ({}^\vee\mathrm{GL}_N \times {}^\vee\mathrm{GL}_N) \rtimes \langle {}^\vee\delta_1 \rangle$, ${}^\vee\delta_1 =$ swap
- Define the involution $\vartheta \in \mathrm{Aut}(\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N)$ by

$$\vartheta(g_1, g_2) = (J(g_2^{-1})^\intercal J^{-1}, J(g_1^{-1})^\intercal J^{-1}), \quad g_1, g_2 \in \mathrm{GL}_N$$

- $\mathrm{GL}_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$ is a disconnected group with non-identity component $\mathrm{GL}_N(\mathbb{C}) \rtimes \vartheta$.

Twisted endoscopic data II

- We identify ϑ with its dual automorphism

$$\vartheta(g_1, g_2) = (J(g_2^{-1})^{\intercal} J^{-1}, J(g_1^{-1})^{\intercal} J^{-1}), \quad g_1, g_2 \in {}^{\vee}\mathrm{GL}_N$$

-

$$({}^{\vee}\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N)^{\vartheta} = \{(g_1, J(g_1^{-1})^{\intercal} J^{-1})\} \cong {}^{\vee}\mathrm{GL}_N = {}^{\vee}G$$

- $\epsilon : {}^{L}G \hookrightarrow {}^{L}\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$, with $\epsilon({}^{\vee}G) = ({}^{\vee}\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N)^{\vartheta}$ and $\epsilon({}^{\vee}\delta_0) = {}^{\vee}\delta_1$.
- In this way G is an *endoscopic group* for the pair $(\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N, \vartheta)$.

A-parameters and endoscopy

- Composing ψ_G with $\epsilon : {}^L G \hookrightarrow {}^L \mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$ yields an A-parameter

$$\psi := \epsilon \circ \psi_G$$

- $$\Pi_\psi = \Pi_{\phi_\psi} = \{\pi_\psi\}$$

  is a single representation of $\mathrm{GL}_N(\mathbb{C})$.
- The ϑ-stability of ψ implies $\pi_\psi \circ \vartheta \cong \pi_\psi$.

Twisted characters

- If π is an irrep of $\mathrm{GL}_N(\mathbb{C})$ such that $\pi \circ \vartheta \cong \pi$ then there are two inequivalent extensions of π to $\mathrm{GL}_N(\mathbb{C}) \rtimes \langle\vartheta\rangle$.
- Fix a Whittaker datum for $\mathrm{GL}_N(\mathbb{C})$. Then we obtain a preferred extension π^+ of π.
- The twisted character $\mathrm{Tr}_\vartheta \pi$ is defined by

$$\mathrm{Tr}_\vartheta \pi(f) := \mathrm{Tr} \int_{\mathrm{GL}_N(\mathbb{C})} \pi^+(g\vartheta)\, f(g\vartheta)\, dx, \quad f \in C_c^\infty(\mathrm{GL}_N(\mathbb{C}) \rtimes \vartheta)$$

Character decompositions

- Let $K\Pi(\mathrm{GL}_N(\mathbb{C}))$ be the Grothendieck group of admissible representations of $\mathrm{GL}_N(\mathbb{C})$.
- The set of irreducible representations $\{\pi\}$ is a $\mathbb{Z}$-basis for $K\Pi(\mathrm{GL}_N(\mathbb{C}))$.
- The set of *standard* representations $\{M\}$ is also a $\mathbb{Z}$-basis for $K\Pi(\mathrm{GL}_N(\mathbb{C}))$.
- Both irreducible and standard representations have preferred extensions to $\mathrm{GL}_N(\mathbb{C}) \rtimes \langle\vartheta\rangle$.
- It follows that

$$\mathrm{Tr}_\vartheta \pi_\psi = \sum_j n_j \, \mathrm{Tr}_\vartheta M_{\phi_j}$$

where M_{ϕ_j} are standard representations indexed by L-parameters.

The definition of $\Pi_{\psi_G}^{\mathrm{Mok}}$

- $\mathrm{Tr}_\vartheta \pi_\psi = \sum_j n_j \, \mathrm{Tr}_\vartheta M_{\phi_j}$

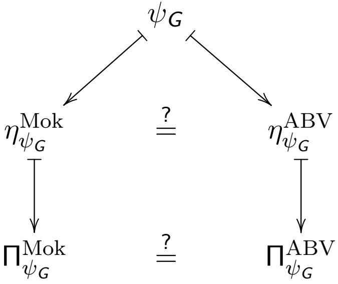
Theorem (Mok, AMR, M.)

$$\mathrm{Tr}_\vartheta \pi_\psi = \mathrm{Trans}\left(\sum_j n_j \, \mathrm{Tr} \tilde{\Pi}_{\phi_{G,j}} \right)$$

where $\phi_j = \epsilon \circ \phi_{G,j}$ *and* $\tilde{\Pi}_{\phi_{G,j}}$ *is the pseudo L-packet of* $\phi_{G,j}$*.*

- Trans denotes the endoscopic transfer map from stable virtual characters of $G(\mathbb{R})$ to virtual characters of $\mathrm{GL}_N(\mathbb{C})$.
- $\eta_{\psi_G}^{\mathrm{Mok}} = \sum_j n_j \, \mathrm{Tr} \tilde{\Pi}_{\phi_{G,j}}$ is stable and $\Pi_{\psi_G}^{\mathrm{Mok}}$ is the set of irreducible representations occurring in it with non-zero multiplicity.

A-packets


ψ_G
$\eta_{\psi_G}^{\mathrm{Mok}}$
$\stackrel{?}{=}$
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Two themes in ABV

In this section G is any connected reductive algebraic group defined over $\mathbb{R}$.

1. Pair representations of $G(\mathbb{R})$ with sheaves on a topological space X.
2. Do interesting work on the sheaves and transport back to representations using the pairing.

The space X of geometric parameters

- $X = \coprod_\lambda X(\lambda)$
- $\lambda \in {}^\vee\mathfrak{g}$ are representatives of infinitesimal characters
- $X(\lambda)$ is a smooth complex variety
- ${}^\vee G$ acts on $X(\lambda)$ with finitely many orbits
- The ${}^\vee G$-orbits on $X(\lambda)$ are in natural bijection with ${}^\vee G$-orbits of (quasisplit) L-parameters with infinitesimal character λ

Complete geometric parameters

- Let $x \in X$, $S = {}^\vee G \cdot x \subset X$, and τ be an irrep of ${}^\vee G_x/({}^\vee G_x)^0$.
- The pair (S, τ) is a *complete geometric parameter*.
- Let $\Xi({}^L G) = \{(S, \tau)\}$ be the set of complete geometric parameters.
- The local Langlands correspondence takes the form of a bijection

$$\Xi({}^L G) \longleftrightarrow \Pi(G/\mathbb{R})$$

$$\xi = (S, \tau) \longmapsto \pi(\xi)$$

- The irreducible representation $\pi(\xi)$ is the Langlands quotient of a *standard* representation $M(\xi)$.

Sheaves on X

- $\xi = (S, \tau)$ defines a ${}^\vee G$-equivariant local system of $\mathbb{C}$-vector spaces on $S \subset X$.
- Extend this local system *by zero* to $\bar{S}$ and take the direct image to obtain an irreducible equivariant *constructible* sheaf $\mu(\xi)$.
- Alternatively, take the *intermediate extension* to $\bar{S}$ and the direct image to obtain an irreducible equivariant *perverse* sheaf $P(\xi)$.

Grothendieck groups again

- The Grothendieck group of the equivariant constructible sheaves is isomorphic to the Grothendieck group of the equivariant perverse sheaves.
- Identify the two Grothendieck groups and denote them by $KX({}^L G)$.
- $KX({}^L G)$ has two $\mathbb{Z}$-bases: $\{\mu(\xi)\}$ and $\{P(\xi)\}$.
- $K\Pi(G/\mathbb{R})$ also has two $\mathbb{Z}$-bases: $\{\pi(\xi)\}$ and $\{M(\xi)\}$.

Theorem (ABV)

There is a perfect pairing

$$K\Pi(G/\mathbb{R}) \times KX({}^L G) \to \mathbb{Z}$$

such that for representations of the quasisplit form

$$\langle M(\xi), \mu(\xi')\rangle = \delta_{\xi,\xi'} \text{ and } \langle \pi(\xi), P(\xi')\rangle = (-1)^{\dim S_\xi}\, \delta_{\xi,\xi'}$$

The ϑ-twisted pairing

- Here take $G = \mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$.
- ϑ acts on the reps of $\mathrm{GL}_N(\mathbb{C})$ and the sheaves on X.
- One may define $\mathbb{Z}$-modules for the twisted characters on $\mathrm{GL}_N(\mathbb{C})$, and "twisted sheaves",

$$K\Pi(\mathrm{GL}_N(\mathbb{C}), \vartheta) \text{ and } KX({}^L\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N, \vartheta)$$

Theorem (Adams, Arancibia, M.)

There is a natural pairing $K\Pi(\mathrm{GL}_N(\mathbb{C}), \vartheta) \times KX({}^L\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N, \vartheta) \to \mathbb{Z}$ *such that*

$$\langle M(\xi)^+, \mu(\xi')^+\rangle = \delta_{\xi,\xi'} \text{ and } \langle \pi(\xi)^+, P(\xi')^+\rangle = (-1)^{\dim S_\xi}\,\delta_{\xi,\xi'}$$

(Depends on choices of extensions.)

Endoscopic transfer à l'ABV

- Now take $G(\mathbb{R})$ to be a quasisplit unitary group again. Recall $\epsilon : {}^L G \hookrightarrow {}^L\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N$.
- Let $\epsilon^* : KX({}^L\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N, \vartheta) \to KX({}^L G)$ be the inverse image functor.
- Define $\epsilon_* : K\Pi(G(\mathbb{R})) \to K\Pi(\mathrm{GL}_N(\mathbb{C}), \vartheta)$ by

$$\langle \epsilon_* M(\xi), \mu(\xi')^+ \rangle = \langle M(\xi), \epsilon^* \mu(\xi')^+ \rangle$$

- The endoscopic transfer map is the restriction of ϵ_* to the stable virtual characters.
- It agrees with Trans.

A big leap

- ABV define stable virtual characters $\eta_{\psi_G}^{\mathrm{ABV}}$ and $\eta_{\psi}^{\mathrm{ABV}}$ in terms of *characteristic cycles*.
- These define Π_{ψ_G} and $\Pi_{\psi} = \{\pi_{\psi}\}$.

Comparison of the two stable virtual characters

- Recall that $\Pi^{\mathrm{Mok}}_{\psi_G}$ are the irreps occurring in the stable virtual character $\eta^{\mathrm{Mok}}_{\psi_G} = \sum_j n_j \, \mathrm{Tr}\tilde{\Pi}_{\phi_{G,j}}$
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$$\begin{aligned} \epsilon_*(\eta^{\mathrm{ABV}}_{\psi_G}) &= \eta^+_\psi \quad \text{Deep theorem in ABV} \\ &= \mathrm{Tr}_\vartheta \pi_\psi \\ &= \sum_j n_j \, \mathrm{Tr}_\vartheta M_{\phi_j} \\ &= \epsilon_* \left(\eta^{\mathrm{Mok}}_{\psi_G}\right) \end{aligned}$$

- Injectivity of $\epsilon_* \Rightarrow \eta^{\mathrm{ABV}}_{\psi_G} = \eta^{\mathrm{Mok}}_{\psi_G} \Rightarrow \Pi^{\mathrm{Mok}}_{\psi_G} = \Pi^{\mathrm{ABV}}_{\psi_G}$