Fixed points for group actions on 2-dimensional buildings

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BIRS Workshop
Totally Disconnected Locally Compact Groups via Group Actions
16–20 August 2021

Introduction

This is joint work with Jeroen Schillewaert and Koen Struyve.

Today:

1. Trees
2. Buildings
3. Proof ideas

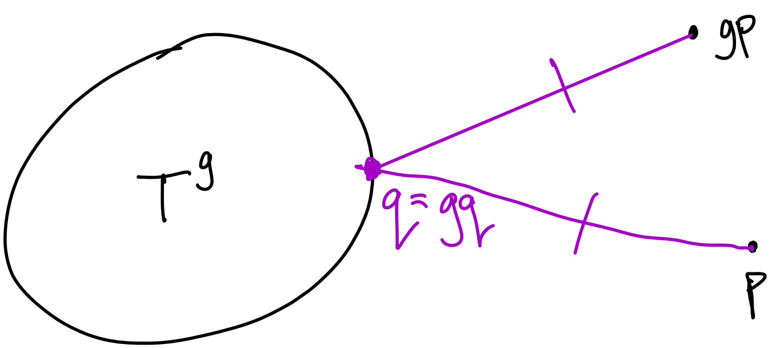
Trees

Theorem (Serre 1977)

Let G be a finitely generated group acting on a tree T without inversions. If every element of G fixes a point of T, then G has a global fixed point.

Lemma

Suppose g fixes a point in T. Let p be any point of T such that $gp \neq p$. Then g fixes the midpoint of the geodesic $[p, gp]$.



Trees

Theorem (Serre 1977)

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Now by induction it's enough to show that if $G = \langle A, B \rangle$ with $A = \langle a_i \rangle$ and $B = \langle b_j \rangle$, and the fixed sets T^A, T^B and $T^{a_i b_j}$ are all nonempty, then $T^G \neq \emptyset$.

Trees

Theorem (Serre 1977)

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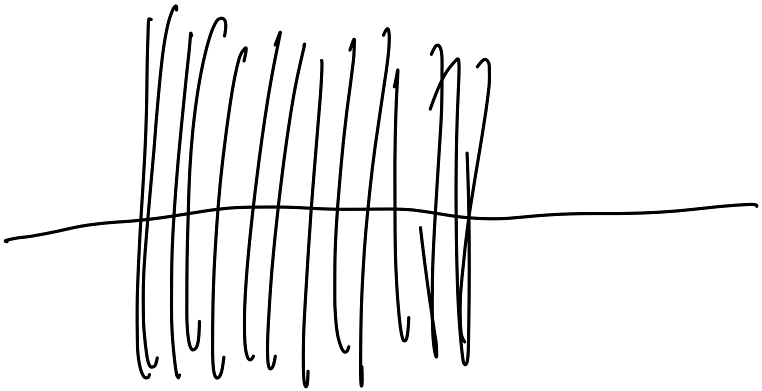
Morgan–Shalen (1984) extended this to $\mathbb{R}$-trees.



Trees

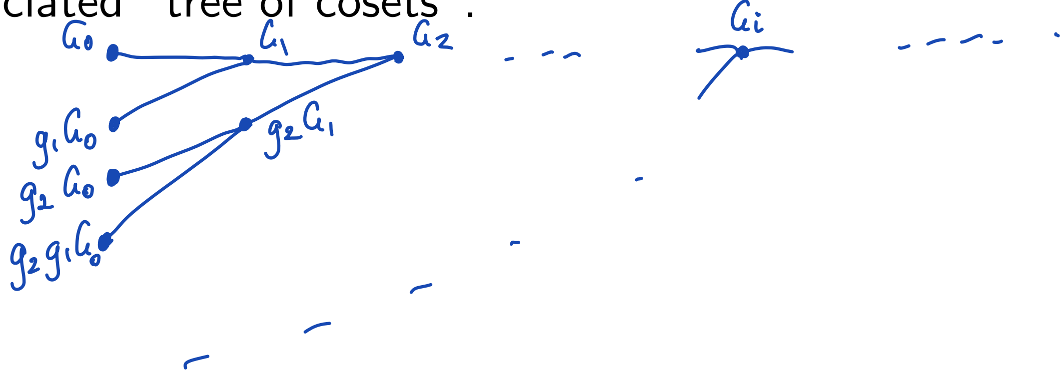
Theorem (Serre 1977)

Let G be a finitely generated group acting on a tree T without inversions. If every element of G fixes a point of T, then G has a global fixed point.

Finite generation is essential!

Example (Bridson–Haefliger)

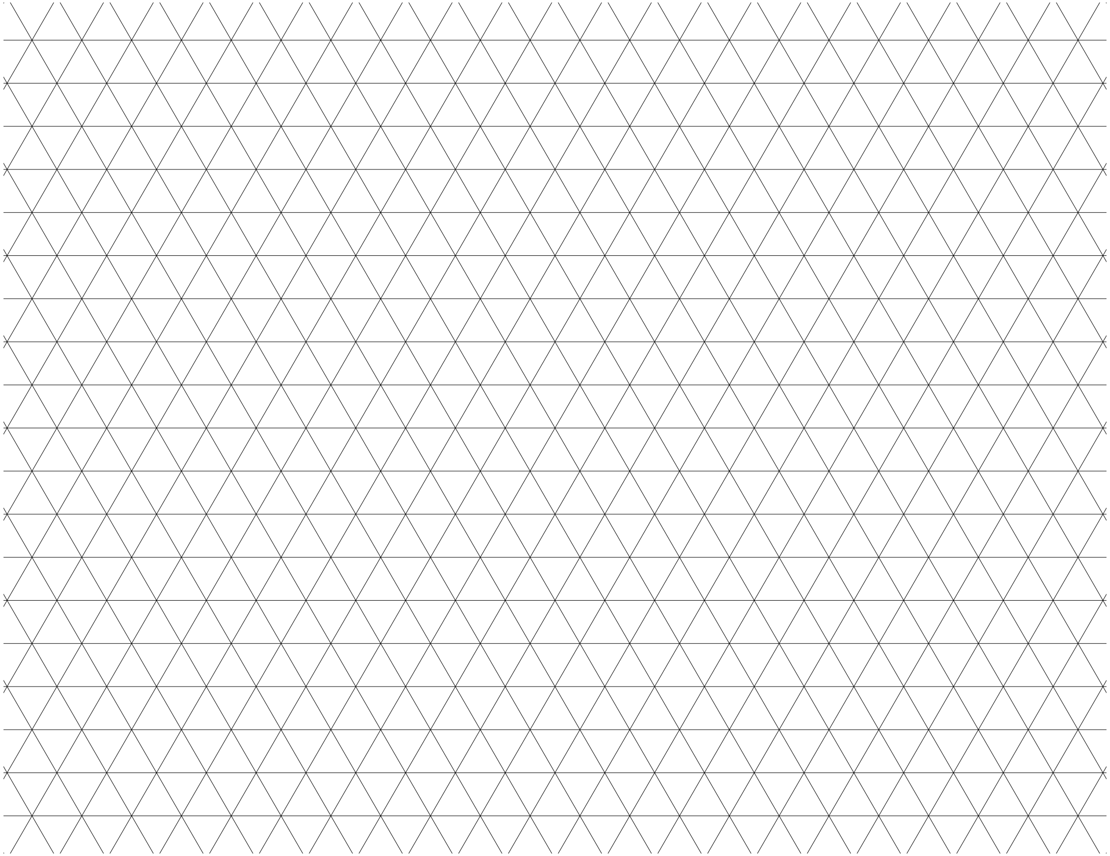
Let $G_0 \lneq G_1 \lneq G_2 \lneq \cdots \lneq G_i \lneq \cdots$ be finite groups and let T be the associated "tree of cosets".



Then $G = \cup G_i$ acts on T without inversions and every element of G fixes a point, but G has no global fixed point.

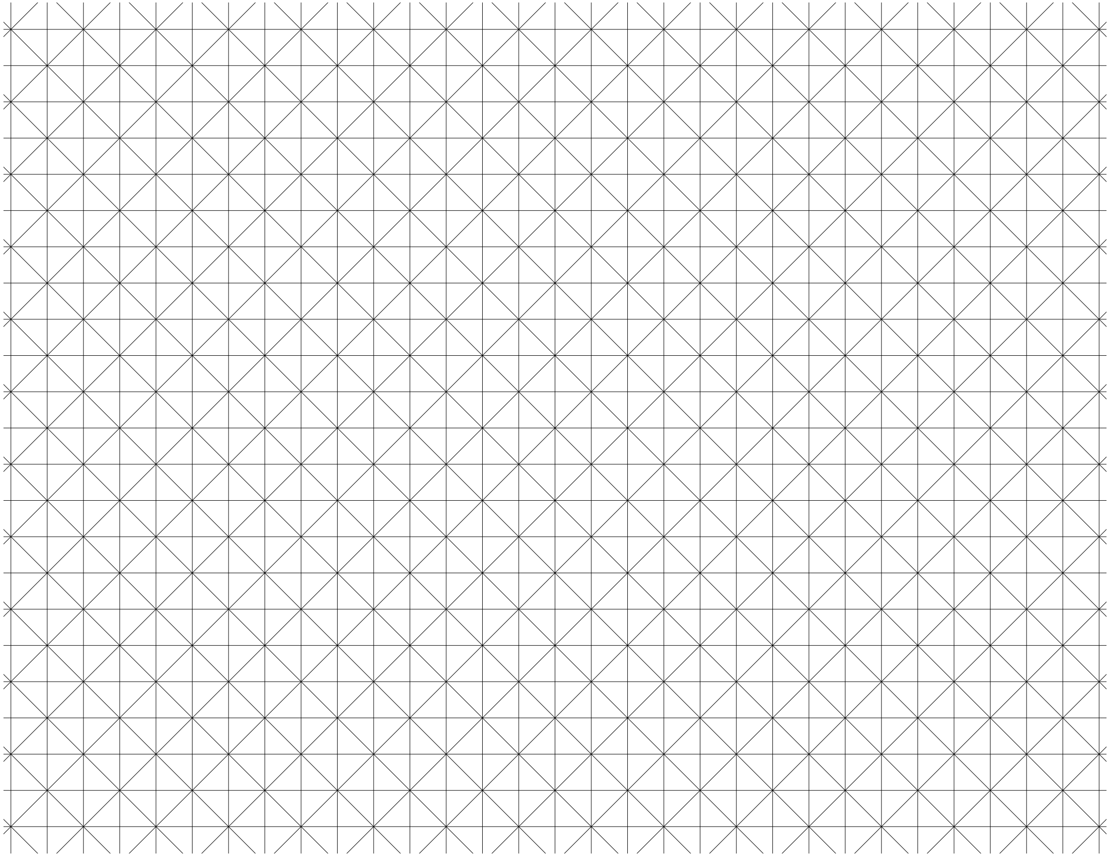
Buildings

Apartment of type $\tilde{A}_2$



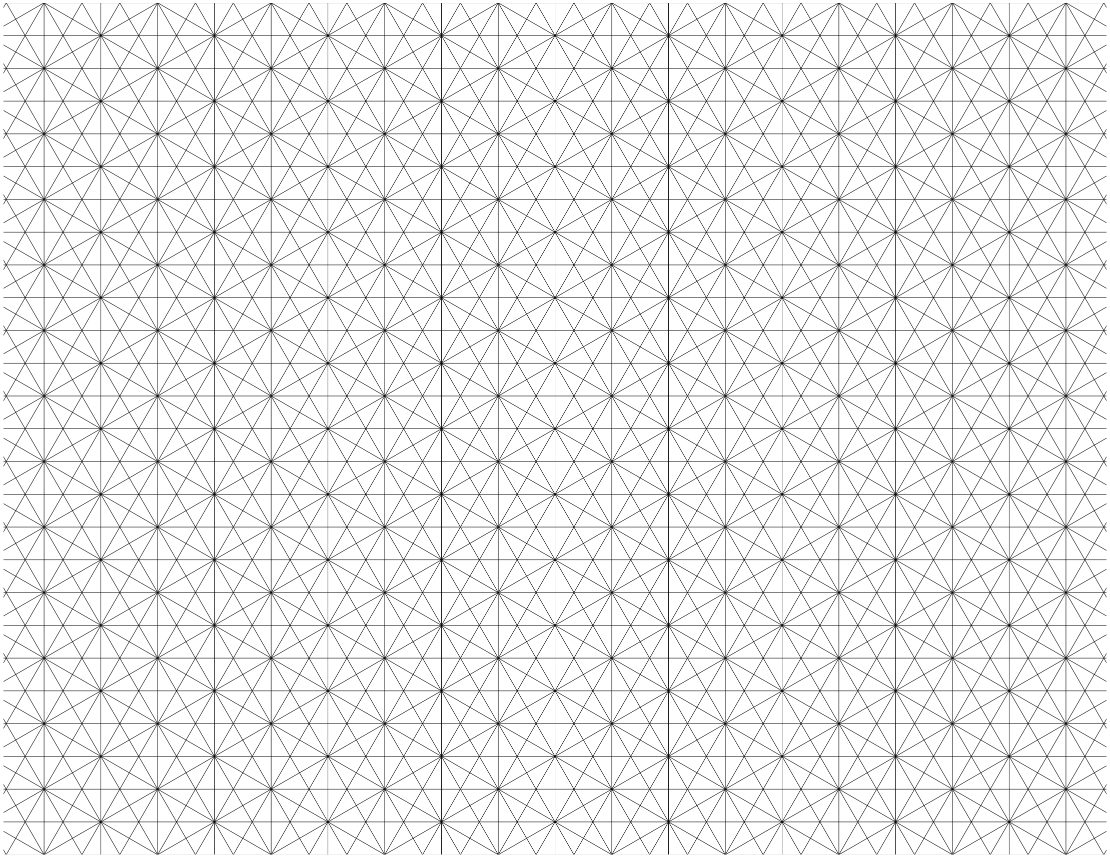
Buildings

Apartment of type $\tilde{C}_2$



Buildings

Apartment of type $\tilde{G}_2$



Buildings

Let $\mathbb{A}$ be a 2-dimensional real vector space.
Let W be a group of affine isometries of $\mathbb{A}$ such that

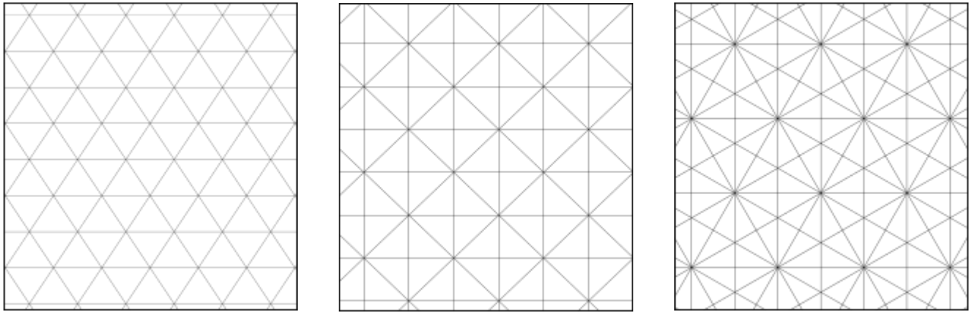
$$W = T \rtimes W_0$$

where

- $W_0 = \mathrm{Stab}_W(0)$ is a finite (linear) reflection group
- $T = W \cap \{\text{translations}\}$

Facts

1. *T is either discrete or dense in $\mathbb{R}^2$*
2. *If T is discrete, then W is crystallographic i.e. of type $\tilde{A}_2$, $\tilde{C}_2$ or $\tilde{G}_2$.*



Buildings

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Definition

Let X be a set and $\mathcal{F}$ a family of injections $f : \mathbb{A} \to X$. Each image $f(\mathbb{A})$ is an apartment. We say X equipped with $\mathcal{F}$ is an affine building of type W if:

(A1) $\mathcal{F}$ is invariant under precomposition by elements of W

(A2) For all $f, g \in \mathcal{F}$, $g^{-1}(f(\mathbb{A}))$ is a closed convex subset C of $\mathbb{A}$, and $g^{-1} \circ f|_C = w|_C$ for some $w \in W$

(A3) Any two points of X are contained in some apartment

...

Main result

Theorem (Schillewaert–Struyve–T 2021)

Let G be a finitely generated group acting on a 2-dimensional affine building X of type $\tilde{A}_2$ or $\tilde{C}_2$. If every element of G fixes a point of X, then G has a global fixed point.

Corollaries

Corollary 1

Suppose a group G acts on a complete 2-dimensional affine building X of type $\tilde{A}_2$ or $\tilde{C}_2$ such that every element of G fixes a point of X. Then G fixes a point in $\overline{X} = X \cup \partial X$.

Proof of Corollary 1.

Consider finitely generated subgroups of G, and apply theorem of Caprace–Lytchak (2010). □

Corollaries

Corollary 2

If a finitely generated group G acts without a global fixed point on a complete 2-dimensional affine building X of type $\tilde{A}_2$ or $\tilde{C}_2$, then G contains a hyperbolic element, in particular $\mathbb{Z} < G$.

Corollaries

Corollary 3

If a finitely generated infinite torsion group G acts on a discrete 2-dimensional affine building of type $\tilde{A}_2$ or $\tilde{C}_2$, then G has a global fixed point.

Previous local-to-global results

Theorem (Parreau 2003)

Let Γ *be a boundedly generated subgroup of* $\mathcal{G}(F)$*,* X *the Bruhat–Tits building for* $\mathcal{G}(F)$ *and* $\overline{X}$ *the Cauchy completion of* X*. If every element of* Γ *fixes a point of* $\overline{X}$ *then* $\overline{X}^{\Gamma} \neq \emptyset$*.*

Theorem (Breuillard–Fujiwara 2018)

Quantitative version of Parreau's result, for discrete Bruhat–Tits buildings.

Theorem (Leder–Varghese 2019, using work of Sageev 1995)

Finite-dimensional CAT(0) *cube complexes.*

Theorem (Norin–Osajda–Przytycki 2019)

Let G *be a finitely generated group acting on a* CAT(0) *triangle complex* X *such that either every element of* G *fixing a point in* X *has finite order, or* X *is locally finite, or* X *has rational angles. If every element of* G *fixes a point of* X*, then* G *has a global fixed point.*

Local-to-global for actions of finitely generated groups on nonpositively curved spaces?

There are many results in this direction, for various notions of nonpositive curvature.

However, Osajda (2018) constructed an action of a finitely generated infinite torsion group on an infinite-dimensional CAT(0) cube complex with no global fixed point.

Proof ideas

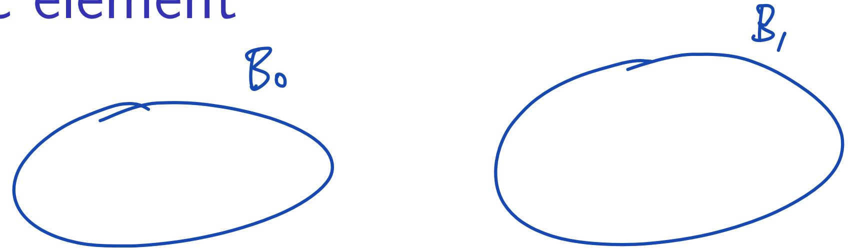
1. Reductions
2. Proof by contradiction: construct a hyperbolic $g \in G$

Reductions

We may assume:

- X is an $\mathbb{R}$-building with each point a special vertex (standard)
- X is metrically complete (uses ultrapower of X and theorems of Kleiner–Leeb (1997) and Struyve (2011))
- G is type-preserving (easy)

Construction of hyperbolic element



Assuming our reductions, we prove:

Proposition (SST)

Suppose G_0 and G_1 are proper finitely generated subgroups of G so that $B_0 := X^{G_0}$ and $B_1 := X^{G_1}$ are nonempty and disjoint.
Then G contains a hyperbolic element.

Proof of Theorem, assuming this Proposition.

Let $G = \langle s_1, \ldots, s_n \rangle$ and induct on n. $\square$

Construction of hyperbolic element

Proposition (SST)

Suppose G_0 and G_1 are proper finitely generated subgroups of G so that $B_0 := X^{G_0}$ and $B_1 := X^{G_1}$ are nonempty and disjoint. Then G contains a hyperbolic element.

We construct a hyperbolic $g \in G$ using:

- general results for complete CAT(0) spaces (Bridson–Haefliger)
- specific building-theoretic arguments for X and its vertex links

Construction of hyperbolic element

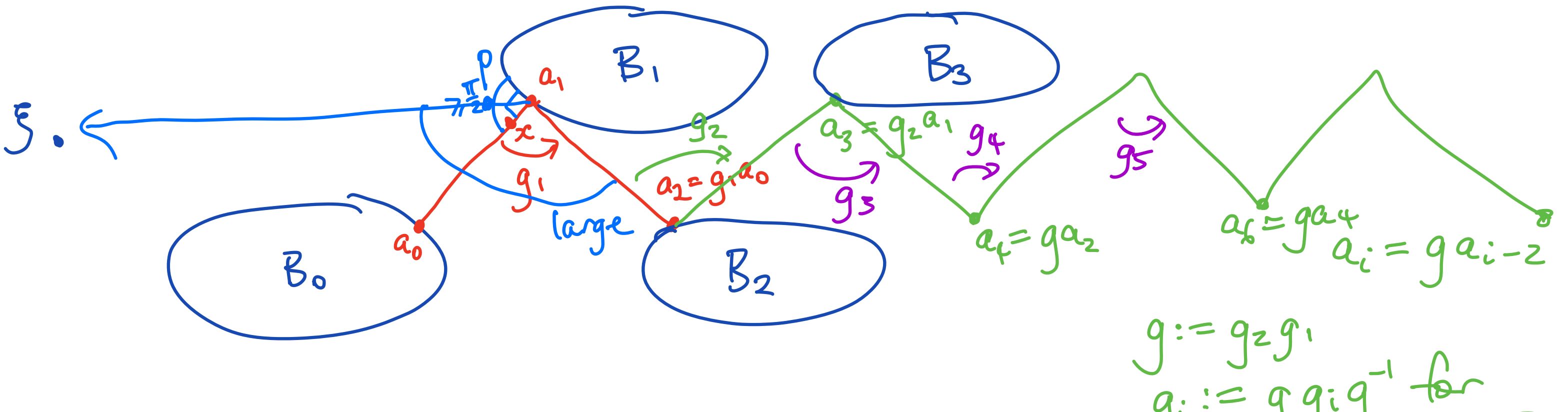
We construct:

- an element $g \in G$
- a sequence $\{a_i\}$ in X, with $a_{2k} = g^k a_0$ for all $k \geq 1$
- a point $\xi \in \partial X$, so that $d(a_{2k}, \xi) \to \infty$

If g is elliptic then all its orbits are bounded, so g must be hyperbolic.

Construction of hyperbolic element

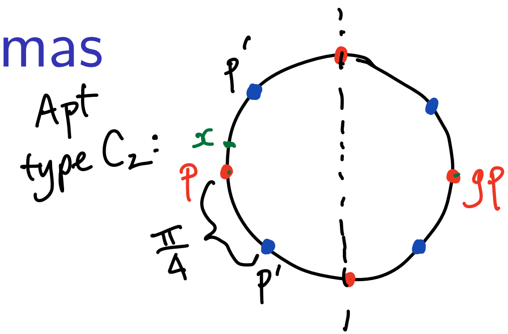
Suppose $B_0 := X^{G_0}$ and $B_1 := X^{G_1}$ are nonempty and disjoint.

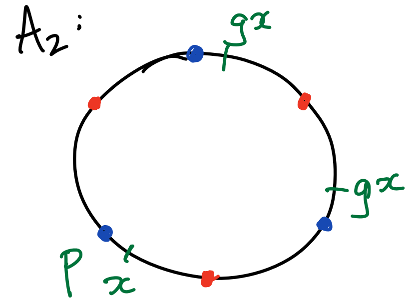


Ingredients to show that this actually happens:

- properties of complete CAT(0) spaces: B_0, B_1 are closed and convex; closest-point projections
- prove $d(B_0, B_1)$ is realised (for nondiscrete buildings)
- "local lemmas" for spherical buildings, to show $\exists g_1, g_2$
- existence of apartments in X, using opposite sectors
- basic Euclidean geometry within apartments of X
- properties of retractions of X
- properties of Busemann functions

Local lemmas





Lemma

Let Δ be a building of type C_2 and let G be a group of type-preserving automorphisms of Δ. If x is a point of Δ (not necessarily a panel) and p is a panel of Δ at minimum distance from x, then at least one of the following must hold:

1. *There is an element $g \in G$ mapping p to a panel opposite p, in which case $d(p, gx) \geq \frac{7\pi}{8}$.*
2. *There is a panel p' of Δ which is fixed by G such that $d(p', x) < \frac{\pi}{2}$.*