Jenny Wilson 4 October 2021

The high-degree cohomology of the special linear groups.

joint w Benjamin Brück, Sander Kupers, Jeremy Miller, Peter Patzt, Robin Sroka, Dan Yasaki

Notation $n \in \mathbb{Z}$, $n \geq 2$.

F - number field

R - ring of integers in F.

Eg

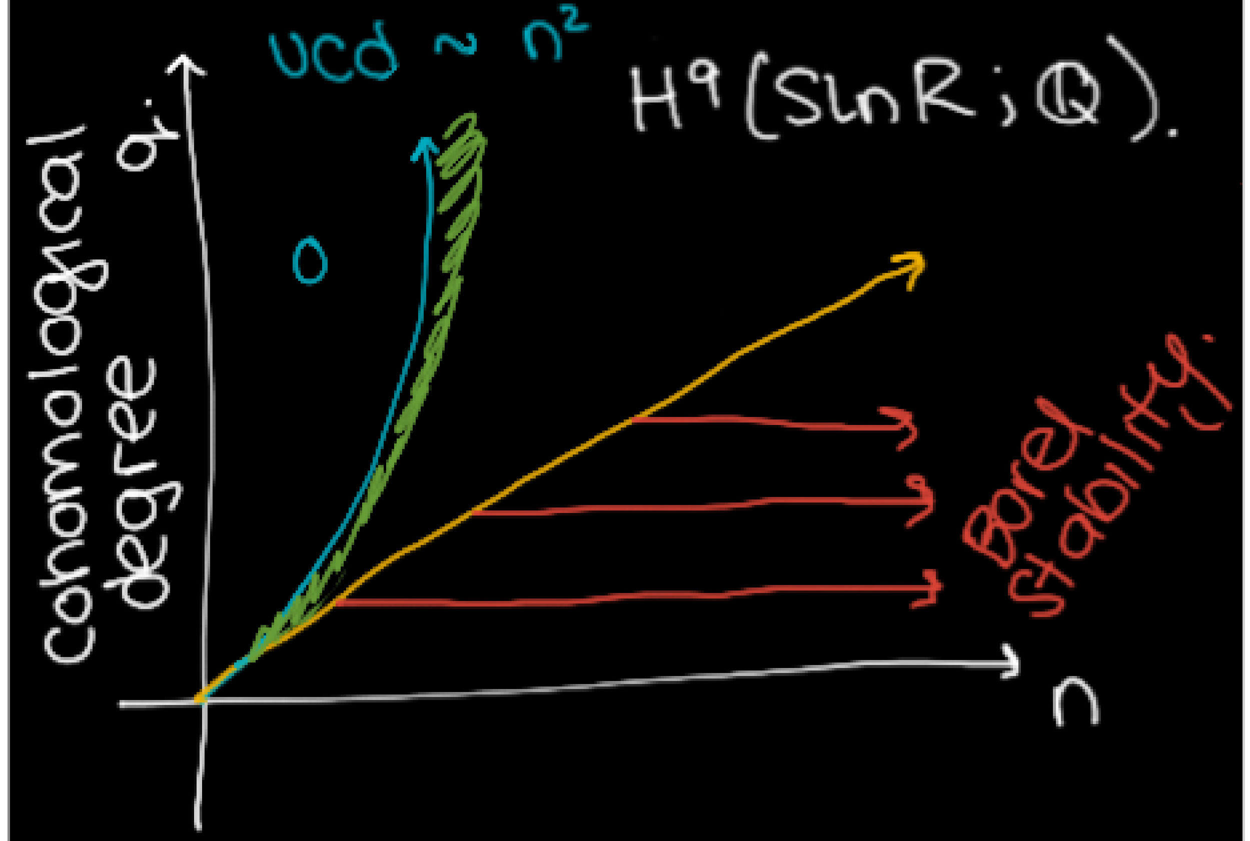
R	$\subseteq$	F
$\mathbb{Z}$	$\subseteq$	$\mathbb{Q}$
$\mathbb{Z}[i]$	$\subseteq$	$\mathbb{Q}(i)$
$\mathbb{Z}\left[\frac{1}{2}(1+\sqrt{-3})\right]$	$\subseteq$	$\mathbb{Q}(\sqrt{-3})$

etc.

(Borel-Serre) $r = \#$ real embeddings, $c = \#$ pairs complex embeddings.

$$\mathrm{vcd} = r\frac{(n+1)n}{2} + c(n^2-1) + n + 1$$

$H^q(SL_n R; \mathbb{Q})$.



Thm (Borel)

$$SL_n R \longrightarrow SL_{n+1} R$$
$$A \longmapsto \left[\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right]$$

induces iso on H^* for $n \gg q$.

Our goal: Understand $H^q(SL_n R; \mathbb{Q})$ for $q \gg n$.

Motivating Question:

What is the largest q s.t. $H^q(SL_n R; \mathbb{Q})$ is nonzero?

How does the answer depend on the ring theory of R?

Conjecture (Church-Farb-Putman)

$R = \mathbb{Z}$

$$H^{\mathrm{vcd}-i}(SL_n \mathbb{Z}; \mathbb{Q}) = 0 \quad \forall n \geq i+2.$$

Known Results

Codimension 0 $\quad q = \text{vcd}$

R Euclidean Eg R = $\mathbb{Z}$, $\mathbb{Z}[i]$	Thm (Lee-Szczarba) $H^{\text{vcd}}(SL_n R; \mathbb{Q}) = 0$
R non-Euclidean PID Thm (Weinberger) Assuming GRH, $F = \mathbb{Q}(\sqrt{d})$, d in $\{-19, -43, -67, -163\}$	Thm (Miller-Patzt-W-Yasaki) $d = -43, -67, -163$ $H^{\text{vcd}}(SL_{2n}(R); \mathbb{Q}) \neq 0$ ↑ even.
R not a PID Eg. R = $\mathbb{Z}[\sqrt{-5}]$	Thm (Church-Farb-Putman) $H^{\text{vcd}}(SL_n R; \mathbb{Q}) \neq 0$

Codimension 1 $\quad q = \text{vcd} - 1$

R = $\mathbb{Z}$ (Euclidean)	Thm (Church-Putman) $H^{\text{vcd}-1}(SL_n \mathbb{Z}; \mathbb{Q}) = 0$ $\forall n \geq 3$.
R = $\mathbb{Z}[i]$ Gaussian, $\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$ Eisenstein. (Euclidean)	Thm (Kupers-Miller-Patzt-W) $H^{\text{vcd}-1}(SL_n R; \mathbb{Q}) = 0$ $\forall n \geq 3$.

Work in progress on codimension 2....

Key facts: Virtual Bieri-Eckmann duality.

$H^{vcd-i}(SL_n R; \mathbb{Q})$

$\cong H_i(SL_n R; \mathbb{Q} \otimes St_n(F))$

"twisted Poincaré duality"

Dualizing module:

$St_n(F)$ – Steinberg module.

defined in terms of the Tits building $T_n(F)$.

Defn $T_n(F)$ simplicial complex

vertices

$\leftrightarrow$ nonzero proper subspaces $U \subseteq F^n$

p-simplices

$\leftrightarrow$ flags

$0 \subsetneq U_0 \subsetneq \cdots \subsetneq U_p \subsetneq F^n$

Thm (Solomon-Tits)

$T_n(F) \simeq \bigvee S^{n-2}$

wedge of $(n-2)$-spheres

Defn $St_n(F) := \tilde{H}_{n-2}(T_n(F))$

Upshot: To compute

$H^{vcd-i}(SL_n(R); \mathbb{Q})$

$\cong H_i(SL_n(R); \mathbb{Q} \otimes St_n(F))$:

- Take a resolution of $\mathbb{Q} \otimes St_n(F)$ by flat $\mathbb{Q}[SL_n R]$-modules:

$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathbb{Q} \otimes St_n(F)$

- Take $SL_n R$-coinvariants

$\longrightarrow (F_1)_{SL_n R} \longrightarrow (F_0)_{SL_n R} \longrightarrow 0$

$M_G := M / \langle gm - m \mid m \in M, g \in G \rangle$

quotient by G-action.

- Take homology.

Goal restated:

Construct a flat resolution of the Steinberg module that is "small enough" to compute SL_nR-coinvariants.

Strategy:

Construct simplicial complexes related to $T_n(F)$ with homology vanishing in a range, use their (relative) chain groups to construct a resolution for $St_n(F) \otimes \mathbb{Q}$

- K_n – contractible (or suitably highly connected). simplicial complex.
- $L_n \subseteq K_n$ subcomplex
  - agree on $(n-2)$-skeletons.
  - $\tilde{H}_i(L_n) \cong \tilde{H}_i(T_n(F))$ (in a range).

Then use the relative chains $C_{q+n}(K_n, L_n)$ to construct res. of $St_n(F) = \tilde{H}_{n-2}(T_n(F))$.

LES of a pair:

$$H_{i+1}(K_n, L_n) \cong \tilde{H}_i(L_n) \underset{\text{in a range}}{\cong} \tilde{H}_i(T_n(F)).$$

$$\begin{array}{l} \downarrow \\ C_{q+n}(K_n, L_n) \\ \downarrow \\ \vdots \\ \downarrow \\ C_{n-1}(K_n, L_n) \\ \downarrow \\ H_{n-1}(K_n, L_n) \cong \tilde{H}_{n-2}(L_n) \\ \qquad\qquad\qquad\quad \cong St_n(F) \\ \downarrow \\ 0 \end{array}$$

exact by Solomon-Tits.

1976 Lee-Szczarba constructed the Sharbly resolution.

Assume R a PID.

- K_n
  vertices – unimodular $v \in R^n$
  simplices – all finite subsets.

- $L_n \subseteq K_n$
  – all vertices –
  – $\{v_0, \dots, v_p\}$ simplex iff lie in a proper direct summand of R^n.

$C_{q+n}(K_n, L_n)$ = free ab gp on $(q+n+1)$-tuples of unimodular vectors with F-span F^n.

But resolution is "too big" to compute SL_nR-coinvariants for $q \geq 0$.

"Smaller" resolutions for $R = \mathbb{Z}$, $F = \mathbb{Q}$

Defn B_n – complex of partial frames for $\mathbb{Z}^n$

- vertices – lines in $\mathbb{Z}^n$ (represented by spanning vectors $v \in \mathbb{Z}^n$)
- simplices $\{v_0, \dots, v_p\}$ $\longleftrightarrow$ subsets of a basis for $\mathbb{Z}^n$.

Church-Putman 2016
proof of Bykovski presentation.

$K_n = BA_n$
complex of augmented partial frames for $\mathbb{Z}^n$.

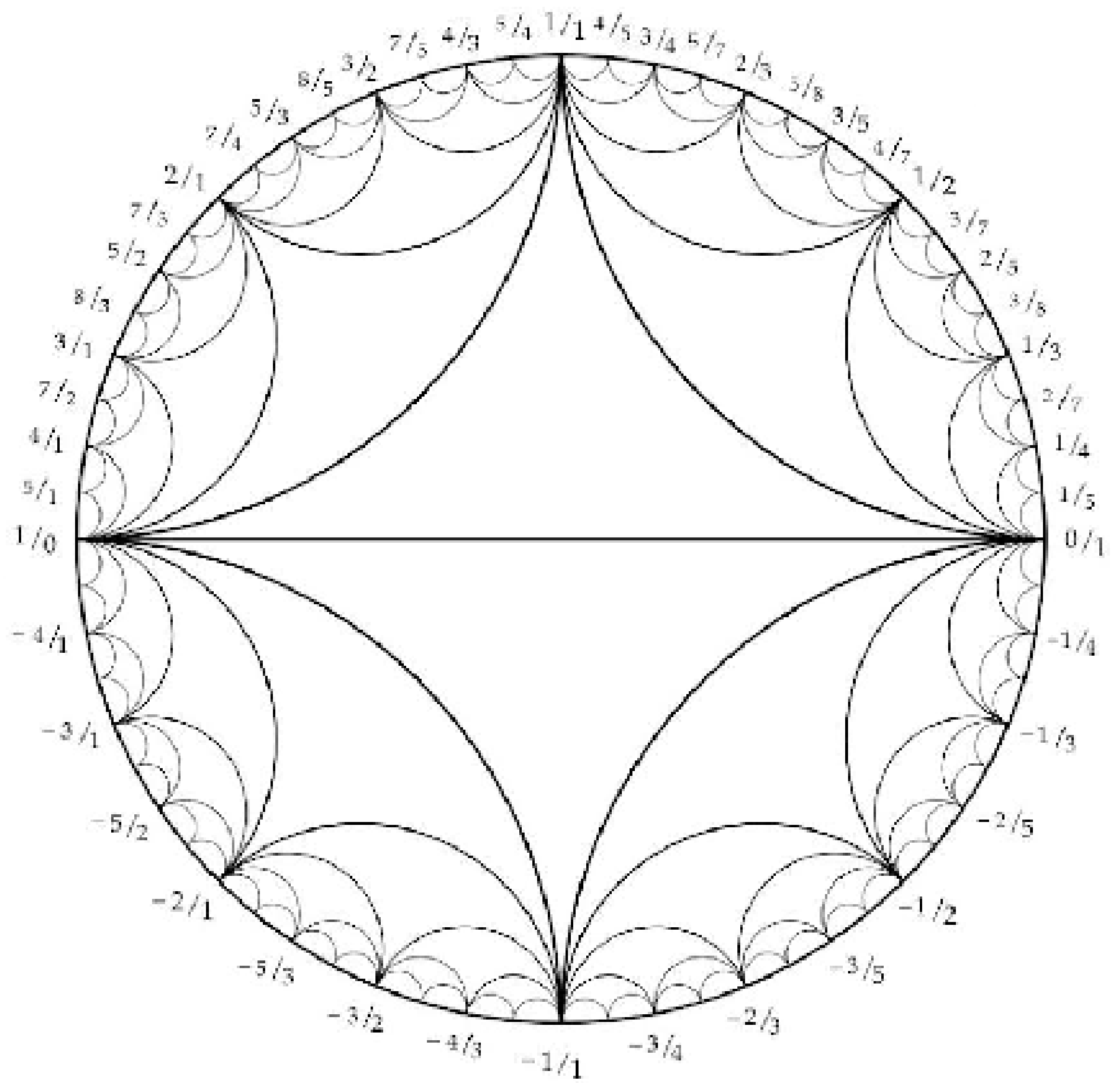
B_n + extra Simplices

$\{v_1+v_2, \underbrace{v_1, v_2, \ldots, v_p}_{\text{partial basis.}}\}$

Eg $n=2$.
line $\mathbb{Z}\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{Z}^2 \longleftrightarrow \frac{a}{b}$

$B_2 \longleftrightarrow$ Farey graph.

$BA_2 \rightsquigarrow$ glue in 2-simplices.



$L_n = BA_n' \subseteq BA_n$
simplices $\{v_0, \ldots, v_p\}$ spanning a proper summand.

$\rightsquigarrow$ partial resolution

$$C_n(BA_n, BA_n') \otimes \mathbb{Q}$$
$$\downarrow$$
$$C_{n-1}(BA, BA_n') \otimes \mathbb{Q}$$
$$\downarrow$$
$$H_{n-1}(BA_n, BA_n') \cong St_n(F) \otimes \mathbb{Q}$$
$$\downarrow$$
$$0$$

$\rightsquigarrow H^{vcd-1}(SL_n\mathbb{Z}; \mathbb{Q}) = 0$

work in progress:
(Brück–Miller–Patzt–Sroka–W)

- $K_n = BAA_n$

BA_n + add simplices

$\{v_1+v_2, v_2+v_3, v_1, \dots, v_p\}$

$\{v_1+v_2, v_3+v_4, v_1, \dots, v_p\}$

where $v_1, \dots, v_p$ is a partial basis for $\mathbb{Z}^n$.

$L_n = BAA'_n \subseteq BAA_n$
simplices spanning a proper summand.

$\rightsquigarrow$ partial resolution

$$C_{n-3}(BAA, BAA') \otimes \mathbb{Q} \to C_{n-2}(BAA, BAA') \otimes \mathbb{Q} \to C_{n-1}(BAA, BAA') \otimes \mathbb{Q} \to St_n(\mathbb{Q}) \otimes \mathbb{Q} \to 0$$

$C_{n-3}(BAA, BAA')$: generated by $[v_{-1}, v_0, v_1, \dots, v_n]$ s.t. (after reordering) $v_1, \dots, v_n$ a basis, $v_0 = \pm v_1 \pm v_2$, $v_{-1} = \pm v_2 \pm v_3$ or $\pm v_3 \pm v_4$ & relations (1), (2).

$C_{n-2}(BAA, BAA')$: generated by $(v_0, \dots, v_n]$ s.t. (after reordering) $v_1, \dots, v_n$ a basis, $v_0 = \pm v_1 \pm v_2$ or $\pm v_1 \pm v_2 \pm v_3$. & relations (1), (2).

$C_{n-1}(BAA, BAA')$: generated by symbols $[v_1, \dots, v_n]$ s.t. $v_1, \dots, v_n$ basis for $\mathbb{Z}^n$

(1) $[v_1, \dots, v_n] = [-v_1, \dots, v_n]$

(2) $[v_{\sigma(1)}, \dots, v_{\sigma(n)}] = \mathrm{sgn}(\sigma) [v_1, \dots, v_n]$

$SL_n R$-coinvariants vanish

$\Rightarrow$ **Thm** $H^{vcd-2}(SL_n \mathbb{Z}; \mathbb{Q}) = 0$ $\forall n \geq 4$.