# Tautological classes and symmetry in Khovanov-Rozansky homology

Eugene Gorsky (University of California, Davis)

joint with Matt Hogancamp and Anton Mellit, arXiv:2103.01212

For any link $L$, Khovanov and Rozansky defined **triply graded homology** $\mathrm{HHH}_{i,j,k}(L)$ categorifying the unreduced HOMFLY-PT polynomial

$$P(a,q) = \sum_{i,j,k} a^i q^j (-1)^k \dim \mathrm{HHH}_{i,j,k}(L).$$

For knots, there is a reduced version $\overline{\mathrm{HHH}}_{i,j,k}(K)$ which yields a finite dimensional vector space.

**Theorem (GHM)**

*For any knot $K$, the reduced Khovanov-Rozansky homology is symmetric:*

$$\overline{\mathrm{HHH}}_{i,-2j,k}(K) = \overline{\mathrm{HHH}}_{i,2j,k+2j}(K).$$

This was conjectured by Gukov, Dunfield and Rasmussen in 2005:

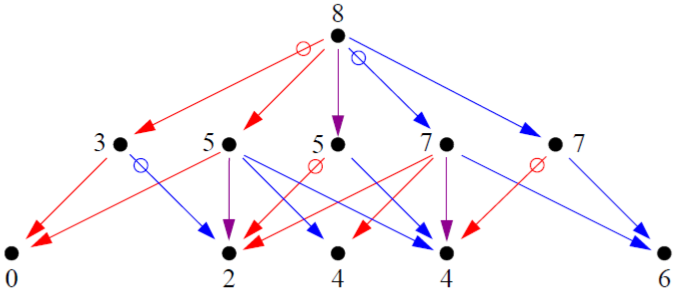


FIGURE 3.7. Differentials for $T_{3,4}$.The bottom row of dots has $a$-grading 6. The leftmost dot on that row has $q$-grading $-6$, which you can determine by noting that the vertical axis of symmetry corresponds to the line $q = 0$.

Other approaches:

- Oblomkov, Rozansky: use matrix factorizations over Hilbert schemes of points on the plane.
- Galashin, Lam (for knots related to Richardson varieties): use graded Koszul duality for category $\mathcal{O}$ developed by Bezrukavnikov-Yun.

Both approaches use very heavy machinery of geometric representation theory. Our proof is more straightforward, and generalizes to links.

**Problems** for links:

- No good reduced homology
- There is an action of a polynomial ring $\mathbb{C}[x_1, \ldots, x_c]$ on the unreduced homology $\mathrm{HHH}(L)$, where $c$ is the number of components of $L$
- The symmetry does not preserve the degrees of $x_i$.

**Solution**: use "$y$-ified" homology $\mathrm{HY}(L)$ defined by G.-Hogancamp. It is naturally a module over $\mathbb{C}[x_1, \ldots, x_c, y_1, \ldots, y_c]$. The symmetry would exchange $x_i$ with $y_i$.

**Theorem (G.,Hogancamp)**

*For all $n, k \geq 0$ the homology of the $(n, kn)$ torus link with $n$ components is given by:*

$$\mathrm{HY}(T(n, kn)) = \bigcap_{i \neq j} (x_i - x_j, y_i - y_j, \theta_i - \theta_j)^k \subset \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n, \theta_1, \ldots, \theta_n] = \mathrm{HY}(\mathit{unlink}).$$

# Main result

**Theorem (GHM)**

*For any link $L$, there is an action of operators $F_k$ on $\mathrm{HY}(L)$ satisfying the following relations:*

$$[F_k, F_m] = 0, \quad [F_k, x_i] = 0, \quad [F_k, y_i] = kx_i^{k-1}.$$

*Furthermore, $F_2$ satisfies "hard Lefshetz property":*

$$F_2^j : \mathrm{HY}_{i,-2j,k}(L) \to \mathrm{HY}_{i,2j,k+2j}(L)$$

*is an isomorphism, and $F_2$ extends to an action of $\mathfrak{sl}_2$ on $\mathrm{HY}(L)$.*

For knots, $\mathrm{HY}(K) = \overline{\mathrm{HHH}}(K) \otimes \mathbb{C}[x, y]$ and the symmetry of $\mathrm{HY}(K)$ implies the symmetry of $\overline{\mathrm{HHH}}(K)$.

# Definition of Khovanov-Rozansky homology

Let $R = \mathbb{C}[x_1, \dots, x_n]$.

- Define the $R - R$-bimodules $B_i = R \otimes_{R^{(i\ i+1)}} R$.
- To a simple crossing, associate *Rouquier complexes*

$$T_i = [B_i \to R],\ T_i^{-1} = [R \to B_i]$$

- To any braid $\beta$, associate the product $T_\beta$ of $T_i, T_i^{-1}$. It is naturally a complex of $R - R$ bimodules.
- The braid closure corresponds to the Hochschild homology $\mathrm{HH}(T_\beta)$.

**Theorem (Khovanov,Rozansky)**

*The output of this construction is a topological invariant of the closure of $\beta$.*

# Dg algebra $\mathcal{A}$

Let

$$B = \frac{\mathbb{C}[x_1, \ldots, x_n, x_1', \ldots, x_n']}{f(x_1, \ldots, x_n) = f(x_1', \ldots, x_n') \text{ for any symmetric function } f}.$$

We define a dg algebra $\mathcal{A}$ as follows:

$$\mathcal{A} = B[\xi_1, \ldots, \xi_n, u_1, \ldots, u_n], \ d(\xi_i) = x_i - x_i', \ d(u_k) = \sum_{i=1}^{n} h_{k-1}(x_i, x_i')\xi_i$$

Here $h_{k-1}(x_i, x_i') = x_i^{k-1} + \ldots + (x_i')^{k-1}$ is the complete symmetric function. Note that

$$d^2(u_k) = \sum_{i=1}^{n} h_{k-1}(x_i, x_i')(x_i - x_i') = \sum x_i^k - \sum (x_i')^k = 0.$$

# Dg algebra $\mathcal{A}$: properties

**Theorem**

*The dg algebra $\mathcal{A}$ is a free resolution of $R$ as a $B$-module.*

**Theorem**

*The dg algebra $\mathcal{A}$ has a coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes_R \mathcal{A}$ defined by the equations*

$$\Delta(x_i) = x_i \otimes 1, \ \Delta(x_i') = 1 \otimes x_i', \ \Delta(\xi_i) = \xi_i \otimes 1 + 1 \otimes \xi_i,$$

$$\Delta(u_k) = u_k \otimes 1 + 1 \otimes u_k + \sum_{i=1}^{n} h_{k-2}(x_i, x_i', x_i'')\xi_i \otimes \xi_i.$$

*This coproduct is coassociative up to homotopy.*

# Dg algebra $\mathcal{A}$: modules

The dg algebra $\mathcal{A}$ acts on simple crossings $T_i$, $T_i^{-1}$ as follows: $\xi_i$ are explicit "dot sliding homotopies" and $u_k$ act by 0. By using the coproduct on $\mathcal{A}$, we can extend the action to arbitrary Rouquier complexes and obtain the following:

**Theorem**

*There is an action of $\mathcal{A}$ on the Rouquier complex $T_\beta$ associated to an arbitrary braid $\beta$. This action is invariant under braid relations and unique up to homotopy.*

# $y$-ification

We can use the action of $\xi_i$ to deform the differential on $T_\beta$:

$$D = d + \sum \xi_i y_i$$

The differential $D$ does not square to zero, but $D^2$ vanishes after closing the braid (that is, applying $\mathrm{HH}$), and we can define

$$\mathrm{HY}(\beta) = H(\mathrm{HH}(T_\beta) \otimes \mathbb{C}[y_1, \ldots, y_c], D)$$

**Theorem (G.,Hogancamp)**

*The "y-ified" homology $\mathrm{HY}(\beta)$ is the topological invariant of the closure of $\beta$.*

# Construction of $F_k$

The action of $u_k$ can be used to define the operators

$$F_k = h_{k-1}(x_i, x_i')\frac{\partial}{\partial y_i} + u_k.$$

One can check that $[D, F_k] = 0$ and hence $F_k$ yield well-defined operators on $\mathrm{HY}(\beta)$.

**Theorem (GHM)**

*The action of $F_k$ on $\mathrm{HY}(\beta)$ is a topological invariant of the closure of $\beta$.*

# Lefshetz property

To prove that $F_2$ satisfies "hard Lefshetz property", we use the objects $K_{i,j} = R \xrightarrow{x_i - x_j} R$ which are $\mathcal{A}$-modules as well. The maps in the "skein exact triangle"

$$T_i \to T_i^{-1} \to K_{i,i+1}$$

agree with the action of $\mathcal{A}$ (up to homotopy). This allows us to reduce a complicated Rouquier complex to the ones for unlinks, possibly multiplied by products of $K_{ij}$. We explicitly compute the action of all $F_k$ for such complexes, and verify the hard Lefshetz property for them.

# Geometric motivation

Let me comment on geometric motivation behind the construction of $\mathcal{A}$ and the coproduct. Let $G = GL(n)$, for any symmetric function $Q(x_1, \ldots, x_n)$ of degree $d$ one can construct the following differential forms:

$$\Phi_1(Q) \in \Omega^{2d-1}(G),\ \Phi_2(Q) \in \Omega^{2d-2}(G \times G),\ \ldots \Phi_d(Q) \in \Omega^d(G^d)$$

satisfying equations

$$d\Phi_1(Q) = 0,\ d\Phi_2(Q) = \Phi_1(Q) \otimes 1 + 1 \otimes \Phi_1(Q) - m^*(\Phi_1(Q)),\ \ldots$$

where $m : G \times G \to G$ is the multiplication map on $G$.
For example, for $Q = \sum x_i^2$ we get a 3-form on $G$ and a 2-form on $G \times G$.

# Geometric motivation cont'd

Atiyah, Bott, Jeffrey and others used these forms to construct interesting cohomology classes on character varieties.

Suppose that $f : X \to G$ and $g : Y \to G$ are matrix-valued functions such that $f^*\Phi_1(Q) = d\omega_X$ and $g^*\Phi_1(Q) = d\omega_Y$ for some forms $\omega_X$ and $\omega_Y$, then we can define

$$f \cdot g : X \times Y \to G, \ \omega_{X\times Y} = \omega_X \otimes 1 + 1 \otimes \omega_Y + (f \times g)^*\Phi_2(Q)$$

such that

$$d(\omega_{X\times Y}) = (f \cdot g)^*(\Phi_1(Q)).$$

This is very similar to our coproduct, where $\omega_X$ play the role of $u_k$. Note that even if $\omega_X = \omega_Y = 0$, $\omega_{X\times Y}$ could be nontrivial.

# Geometric motivation: braid varieties

Given a positive braid $\beta = \sigma_{i_1} \cdots \sigma_{i_k}$, Mellit defined the *braid variety*

$$X(\beta) = \{z_1, \ldots, z_k : B_{i_1}(z_1) \cdots B_{i_k}(z_k) \text{ is upper-triangular}\}$$

where $B_i(z)$ are certain explicit matrices.

**Theorem (Mellit)**

*There is an algebraic closed 2-form $\omega$ on $X(\beta)$. The cup product with $\omega$ satisfies "curious hard Lefshetz property" with respect to the weight filtration in $H^*(X(\beta))$.*

The form $\omega$ corresponds to the symmetric function $Q = \sum x_i^2$.

**Theorem (Casals,G.,M.Gorsky,Simental)**

*If $\beta = \gamma\Delta^2$ then $X(\beta)$ is smooth and $\omega$ is holomorphic symplectic.*

# Example

**Theorem (Galashin, Lam)**

*The braid variety for the $(m, n)$ torus knot is isomorphic to the open positroid stratum in $Gr(m, m+n)$, up to a free action of a certain torus.*

For example, $(3, 4)$ torus knot corresponds to the open positroid stratum in $Gr(3, 7)$, also known as $E_6$ cluster variety. The weight filtration in its cohomology was computed by Lam and Speyer:

$E_6$ :

| | $H^0$ | $H^1$ | $H^2$ | $H^3$ | $H^4$ | $H^5$ | $H^6$ |
|---|---|---|---|---|---|---|---|
| $k-p=0$ | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| 1 | | | | | 1 | | |

The 2-form $\omega$ generates the second cohomology group of this variety, and lifts to an action of $\mathfrak{sl}_2$. The symmetric function $Q = \sum x_i^3$ corresponds to an interesting generator of $H^4$.

One can compare this with the bottom row of the HOMFLY-PT homology:

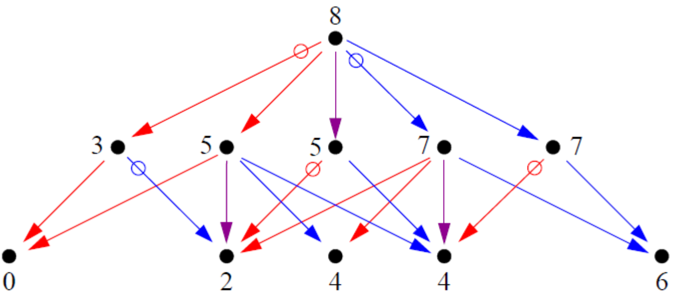


FIGURE 3.7. Differentials for $T_{3,4}$.The bottom row of dots has $a$-grading 6. The leftmost dot on that row has $q$-grading $-6$, which you can determine by noting that the vertical axis of symmetry corresponds to the line $q = 0$.

$E_6$ :

| | $H^0$ | $H^1$ | $H^2$ | $H^3$ | $H^4$ | $H^5$ | $H^6$ |
|---|---|---|---|---|---|---|---|
| $k - p = 0$ | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| 1 | | | | | 1 | | |

# Picture credits:

1. N. Dunfield, S. Gukov, J. Rasmussen. The superpolynomial for knot homologies. Experiment. Math. 15 (2006), no. 2, 129–159.
2. T. Lam, D. Speyer. Cohomology of cluster varieties. I. Locally acyclic case. 1604.06843

Thank you