Stability patterns in the cohomology of $SL_n(\mathbb{Z})$ and its congruence subgroups

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Groups of interest

Definition

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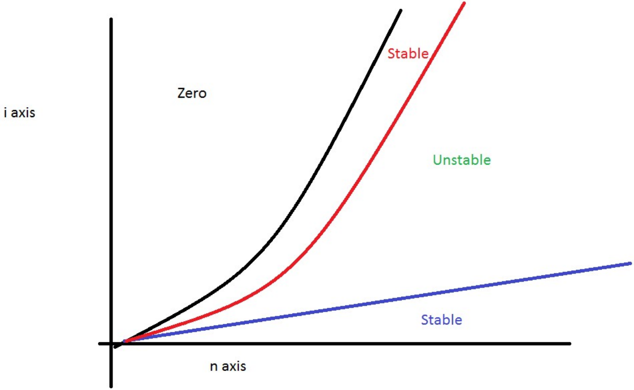
Goal

Study the asymptotic behavior of $H^*(\mathrm{SL}_n(\mathbb{Z}))$ and $H^*(\Gamma_n(L))$ as n gets large.

$H^i(\mathrm{SL}_n(\mathbb{Z}))$ and $H^i(\Gamma_n(L))$.



Stabilization map

There are group homomorphisms: $\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_{n+1}(\mathbb{Z})$ and $\Gamma_n(L) \to \Gamma_{n+1}(L)$ given by:
given by

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix} \mapsto \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n,1} & \cdots & a_{n,n} & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

Question

Do these maps induce stability?

Homological stability

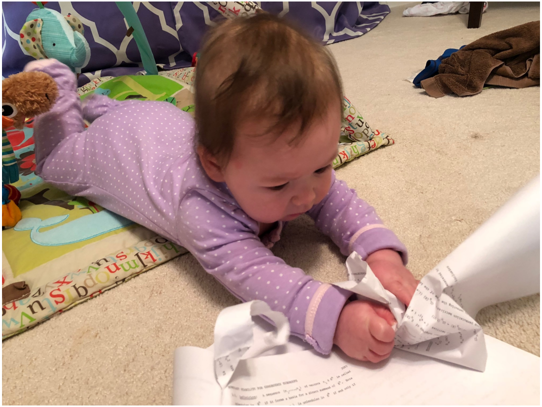
Theorem (Borel, Charney)

The induced map $H_i(\mathrm{SL}_n(\mathbb{Z})) \to H_i(\mathrm{SL}_{n+1}(\mathbb{Z}))$ *is an isomorphism for* $i \leq \frac{n}{3} - 1$.

Theorem (Borel, Charney)

The induced map $H_i(\Gamma_n(L); \mathbb{Z}[1/L]) \to H_i(\Gamma_{n+1}(L); \mathbb{Z}[1/L])$ *is an isomorphism for* $i \leq \frac{n}{2} - 5$.

Relevant paper of Charney.



Stable rational cohomology known

Theorem (Borel)

Let $\Gamma \subset \mathrm{SL}_n(\mathbb{Z})$ *be finite index. Then there is a map* $\bigwedge(x_5, x_9, x_{13}, \ldots) \to H^*(\Gamma; \mathbb{Q})$ *which is an isomorphism for* $* \leq \frac{n}{2} - 5$.

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Corollary

In the stable range, $H_*(\Gamma_n(L); \mathbb{Q}) \to H_*(\mathrm{SL}_n(\mathbb{Z}); \mathbb{Q})$ *is an isomorphism.*

Failure of homological stability

Theorem (Lee–Szczarba)

For $n \geq 3$, $H_1(\Gamma_n(L)) \cong \mathfrak{sl}_n(\mathbb{Z}/L) \cong (\mathbb{Z}/L)^{n^2-1}$.

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Corollary

$H_1(\Gamma_n(L)) \to H_1(\Gamma_{n+1}(L))$ *is not eventually surjective.*

Representation stability

$\mathrm{SL}_n(\mathbb{Z})$ acts on $\Gamma_n(L)$ via conjugation and this action on homology factors through $\mathrm{SL}_n(\mathbb{Z})/\Gamma_n(L) \cong \mathrm{SL}_n(\mathbb{Z}/L)$.

Heuristic: A sequence $\{M_n\}$ of G_n-representations has *representation stability* if M_n has "a description that does not depend on n" for n sufficiently large.

Example (Lee–Szczarba)

For $n \geq 3$, $H_1(\Gamma_n(L))$ is the group of trace zero mod L matrices.

Surjectivity up group actions

Theorem (Putman, Church–Ellenberg–Farb–Nagpal, Church–Ellenberg, Church–M.–Nagpal–Reinhold, Gan–Li)

For $i \leq \frac{n}{4} - 3$, $\mathrm{Ind}_{\mathrm{SL}_{n-1}(\mathbb{Z}/L)}^{\mathrm{SL}_n(\mathbb{Z}/L)} H_i(\Gamma_{n-1}(L)) \to H_i(\Gamma_n(L))$ *is surjective.*

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Corollary

For fixed i *and field* $\mathbb{F}$, $\dim H_i(\Gamma_n(L); \mathbb{F})$ *agrees with a polynomial in* n *of degree* $\leq 2i + 3$ *for* $n \geq 8i + 11$.

Vanishing in high dimensions

Theorem (Borel–Serre)

Let $\Gamma \subset \mathrm{SL}_n(\mathbb{Z})$ be finite index and assume that the torsion primes of Γ are invertible in R. Then $H^i(\Gamma; R) \cong 0$ for $i > \binom{n}{2}$.

Question

Call $H^{\binom{n}{2}-i}(\Gamma)$ the codimension-i cohomology of Γ. Does the codimension-i cohomology of $\mathrm{SL}_n(\mathbb{Z})$ or $\Gamma_n(L)$ stabilize as n tends to infinity?

Conjectural high dimensional stability for $SL_n(\mathbb{Z})$

Conjecture (Church–Farb–Putman)

$H^{\binom{n}{2}-i}(SL_n(\mathbb{Z});\mathbb{Q}) \cong 0$ *for* $n \geq i+2$.

Conjectural high dimensional stability for $\mathrm{SL}_n(\mathbb{Z})$

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Remark

Church–Farb–Putman conjecture would follow if one could show $H^{\binom{n}{2}-i}(\mathrm{SL}_n(\mathbb{Z});\mathbb{Q}) \to H^{\binom{n}{2}-i}(\mathrm{SL}_{n+1}(\mathbb{Z});\mathbb{Q})$ *is surjective for* $n \geq i$.

Known cases of the Church–Farb–Putman conjecture

Theorem

1. $H^{\binom{n}{2}}(\mathrm{SL}_n(\mathbb{Z});\mathbb{Q}) \cong 0$ *for* $n \geq 2$ *[Lee–Szczarba].*
2. $H^{\binom{n}{2}-1}(\mathrm{SL}_n(\mathbb{Z});\mathbb{Q}) \cong 0$ *for* $n \geq 3$ *[Church–Putman].*
3. $H^{\binom{n}{2}-2}(\mathrm{SL}_n(\mathbb{Z});\mathbb{Q}) \cong 0$ *for* $n \geq 4$ *[Brück–M.–Patzt–Sroka–Wilson].*

• Codim 2 case uses computers.

• See Wilson's talk for more details and generalizations to other number rings.

High-dimensional cohomology of congruence subgroups

Theorem (Lee–Szczarba)

$H^{\binom{n}{2}}(\Gamma_n(3)) \cong \mathrm{St}_n(\mathbb{F}_3) \cong \mathbb{Z}^{3^{\binom{n}{2}}}$.

Remark

Complete answer also known for $L = 2$ and 5. $H^{105}(\Gamma_{15}(5)) \cong \mathbb{Z}^{2350867829470159774034814041007591566603522538519291648712545382850352884817741}$.

- See Patzt's talk for more details.

Conjectural stability for high-dimensional cohomology of congruence subgroups

Conjecture (Church–Farb–Putman, M.–Nagpal–Patzt)

$\mathrm{Ind}_{\mathrm{SL}_n(\mathbb{Z}/L)}^{\mathrm{SL}_{n+1}(\mathbb{Z}/L)} H^{\binom{n}{2}-i}(\Gamma_n(L)) \to H^{\binom{n}{2}-i}(\Gamma_{n+1}(L))$ *is surjective for* $n \geq 2i+2$.

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Theorem (M.–Nagpal–Patzt)

The conjecture is true for $i = 0$ *or* $i = 1$ *and* $L = 3$.

Corollary

$$\dim_{\mathbb{F}} H^{\binom{n}{2}-1}(\Gamma_n(3);\mathbb{F}) \leq 227340 \frac{3^{\binom{n-4}{2}} |\mathrm{GL}_n(\mathbb{F}_p)|}{|\mathrm{GL}_{n-4}(\mathbb{F}_p)||\mathrm{GL}_4(\mathbb{F}_p)|}.$$

Steinberg modules

Definition

For $\mathbb{F}$ a field, let $T_n(\mathbb{F})$ denote the realization of the poset of proper, nonzero subspaces of $\mathbb{F}^n$.

Theorem (Solomon-Tits)

$T_n(\mathbb{F}) \simeq \bigvee S^{n-2}$.

Definition

$\mathrm{St}_n(\mathbb{F}) = \widetilde{H}_{n-2}(T_n(\mathbb{F}))$.

Borel–Serre duality

Theorem (Borel–Serre)

Let $\Gamma \subset \mathrm{SL}_n(\mathbb{Z})$ *be finite index and assume that the torsion primes of* Γ *are invertible in* R*. Then* $H^{\binom{n}{2}-i}(\Gamma; R) \cong H_i(\Gamma; \mathrm{St}_n(\mathbb{Q}) \otimes R)$*.*

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Corollary

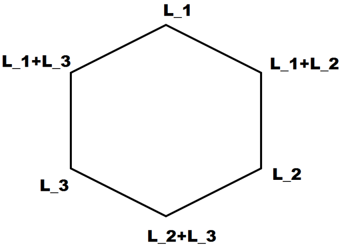
Then $H^i(\mathrm{SL}_n(\mathbb{Z}); \mathbb{Q}) \cong H^i(\Gamma_n(L); \mathbb{Q}) \cong 0$ for $i > \binom{n}{2}$.

Remark

Lee–Szczarba's results can be rephrased as: $(\mathrm{St}_n(\mathbb{Q}) \otimes \mathbb{Q})_{\mathrm{SL}_n(\mathbb{Z})} \cong 0$ for $n \geq 2$ and $\mathrm{St}_n(\mathbb{Q})_{\Gamma_n(3)} \cong \mathrm{St}_n(\mathbb{F}_3)$.

Apartments

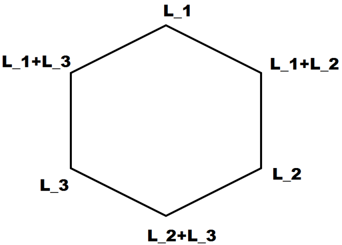
- $T_n(\mathbb{F}) \simeq \bigvee S^{n-2}$.
- Fix $L_1 \oplus \ldots \oplus L_n = \mathbb{F}^n$. Let $S_{\vec{L}}$ be the subposet of $T_n(\mathbb{F})$ of subspaces which are sums of subsets of $\{L_1, \ldots, L_n\}$.
- Let $A_{\vec{L}} = [S_{\vec{L}}] \in \widetilde{H}_{n-2}(T_n(\mathbb{F})) = \mathrm{St}_n(\mathbb{F})$.



Solomon-Tits theorem

Theorem

$\mathrm{St}_n(\mathbb{F})$ *is generated by* $\{A_{\vec{L}}\}$.



Steinberg modules form a ring

Proposition

The association

$$(A_{L_1,\dots,L_n}, A_{L'_1,\dots,L'_m}) \mapsto A_{L_1,\dots,L_n,L'_1,\dots,L'_m}$$

gives a well-defined $\mathrm{GL}_n(\mathbb{F}) \times \mathrm{GL}_m(\mathbb{F})$*-equivariant map* $\mathrm{St}_n(\mathbb{F}) \otimes \mathrm{St}_m(\mathbb{F}) \to \mathrm{St}_{n+m}(\mathbb{F})$ *which makes* $\bigoplus_n \mathrm{St}_n(\mathbb{F})$ *into an associative ring.*

Remark

$\bigoplus_n \mathrm{St}_n(\mathbb{F})$ *is not (skew) commutative.*

GL-modules

Definition

A $\mathrm{GL}(R)$-module is a sequence $M = \{M_n\}$ with M_n a $\mathrm{GL}_n(R)$-representation.

Definition

If $M = \{M_n\}$ and $N = \{N_n\}$ are $\mathrm{GL}(R)$-modules, let $M \otimes_{\mathrm{GL}} N$ be the $\mathrm{GL}(R)$-module with

$$(M \otimes_{\mathrm{GL}} N) = \bigoplus_{a+b=n} \mathrm{Ind}_{\mathrm{GL}_a(R)\times \mathrm{GL}_b(R)}^{\mathrm{GL}_n(R)} M_a \otimes N_b.$$

GL-rings

Definition (Fancy)

A GL(R)-ring is a monoid object in the category of GL-rings with monoidal structure $\otimes_{\mathrm{GL}}$.

Can also make sense of modules over GL-rings, tensor products, Tor-groups, etc.

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Definition (Concrete)

A $\mathrm{GL}(R)$-ring is a sequence $A = \{A_n\}$ with A_n a $\mathrm{GL}_n(R)$-representation and with $\mathrm{GL}_n(R) \otimes \mathrm{GL}_m(R)$-equivariant maps $A_n \otimes A_m \to A_{n+m}$ satisfying certain unitality and associativitiy conditions.

Steinberg modules form a GL-ring

Proposition

The association

$$(A_{L_1,\dots,L_n}, A_{L'_1,\dots,L'_m}) \mapsto A_{L_1,\dots,L_n,L'_1,\dots,L'_m}$$

makes $\mathrm{St}(\mathbb{F}) := \{\mathrm{St}_n(\mathbb{F})\}$ *into a* $\mathrm{GL}(\mathbb{F})$*-ring.*

St-modules

Proposition

The maps

$$H_i(\Gamma_n(p); \mathrm{St}_n(\mathbb{Q})) \otimes H_j(\Gamma_m(p); \mathrm{St}_m(\mathbb{Q})) \to H_{i+j}(\Gamma_{n+m}(p); \mathrm{St}_{n+m}(\mathbb{Q}))$$

make $\{H^{\binom{n}{2}-i}(\Gamma_n(3))\}$ *into a* $\mathrm{St}(\mathbb{F}_3)$*-module.*

This uses $\mathrm{St}(\mathbb{F}_3) = \{H_0(\Gamma_n(3); \mathrm{St}_n(\mathbb{Q}))\}$ and Borel–Serre duality.

Algebraic reformulation

Theorem (M.–Nagpal–Patzt)

$\{H^{\binom{n}{2}-1}(\Gamma_n(3))\}$ *is generated in degree* ≤ 4 *as a* $\mathrm{St}(\mathbb{F}_3)$*-modules*

Algebraic reformulation

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Key lemma:

Theorem (M.–Nagpal–Patzt)

$\mathrm{St}(\mathbb{F})$ *is Koszul.*

See Patzt's talk at next week's workshop for more information.

The end.