A multiplicity formula of K-types

- G connected real reductive group, $K = G^\theta$ a max cpt subgp. π irr. smooth rep'n of G (i.e. irr. Casselman-Wallach rep'n), τ irr. rep'n of K,

$$m(\pi,\tau) = \dim(\mathrm{Hom}_K(\pi,\tau)).$$

Goal: Prove a multiplicity formula

$$m(\pi,\tau) = m_{geom}(\pi,\tau)$$

where $m_{geom}(\pi,\tau)$ is def. via the Harish-Chandra char. Θ_π of π and the char. Θ_τ of τ.

· Local multiplicity of spherical varieties

G conn. red. gp / local field F char. 0, $H \subset G$ spherical subgp.
τ: a f.d. rep'n of H.

$$m(\pi,\tau) = \dim(\mathrm{Hom}_H(\pi,\tau)).$$

One still wants to prove a mult. formula $m(\pi,\tau) = m_{geom}(\pi,\tau)$.

Known cases: Whittaker model, group case (i.e. $G = H \times H$), Gan-Gross-Prasad model, Ginzburg-Rallis model, Galois model, Shalika model ....

- In a previous paper, I gave a def. of $m_{geom}(\pi, \tau)$ for general spherical varieties and I proved that my def. recovers all the known cases. A special case is the case of K-types.

Def. of $m_{geom}(\pi, \tau)$ (general case)

Germ expansion: $x \in G_{ss}$,

$$\theta_\pi(x \exp(Y)) = \sum_{\mathcal{O} \in Nil(\mathfrak{g}_x^*)} c_{\theta_\pi, \mathcal{O}}(x)\, \hat{j}(\mathcal{O}, Y) + O(|Y|)$$

Baby case If G, H are ~~is a~~ finite gp,

$$m_{geom}(\pi, \tau) = \frac{1}{|H|} \sum_{h \in H} \theta_\pi(h) \overline{\theta_\tau(h)}$$

$$= \sum_{H//H} \frac{\theta_\pi(t) \overline{\theta_\tau(t)}}{z_H(t)}$$

General case: $m_{geom}(\pi, \tau) = \int_{\mathcal{T}(G,H)} D^H(t)\, \overline{\theta_\tau(t)} \cdot \frac{d(G_t, H_t, F)}{|Z_H(t) : H_t|\, c(G_t, H_t, F)}$

$\cdot \frac{1}{|\mathcal{N}(G_t, H_t)|} \sum_{\mathcal{O} \in \mathcal{N}(G_t, H_t)} c_{\theta_\pi, \mathcal{O}}(t)\, dt$

- $\mathcal{T}(G,H)$ is a subset of conjugacy classes of H
- $G_t = Z_G(t)^0$, $H_t = Z_H(t)^0$.
- $c(G_t, H_t, F)$, $d(G_t, H_t, F)$ are positive integers asso. to (G_t, H_t).
- $\mathcal{N}(G_t, H_t)$ is a subset of $\mathcal{N}il_{reg}(\mathfrak{g}_t^*)$.

K-type case (G real red. gp, $K = G^\theta$)

- $\mathcal{T}(G,K)$ be the set of conjugacy classes t of K s.t. G_t is split modulo the center of G_t.
- $t \in \mathcal{T}(G,K)$, let $c(G_t, K \cap G_t, \mathbb{R})$ be the number of connected component in $B_t \cap K \cap G_t$ where B_t is any Borel subgp of G_t.
- $\mathcal{N}(G_t, H_t) = \mathcal{N}il_{reg}(\mathfrak{g}_t^*)$.

Def

$$m_{geom}(\theta, \theta_K) = \int^{*}_{\Gamma(G,K)} \frac{1}{c(G_t, K\cap G_t, \mathbb{R}) \cdot |Z_K(t) : K\cap G_t|} D^K(t)\, \theta_K(t)\, c_\theta(t)\, dt$$

average of the regular germs

where θ is a quasi-character on G and θ_K is a smooth function on K that is invariant under conjugation.

$$\int^{*}_{\Gamma(G,K)} \cdots = \lim_{s\to 0^+} \int_{\Gamma(G,K)} \frac{1}{\cdots} D^G(t)^{\frac{1}{2}}\, \theta_K(t)\, c_\theta(t)\, A(t)^{s-\frac{1}{2}}\, dt$$

$$A(t) = D^G(t)\, D^K(t)^{-2}.$$

Locally, $\Gamma(G,K) = \cup\, r.\, \underline{T}//W(T)$ is isom. to $(\mathbb{C}^1)^l$, we give it the measure comes from the Haar measure on $\mathbb{C}^1$ w/ total volume 1.

Thm (W.) For all irr. smooth rep'n π of G and for all irr. rep'n τ of K, we have $m(\pi,\tau) = m_{geom}(\pi,\tau)$.

e.g. $G=GL_2(\mathbb{R})$ $K=O_2(\mathbb{R})$ Three types of conj. classes:

		G_t
$\pm\begin{pmatrix}1&\\&1\end{pmatrix}$		$GL_2(\mathbb{R})$
$\begin{pmatrix}1&\\&-1\end{pmatrix}\epsilon$		$GL_1(\mathbb{R})\times GL_1(\mathbb{R})$
$\rightarrow\begin{pmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{pmatrix}$	$\theta\neq 0,\pi$	$GL_1(\mathbb{C})$

$G=Sp_4(\mathbb{R})$ $K=U_2(\mathbb{R})$

			G_t
	$\pm\begin{pmatrix}1&\\&1\end{pmatrix}$ —		$Sp_4(\mathbb{R})$
	$\nearrow\begin{pmatrix}1&\\&-1\end{pmatrix}$ —		$Sp_2(\mathbb{R})\times Sp_2(\mathbb{R})$
	$\begin{pmatrix}a&\\&a\end{pmatrix}$	$a\in\mathbb{C}^1$, $a\neq\pm1$	$U_2(\mathbb{R})$
$\mathbb{C}^1\sim$	$\begin{pmatrix}a&\\&\bar{a}\end{pmatrix}$	$a\in\mathbb{C}^1$, $a\neq\pm1$ —	$U_{1,1}(\mathbb{R})$
$\mathbb{C}^1\sim$	$\begin{pmatrix}\pm1&\\&b\end{pmatrix}$	$b\neq\pm1$, $b\in\mathbb{C}^1$	$Sp_2(\mathbb{R})\times U_1(\mathbb{R})$
$(\mathbb{C}^1)^2\sim$	$\begin{pmatrix}a&\\&b\end{pmatrix}$	$a,b\in\mathbb{C}^1$, $a,b\neq\pm1$, $a\neq b,\bar{b}$.	$U_1(\mathbb{R})\times U_1(\mathbb{R})$

K-character

Wallach proved that for $\forall\pi$, $\exists$ continuous linear functional $\Theta_{K,\pi}$, on $C^\infty(K)$, called the K-char., s.t.

$$\theta_{K,\pi}(f) = \sum_{\tau \in \mathrm{Irr}(K)} m(\pi,\tau) \int_K \theta_\tau(k) f(k)\, dk.$$

He also showed that $\theta_{K,\pi} = \theta_\pi$ on $K \cap G_{reg}$.

<u>Cor (W.)</u> $\theta_{K,\pi}(f) = m_{geom}(\theta_\pi, \theta_f)$, $\theta_f(k) := \int_K f(k_0^{-1} k k_0)\, dk_0$.

Step 1. To show both $m(\pi,\tau)$ and $m_{geom}(\pi,\tau)$ behaved nicely under parabolic induction.

It's enough to consider elliptic rep'n.

Step 2. To prove a local trace formula

$f \in \mathcal{C}_{scusp}(G)$, $I(f,\tau) = \int_{K\backslash G} \int_K f(g^{-1} k g)\, \overline{\theta_\tau}(k)\, dk\, dg.$

$I_{geom}(f,\tau) = m_{geom}(\theta_f, \overline{\theta_\tau})$, θ_f is q-char. asso. to f via the wted orbital integral.

wted char.

$I_{spec}(f,\tau) = \int_{\mathcal{X}(G)} D(\pi)\, \theta_f(\pi)\, m(\overline{\pi},\tau)\, d\pi.$

Thm (W.) $I(f,\tau) = I_{geom}(f,\tau) = I_{spec}(f,\tau)$.

$\mathcal{C}(G)$ = H-C Schwartz space

$f \in \mathcal{C}(G)$ is strongly cuspidal if $\forall P = MU$, $\int_{U(F)} f(mu)\,du = 0$ $\forall m \in M$.