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1 Overview of the field

The workshop focused on several directions at the intersection of analysis, geometry, and probability, in particular, on stochastic processes in various geometric settings. The topics included diffusions on finite- and infinite-dimensional manifolds (sometimes equipped with degenerate geometries), random matrix theory and non-commutative probability.

These are highly active areas of mathematical research, and the connections between them are some of the most exciting realms of modern research at the interface of stochastics and geometry. In recent years the interplay of analysis, geometry, and stochastics has led to many deep insights and given rise to numerous powerful new developments with important applications in mathematics and mathematical physics. Bringing together leading experts and early career mathematicians from these diverse mathematical fields allowed the exchange of ideas and facilitate collaborations despite the challenges of the COVID-19 pandemic.

The general topic of the workshop was the study of stochastic processes in geometric settings. Unifying themes include heat kernel analysis and related functional inequalities such as Poincaré, Sobolev, logarithmic Sobolev, Bakry-Èmery, and isoperimetric inequalities. Logarithmic Sobolev inequalities have been a central tool in geometric stochastic analysis in infinite dimensions, as well as in random matrix theory. This workshop discussed how these techniques can be used to approach problems in geometric analysis on curved spaces, hypoelliptic operators in finite and infinite dimensions, and other potential applications.

Heat kernel analysis in infinite dimensions

There has been considerable progress in understanding heat kernel measures on infinite-dimensional Lie groups such as path and loop groups, and more recently on Heisenberg-like and other nilpotent groups. These results are of interest in quantum field theory as this research aims at making mathematically rigorous the formal computations done with the Yang-Mills measure in different settings.

The difficulty of the infinite-dimensional setting manifests itself in both the lack of classical notions of Riemannian geometry and much more intricate analysis. For example, there is no natural reference measure, no well-developed PDE theory, and the choice of a Riemannian connection usually often involves topological considerations. This area goes back to earlier work of L. Gross on loop groups and log Sobolev inequalities, as well as work of P. Malliavin on loop groups and hypoellipticty in infinite dimensions. B. Driver connected stochastic analysis on such spaces with their own geometry, and these ideas were further developed by S. Aida, H. Airault, A. Cruzeiro, S. Fang, E. Hsu, A. Thalmaier, et al.

This remains a very active area of research; moreover, recent work by R. Haslhofer and A. Naber revitalized interest in these problems. This work characterizes Ricci curvature bounds on (smooth and non-smooth) spaces in terms of functional inequalities on their path spaces. More precisely, they extend the classical Bochner formula for the heat flow on a Riemannian manifold M to martingales on the path space over M. Their Bochner formula is related to two-sided bounds on Ricci curvature on the path space, similarly to how the classical Bochner formula on M is related to lower bounds on Ricci curvature. This new approach to stochastic geometry on path spaces provides an exciting new tool for proving spectral gap and log Sobolev inequalities, bringing a new, potentially powerful, perspective on this difficult area. Several survey talks gave an overview of this field.

Sub-Riemannian/degenerate/hypoelliptic diffusions

A related area of research is analysis on sub-Riemannian geometries in both finite and infinite dimensions. The main application of such analysis is to degenerate elliptic and parabolic PDEs, control theory, and geometric measure theory. The classical curvature conditions which are a backbone to the theory in the Riemannian/elliptic setting are problematic in the sub-Riemannian setting. But Poincaré, integrated Harnack, log Sobolev, and other functional inequalities are still available (if harder to obtain) and remain very powerful, as shown by the work by F. Baudoin–N. Garofalo, N. Eldredge, E. Grong–A. Thalmaier, and J. Inglis–V. Kontis–B. Zegarliński.

One of the potentially fruitful directions to explore during this workshop is the optimal transport approach of Lott–Villani–Sturm for metric measure spaces. Their "synthetic" lower Ricci curvature bounds have been the subject of extensive research over the last several years. In particular, the work of R. Haslhofer and A. Naber's shows Ricci bounds for path spaces for which the underlying space has a lower Ricci bound in this sense. While there have been several successful attempts to use these methods in the sub-Riemannian setting, this approach is still largely undeveloped here.

There has also been recent progress in the analysis of subelliptic Laplacians on infinite-dimensional Heisenberg groups (F. Baudoin–M. Gordina–T. Melcher, B. Driver–N. Eldredge–T. Melcher) and the horizontal path space of totally geodesic Riemannian foliations (F. Baudoin–M. Gordina–Q. Feng). In particular, a number of functional inequalities have been proved in these degenerate settings, allowing for the proof of smoothness properties like quasi-invariance under translations and integration by parts formulae. The techniques used in these cases come from stochastic analysis, Riemannian geometry, and functional inequalities. Together these have provided a number of new examples to now attempt a better understanding of hypoellipticity in infinite dimensions, thus connecting these developments to important earlier results by M. Hairer–J. Mattingly.

Random matrices and non-commutative potential theory

Random matrix theory is a very popular current research area at the confluence of probability theory, statistical mechanics, functional analysis, and theoretical computer science, with real world applications in statistics and data science driving much of its popularity. In recent years, many difficult problems in the field have been solved by adding noise: adding a perturbation matrix with Brownian motion entries, and tracking the effect on the dynamics of the perturbed eigenvalues, singular values, and eigenvectors. (Dynamic methods played a key role in the works of L. Erdös, B. Schlein, and H. T. Yau on the circular law, local limits, and delocalization for Wigner matrices, for example.)

In the last decade, there has been considerable success in understanding the dynamics of eigenvalues and singular values in several *multiplicative* regimes. More generally, the questions (related to the rest of this proposal) deal with the dynamics of eigenvalues of random matrices sampled from diffusion processes on Lie groups. The case of unitary groups was understood, in the bulk, by P. Biane in the late 1990s, with follow-up work on fluctuations and local limits by T. Lévy and M. Maïda in 2010. The finer question of the behavior of extremal and outlier eigenvalues of unitary Brownian motion was solved by B. Collins, A. Dahlqvist, and T. Kemp in 2018, using a combination of stochastic and tensor algebraic methods; more quantitative bounds were later found by E. Meckes and T. Melcher. Some of these results were recently extended to Brownian motions on other classical compact groups by A. Chan.

Beyond the compact case, the situation is quite different, owing to the generic non-normality of the random matrices involved. The most substantial progress was made very recently by B. Driver, B. Hall, and T. Kemp, in computing the bulk large-N limit distribution of eigenvalues for the Brownian motion on $GL(N, \mathbb{C})$. This work uses stochastic methods, sub-Riemannian geometric methods, and PDE (notably Hamilton–Jacobi theory). The result yields a key insight on the relationship between the unitary and general linear cases (mirroring a "conformal shadow" phenomenon from Ginibre and Wigner matrices). This latter point was extended significantly in follow-up work by C. W. Ho and P. Zhong, extending the theory to more general starting points for the multiplicative diffusion. A natural next step is to consider low-rank perturbations (in multiplicative form) of this setup, which could yield similar (and possibly more powerful) tools to study fine statistics in data analytic methods like principal component analysis, cf. the BBP (J. Baik, G. Ben Arous, S. Péché) phase transition. This is a current very active angle of research, at the intersection of stochastic geometry and PDE; studying this method's potential application to a wide array of problems in (non-normal) random matrix theory is one major prong of the intended workshop.

There have also been numerous recent advances in studying Markov semigroups and Dirichlet forms on non-commutative C^*-algebras, a subject that goes back to the work of S. Albeverio and R. Høegh-Krohn. One direction explored by E. Carlen, J. Maas, et al. is the study of quantum Markov semigroups on C^*-algebras. These ergodic semigroups have a unique stationary state, and their evolution can be described as a gradient flow for the relative entropy in a Riemannian metric which is a non-commutative analog of the Wasserstein metric. Moreover, in some cases the relative entropy is strictly and uniformly convex with respect to this Riemannian metric, which makes it analogous to the work of Otto on gradient flows with respect to the classical Wasserstein metric. In another direction, F. Cipriani, J.L. Sauvageot, et al. have considered extensions of the potential theory of Dirichlet forms from locally compact spaces to noncommutative C^*-algebras. This allows for studying geometric aspects of these spaces, such as carré du champ operators with applications to A. Connes' noncommutative geometry, the construction and analysis of quantum Lévy processes on compact quantum groups, potential theory on the Clifford algebra over a Riemannian manifold, Dirichlet forms in free probability, and potential theory of fractals.

2 Recent Developments

In addition to the survey talks described in the presentation highlights, the workshop hosted an informal session focused on degenerate diffusions. This session featured three talks on more current work in this area, more particularly, on convergence to equilibrium for underdamped Langevin dynamics and the Nosé-Hoover equation under Brownian heating; behavior of Brownian motion conditioned to have trivial signature and on fluctuations for Brownian bridge expansions; and geometric convolution and non-Gaussian kernels for hypoelliptic diffusions. Each of the talks presented recent results in the study of degenerate diffusions in settings where standard geometric methods are not easily applicable, as well as presenting open problems. The speakers for this session were three early-mid career researchers *Karen Habermann*, *David Herzog*, and *Pierre Perruchaud*.

3 Early career participants' presentations

One of the goals of the workshop in the changed format was to facilitate interactions between leading experts in the field and current and recent PhDs. In addition to several opportunities to interact outside of lectures, the workshop held two sessions with shorter presentations by early career mathematicians. These talks were very well attended and led to a number of discussions during and after sessions.

A total of ten talks were given in the short talk sessions. The topics included regularity of heat kernels and geometry in Dirichlet spaces with talks given by *Qi Hou* and *Chiara Rigoni*. Connections between heat kernel analysis and geometric properties of the underlying spaces were presented by *Gunhee Cho*, *Timothy Buttsworth*, *Chiara Rigoni*, and *Gianmarco Vega-Molino*. A number of talks dealt with hypoelliptic diffusions and their applications to functional inequalities and ergodicity, including talks by *Marco Carfagnini*, *Liangbing Luo*, *Qi Feng*, and *Evan Camrud*. A new approach to functional inequalities in a non-commutative setting were presented by *Li Gao*.

4 Presentation highlights

When the format of the workshop was changed to virtual, the organizers scheduled three days of survey talks by leading experts in the fields, with the goal of introducing early career participants to different directions at the intersection of stochastics and geometry. In particular, Mylène Maïda gave an overview on *Mathematical aspects of two-dimensional Yang-Mills theory*, while later Brian Hall talked about *Partial differential equations in random matrix theory*. Several survey talks described recent progress on functional inequalities in geometric settings. Log Sobolev inequalities, quasi-invariance, and integration by parts formulae were central in talks by Fabrice Baudoin on *On log-Sobolev inequalities and their applications*, Elton Hsu on *Stochastic analysis on Riemannian manifolds*, and Robert Haslhofer on *Analysis on path space, Einstein metrics and Ricci flow*. Another application in physics, namely, stochastic Hamiltonian formulation of the momentum map dynamics, was described by Ana Bela Cruzeiro in her talk *On stochastic Clebsch variational principles*. Two very different overviews were given by Ismael Bailleul in his talk *Gardening in the field of stochastic differential geometry*, and by Laurent Saloff-Coste in *Thirty-six views of the ubiquitous heat kernel: a personal selection*.

Mylène Maïda's talk on *Mathematical aspects of two-dimensional Yang-Mills theory* recalled the history of the major breakthrough in quantum field theory which extended the concept of quantum electrodynamics as a gauge theory to non-Abelian groups. The mathematical side of the Yang-Mills theory has been a very active field of research, and the presentation focused on the developments over the last two decades. In particular, the Yang-Mills theory on two-dimensional manifolds with gauge group $U(N)$ or $SU(N)$ relies on the properties of the heat kernel on these groups. The heat kernels correspond to Brownian motion and Brownian bridge processes on these groups. The main results included the construction of the master field arising as the large-N limit for these models, and connections with free probability theory. A different approach to random matrix theory was presented by **Brian Hall** who described how tools from the theory of partial differential equations can be used to compute the eigenvalue distribution of large random matrices. The PDE approach was introduced in work by Brian Hall, Bruce Driver, and Todd Kemp. The talk included several examples where this method can be used, in particular, for random matrices of the form $X + iY$, where X is drawn from the Gaussian Unitary Ensemble (GUE) and Y is an arbitrary Hermitian random matrix independent of X. The talk was accessible to a wide audience.

Several survey talks concerned analysis on curved spaces, including the fundamental contributions by Bruce Driver to the subject. **Fabrice Baudoin** gave a brief history and described some applications of logarithmic Sobolev inequalities to partial differential equations, differential geometry, and stochastic analysis on infinite-dimensional path spaces. Another survey of stochastic analysis on Riemannian manifolds was given by **Elton Hsu**. The main object is the path space over a Riemannian manifold equipped with the Wiener measure (the law of a Brownian motion as a random path). The main results in this area include the quasi-invariance of the Wiener measure under the Cameron-Martin flow, integration by parts formulae, and the logarithmic Sobolev inequality, as well as a more general Beckner's inequality, on the path space. The talk also described most recent developments such as sharp constants for functional inequalities and time-dependent Riemannian metrics. In a talk titled *Analysis on path space, Einstein metrics and Ricci flow*, **Robert Haslhofer** gave a survey on how analysis on path space can be used in the study of Ricci curvature. This approach has been motivated by Driver's foundational work on quasi-invariance and integration by parts on path space. A more recent approach has been developed by Haslhofer with Aaron Naber, providing a characterization of solutions to the Einstein equations and the Ricci flow in terms of certain sharp estimates on path space.

The original motivation for most of the mathematics discussed in this workshop came from physics. **Ana Bela Cruzeiro**'s talk on *stochastic Clebsch variational principles* developed a mathematical approach to Hamiltonian fluid dynamics. Namely, in joint work with D.D. Holm and T.S. Ratiu, Cruzeiro described a stochastic Clebsch action principle and derived the corresponding stochastic differential equations. The configuration space in this case is a Riemannian manifold equipped with a Lie group action. **Ismael Bailleul**'s talk surveyed the origins of stochastic differential geometry and emerging directions. This part of the workshop concluded with a beautiful talk by **Laurent Saloff-Coste** giving *Thirty-six views of the ubiquitous heat kernel: a personal selection*. Much of the research described during the workshop revolves around heat kernels, and the talk explored how the heat kernel helps us understand other problems.

5 Scientific progress made

Several new projects have started after the workshop, including some stemming from questions posed to early career presenters. This includes projects on *spectral theory of hypoelliptic diffusions* and *non-commutative log Sobolev inequalities for sub-Laplacians*.

6 Outcome of the meeting

The meeting was attended by nearly 100 participants from around the world including Australia, Austria, Brazil, Canada, China, France, Germany, Italy, Japan, Luxembourg, Portugal, Romania, Switzerland, Turkey, Sweden, United Kingdom, USA. Scientifically the meeting succeeded in having lectures delivered on topics representing a large cross-section of the field involving geometry and stochastics. In the field where there are very few mathematicians form the groups under-represented in mathematics, our meeting had about 30% of presenters from such groups. There was considerable enthusiasm for the subject and new directions presented during the meeting. Many participants expressed their hope to meet in-person at BIRS after the pandemic.