QFT's for Non-Semisimple TQFT's

Tudor Dimofte

(U of Edinburgh & UC Davis)

May 17, 2021

based on [2105.xxxxx]

w/ T. Creutzig, N. Garner, N. Geer / NSF FRG

cf. [Brown-TD-Geer-Garoufalidis '20]

"The ADO Invariants are a q-Holonomic Family"

Apology: no knot homology (for now)

but both knots and homology will appear!

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Chern-Simons QFT ⟷ Rational VOA ⟷ Quantum groups

G (compact)

$k - h \in \mathbb{Z}_{\geq 0}$

$V^k(\mathfrak{g})$

$U_q(\mathfrak{g}) \quad q = e^{\frac{i\pi}{k}}$

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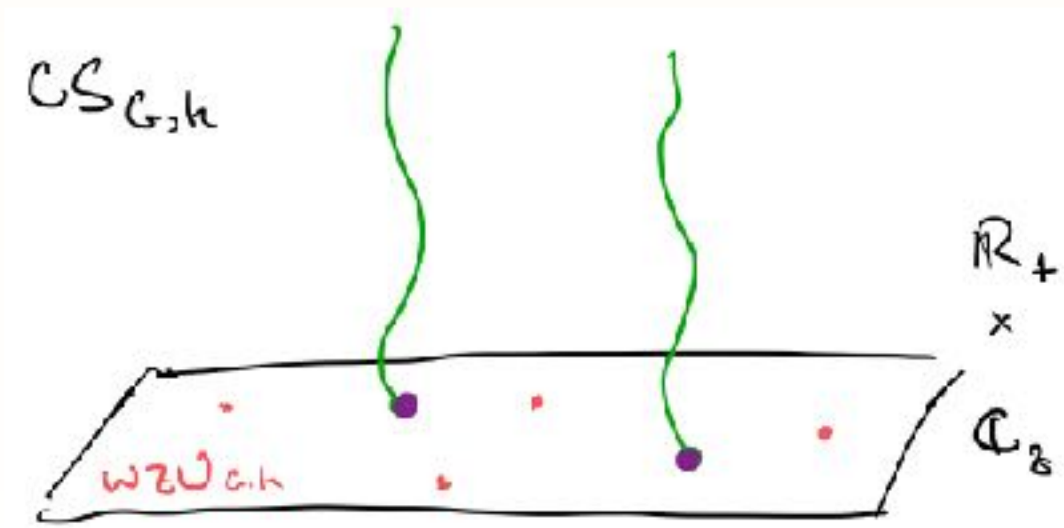

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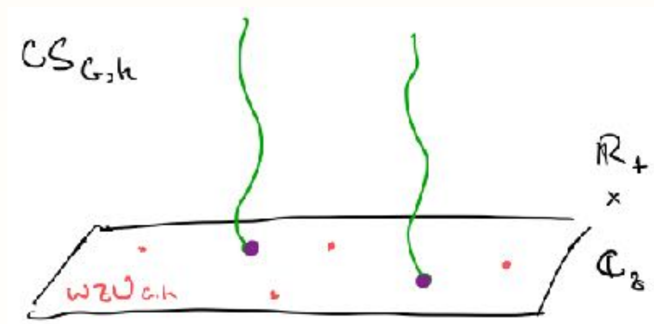
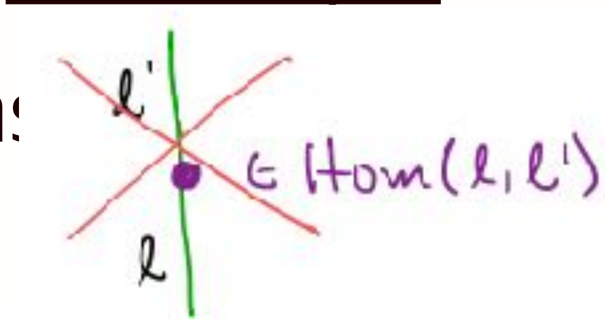


Key property: $\mathcal{C}$ is <u>semisimple</u>

no nontrivial junctions of Wilson lines!



no nontrivial extensions of simple objects

30 years after:

Quantum groups $U_q(\mathfrak{g})\text{-mod} \quad q = e^{\frac{i\pi}{k}}$

non-s.s. rep theory + vanishing quantum dim's

invariants of links/3-manifolds w/ flat $G^\vee_{\mathbb{C}}$ connections

(& spin struc's)

Rational VOA

Chern-Simons QFT

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[Akutsu-Deguchi-Ohtsuki '92] "ADO"

[Lyubashenko '95]

[Kashaev-Reshetikhin '02]

[Geer-Costantino-PatureauMirand '12] "CGP"

[Blanchet-C-G-P '14]

[Blanchet-G-P-Reshetikhin '18]

[deRenzi-Gainutdinov-G-P-Runkel '19, '20]

⋮

unrolling $U_q^H(\mathfrak{g})$

renormalized qu. dim's

Rational VOA

Chern-Simons QFT

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Rational VOA $\mathcal{FT}_k(\mathfrak{g})$

$\rightsquigarrow$ log-VOA's and their module categories

deformations by flat connections

[Kausch '91], [Gaberdiel-Kausch '96] triplet $\mathcal{W}_{1,k} = \mathcal{FT}_k(\mathfrak{sl}_2)$

[Feigin-Tipunin '10]

[Feigin-Gainutdinov-Semikhatov-Tipunin '05, '06]
[Nagatomo-Tsuchiya '09]
[Creutzig-Milas-Rupert '16]
[Creutzig-Gainutdinov-Runkel '17]
[Creutzig-Yang-McRae '20] [Sugimoto '20]
[Creutzig-Lentner-Rupert '21] [Gannon-Negron '21]

Chern-Simons QFT

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Chern-Simons QFT

??

$\rightsquigarrow$ cohomological 3d theories (twists of SUSY) ? [Witten '88]

with flavor symmetry ?

cf. [GHNPPK '20]

[Cheng-Chun-Ferrari-Gukov-Harrison '18]

Our main conjecture:

(for G simply connected ADE…)

QFT ⟷ Log VOA ⟷ Quantum group

$$\mathcal{C} = \text{\{line operators\}} \simeq D^b\mathcal{FT}_k(\mathfrak{g})\text{-mod} \simeq D^b U_q(\mathfrak{g})\text{-mod} \quad q = e^{\frac{i\pi}{k}}$$

* unroll for braiding

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$\mathcal{T}^A_{G,k} = (T[G]/G_k)^A$

- topological A-twist of a 3d N=4 theory cf. [Gaiotto-Witten '08]

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- simpler checks: $K_0(\mathcal{C})$ $\chi[\mathcal{H}(\Sigma_g)]$

via [Benini-Zaffaroni '15,'16], [Closset-Kim '16]

Reps of $U_q(\mathfrak{sl}_2)$ $q = e^{\frac{i\pi}{k}}$

[deConcini-Kac-(Procesi) '91, Beck '94]

$$U_q(\mathfrak{sl}_2) = \mathbb{C}\langle E, F, K \rangle /$$

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$$\begin{array}{c} \mathcal{C} \\ \downarrow \\ PGL_2(\mathbb{C}) \end{array}$$

e.g. $g = \begin{pmatrix} e^\alpha & 0 \\ 0 & 1 \end{pmatrix}$

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Leads to invariants of (spin) 3-manifolds w/ flat $PGL_2(\mathbb{C})$ conn's!

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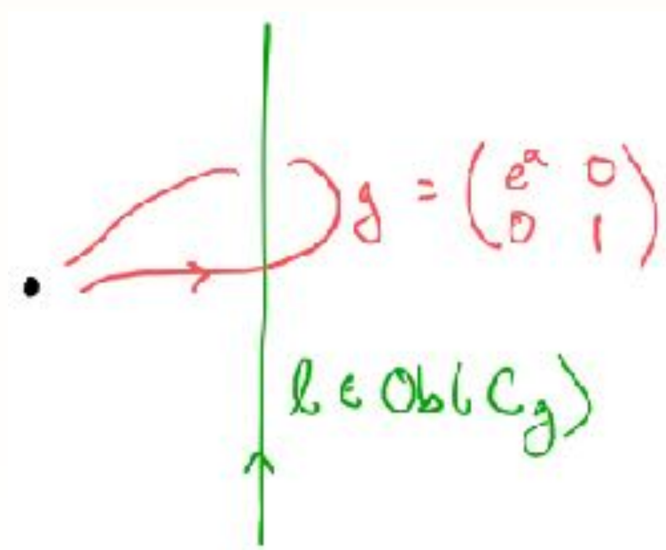

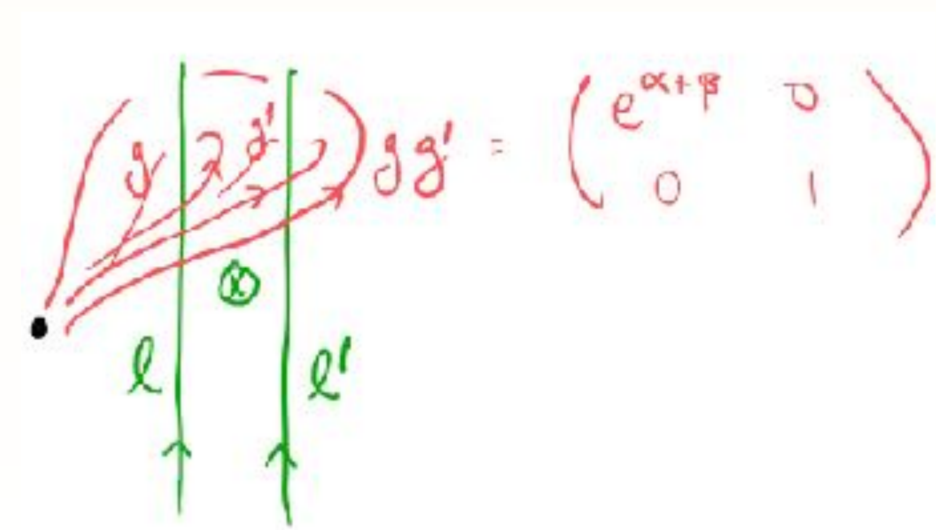

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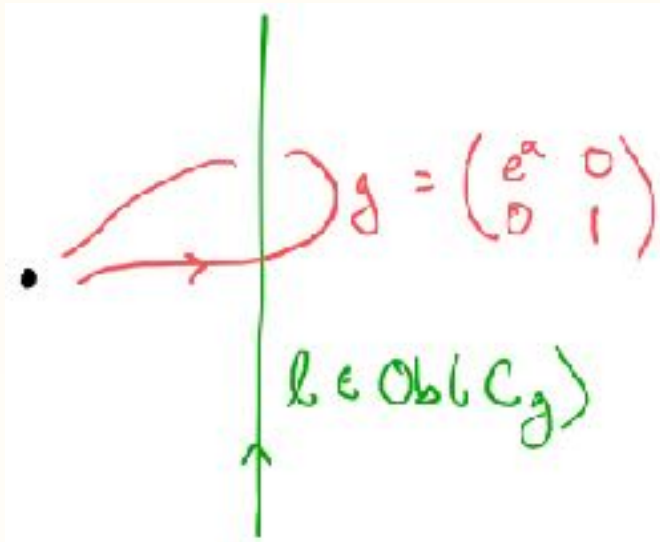

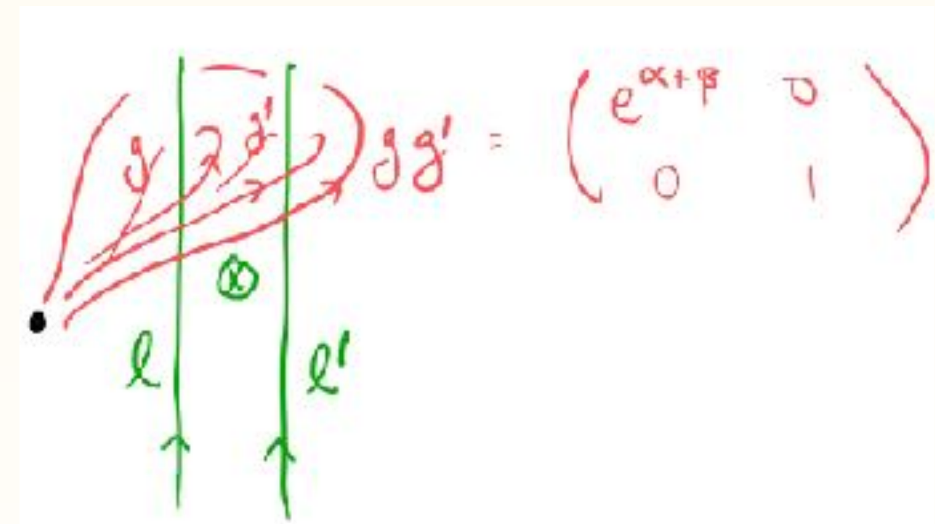

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Physics: expect a topological 3d QFT w/ global symmetry

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flat? complexified?

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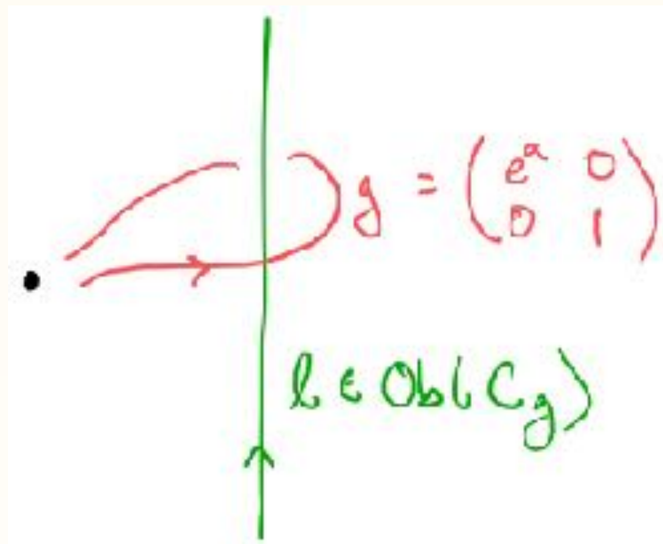

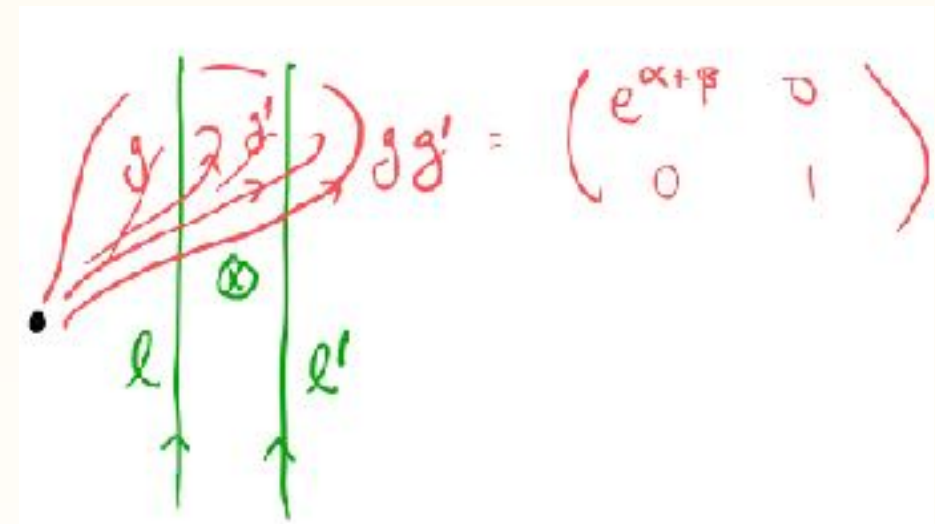

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$\mathcal{C}_g =$ {line ops} in the background of a holonomy defect

cf. [Mikhaylov '15]

$$\mathcal{A} \sim \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} d\theta$$

Reps of $U_q(\mathfrak{sl}_2)$ $q = e^{\frac{i\pi}{k}}$

Generic diagonal holonomy: $g = \begin{pmatrix} e^{\alpha} & 0 \\ 0 & 1 \end{pmatrix}$ $K^{2k} = e^{\alpha}$

$\mathcal{C}_g$ is semisimple, w/ 2k simple objects $\{V^{\pm}_{\alpha,n}\}_{n=0}^{k-1}$

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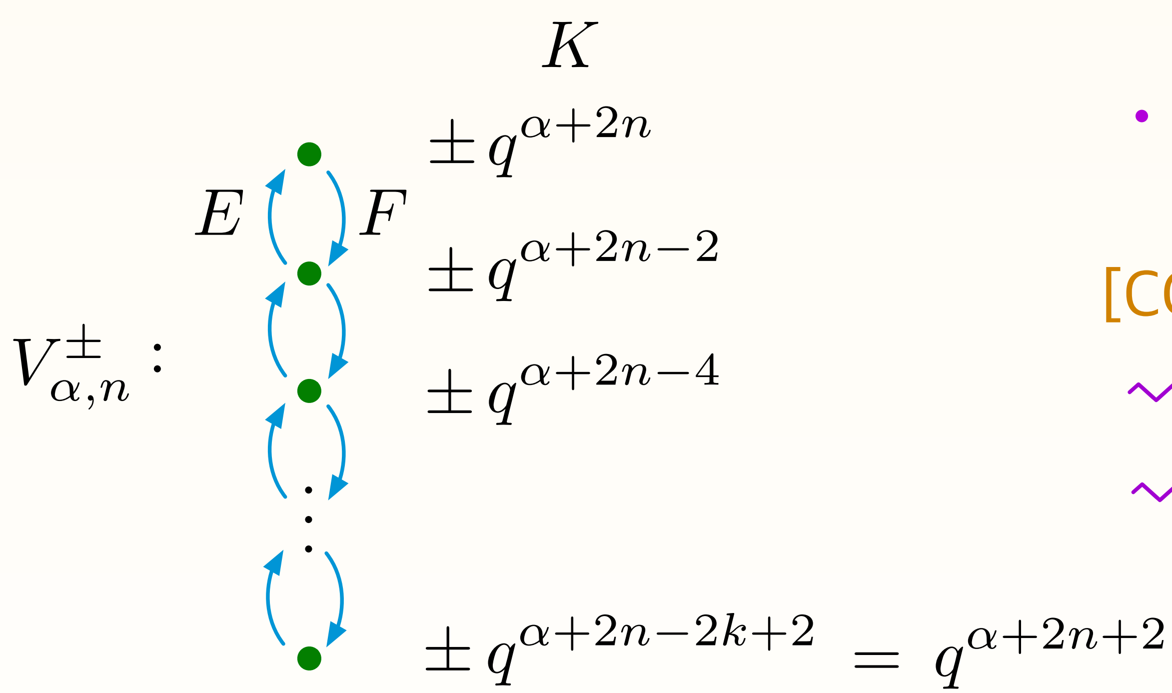

- vanishing quantum dims $\dim V^{\pm}_{\alpha,n} = 2k$

$\dim_q V^{\pm}_{\alpha,n} = 0$

[CGP]

$\rightsquigarrow$ ADO inv't of links in S^3

$\rightsquigarrow$ inv'ts of $(M, \mathcal{A}_{abel})$

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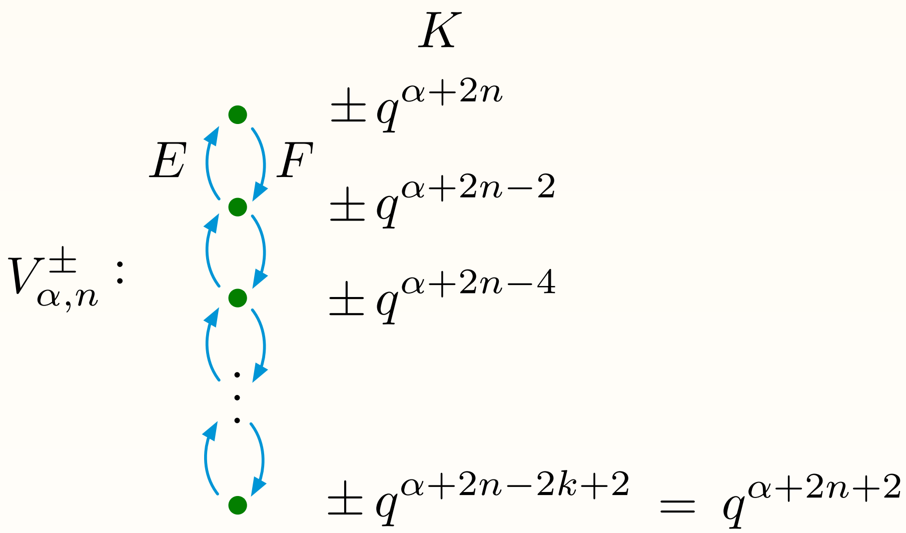

[CGP]

⇝ ADO inv't of links in S^3

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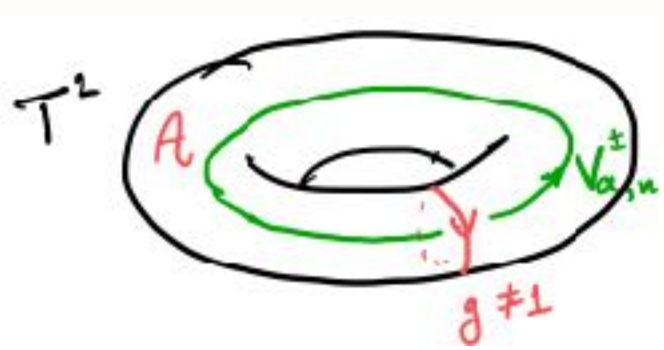
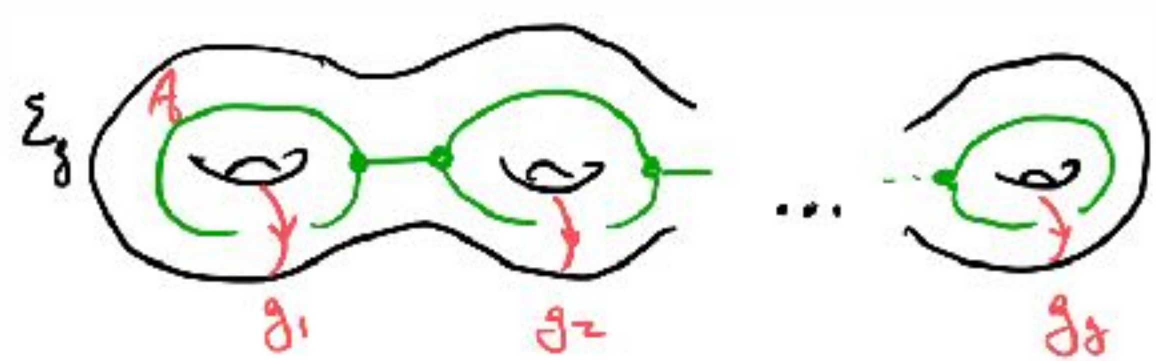
Simple monoidal structure $\quad V^{\epsilon}_{\alpha,n} \otimes V^{\epsilon'}_{\alpha',n'} = \bigoplus_{j=0}^{k-1} V^{\epsilon\epsilon'}_{\alpha+\alpha',n+n'-2j}$

$\Rightarrow \quad \dim \mathcal{H}(T^2, \mathcal{A}_{abel}) = 2k \qquad \dim \mathcal{H}(\Sigma_{g>1}, \mathcal{A}_{abel}) = 2^g k^{3g-3}$





Reps of $U_q(\mathfrak{sl}_2)$ $q = e^{\frac{i\pi}{k}}$

Trivial holonomy: $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $K^{2k} = 1$

small qu. group $u_q(\mathfrak{sl}_2) = U_q(\mathfrak{sl}_2)/(K^{2k} - 1 = E^k = F^k = 0)$

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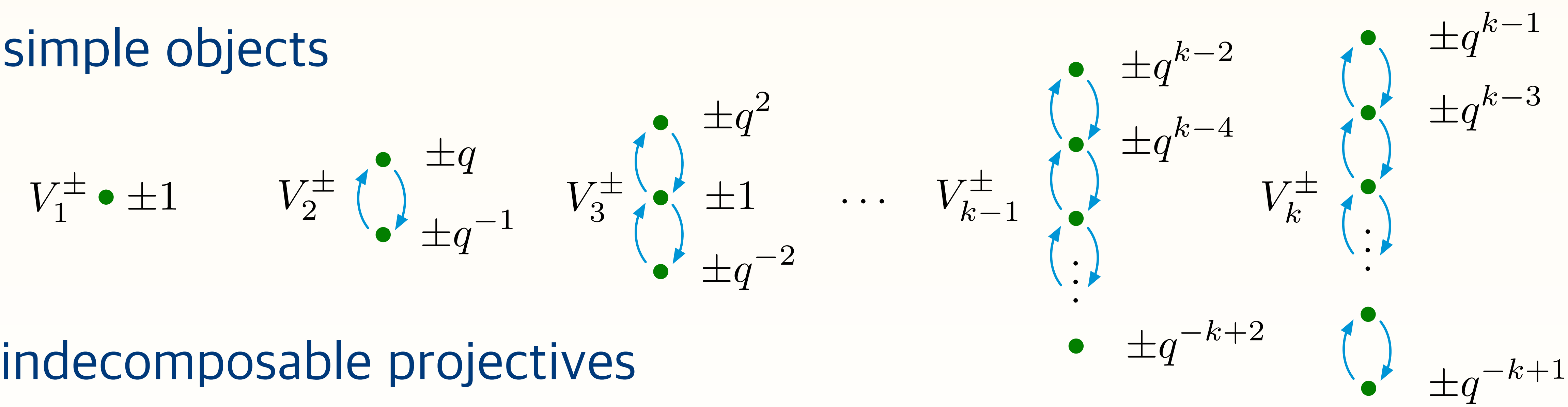
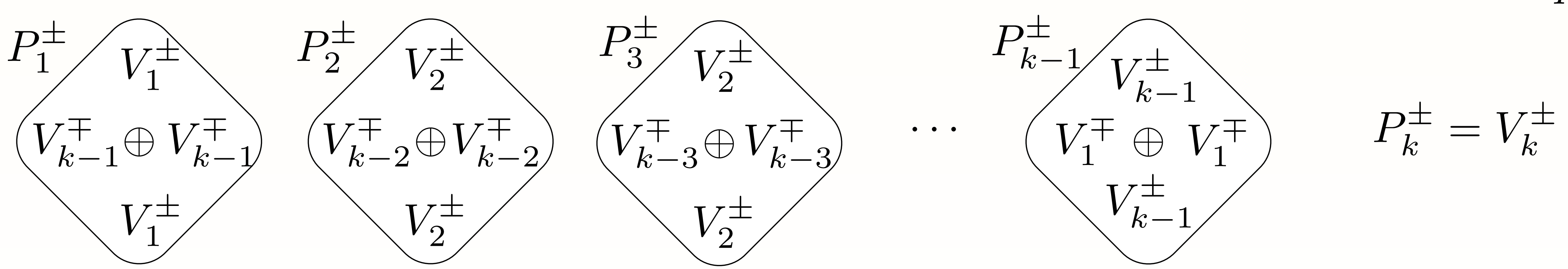
2k simple objects



2k indecomposable projectives



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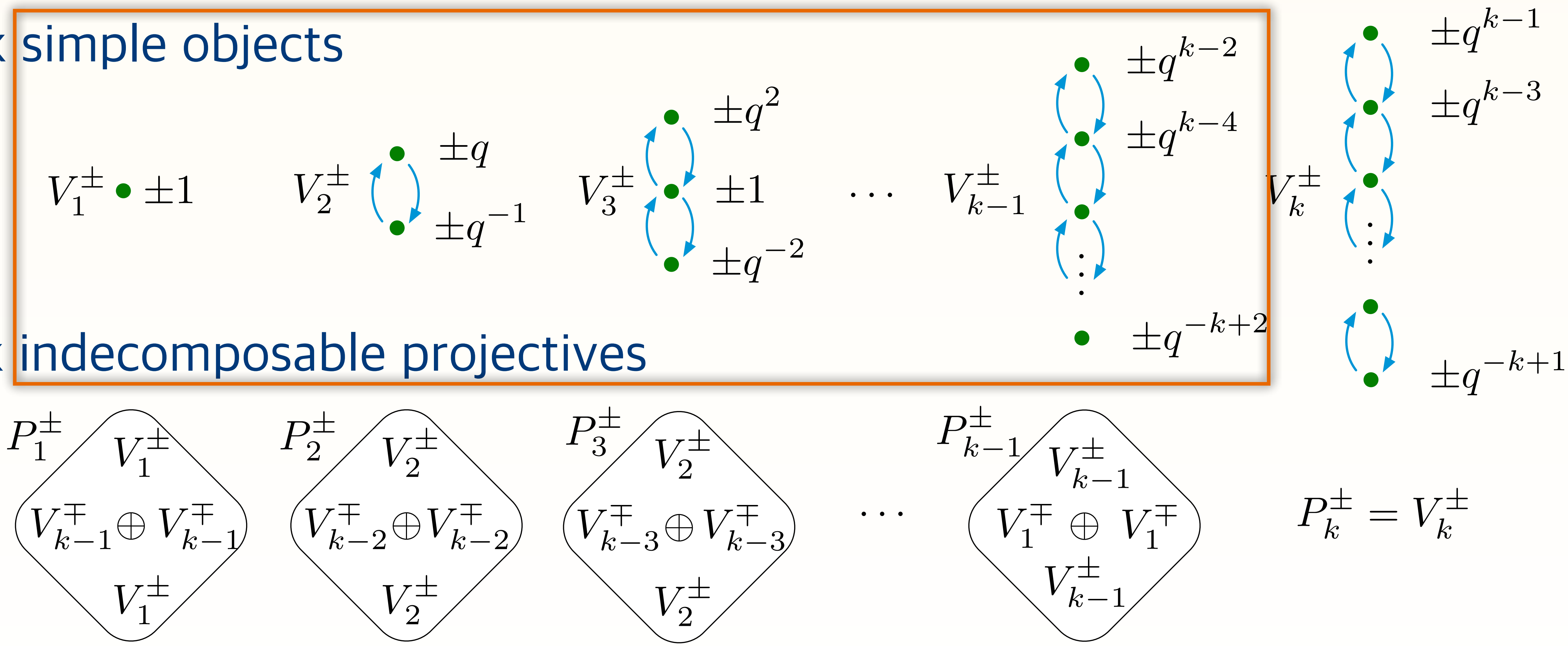
2k indecomposable projectives



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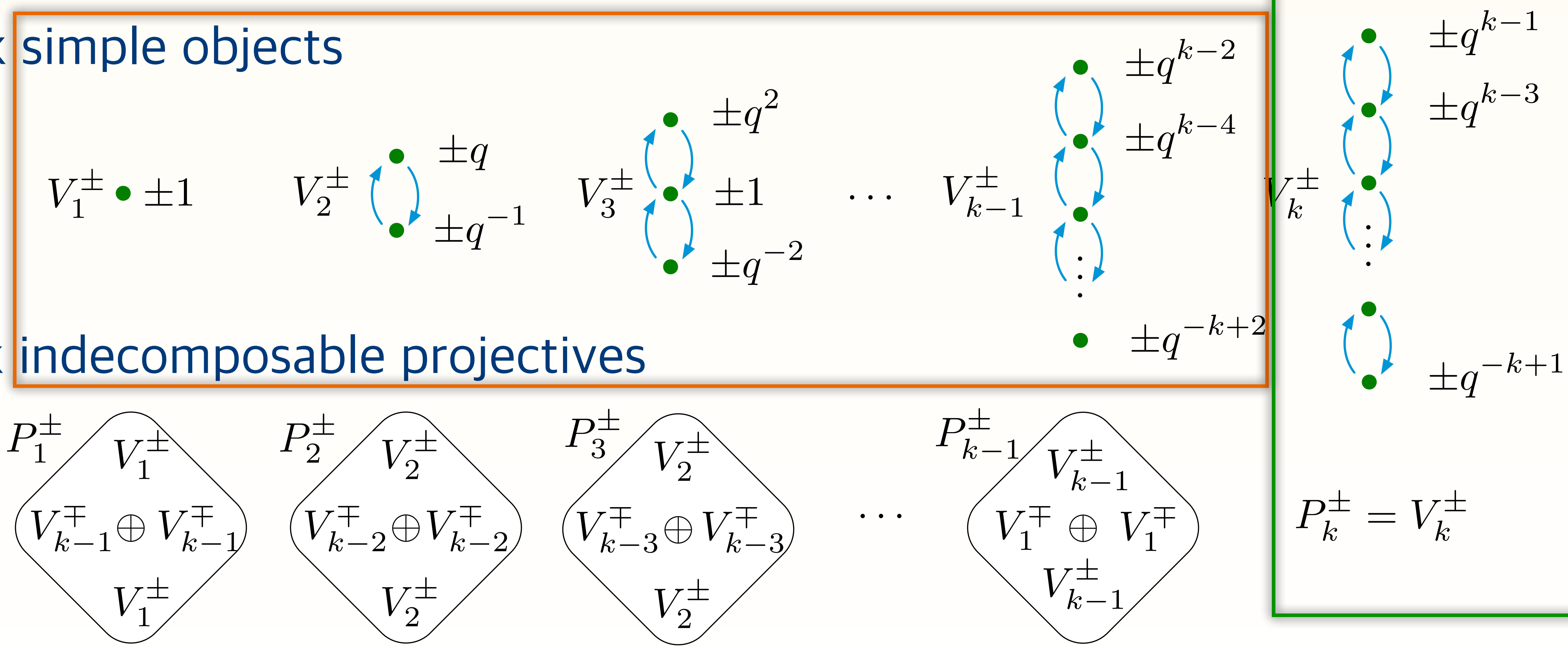

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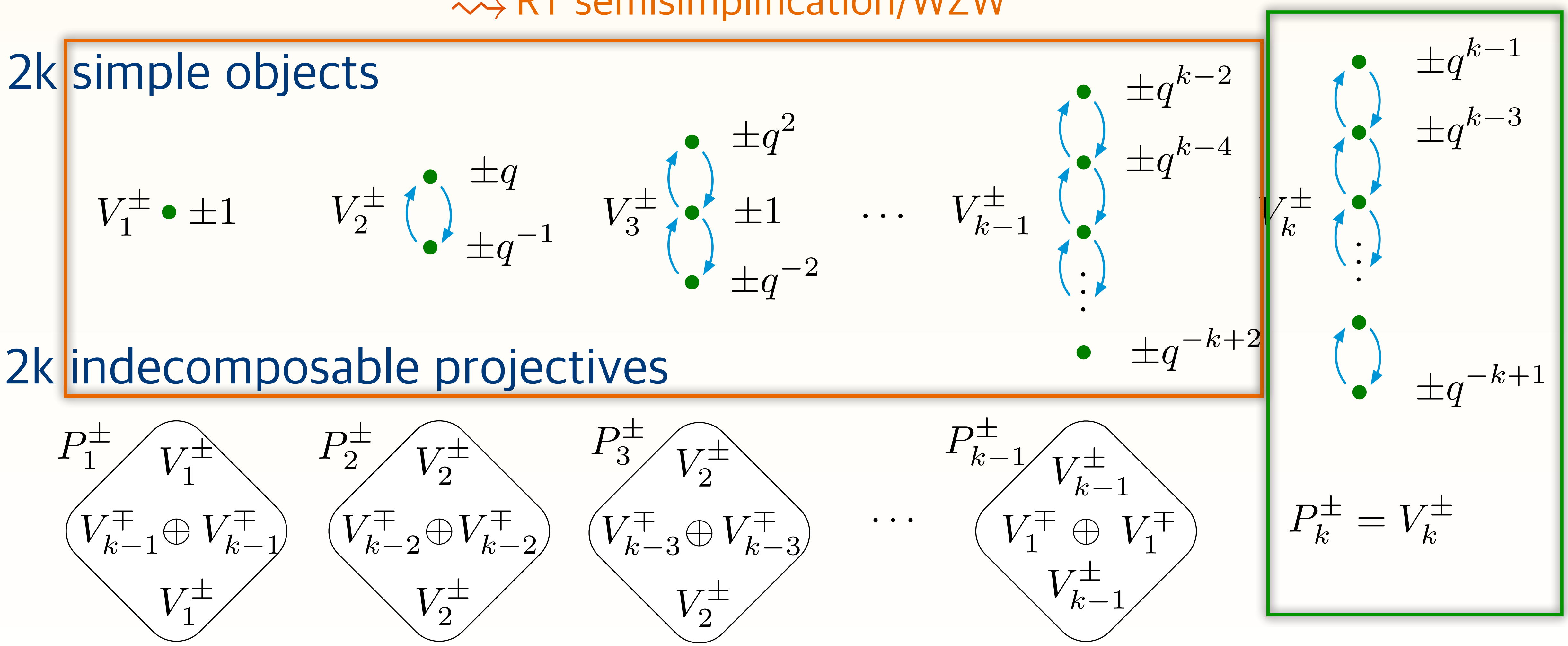

entire category $\rightsquigarrow$ Lyubashenko's TQFT [deRenzi-Gainutdinov-G-P-Runkel '19, '20]

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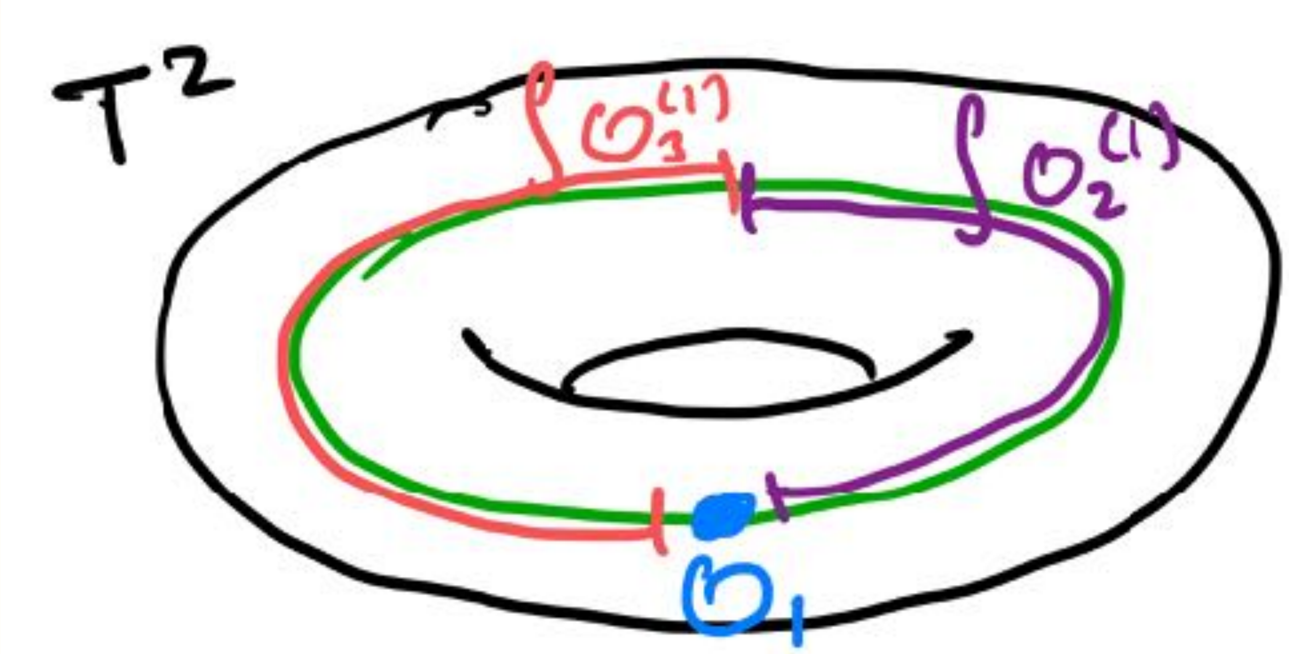

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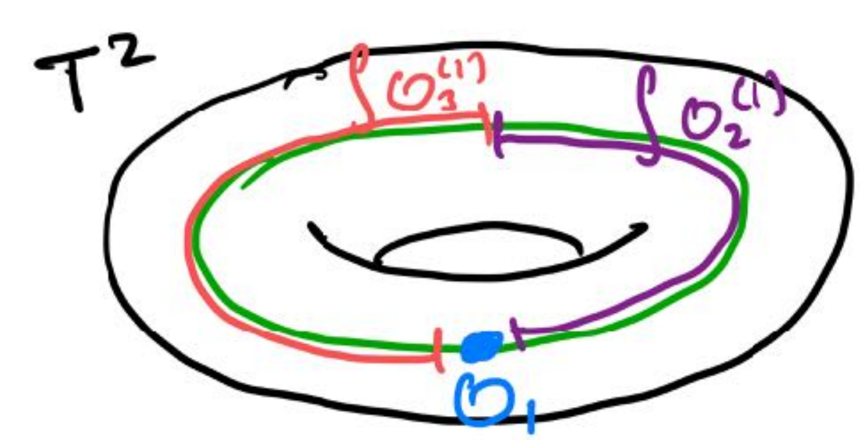

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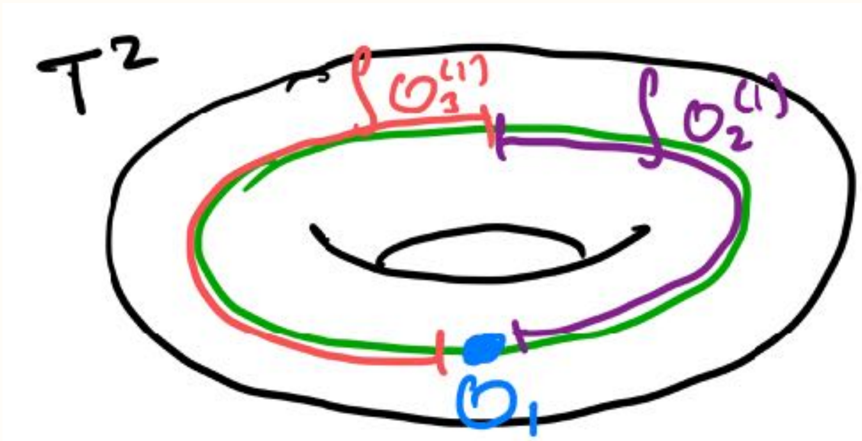

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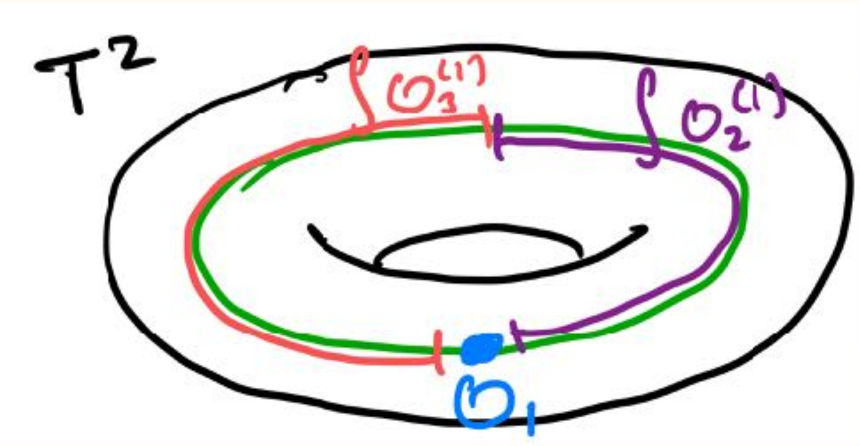

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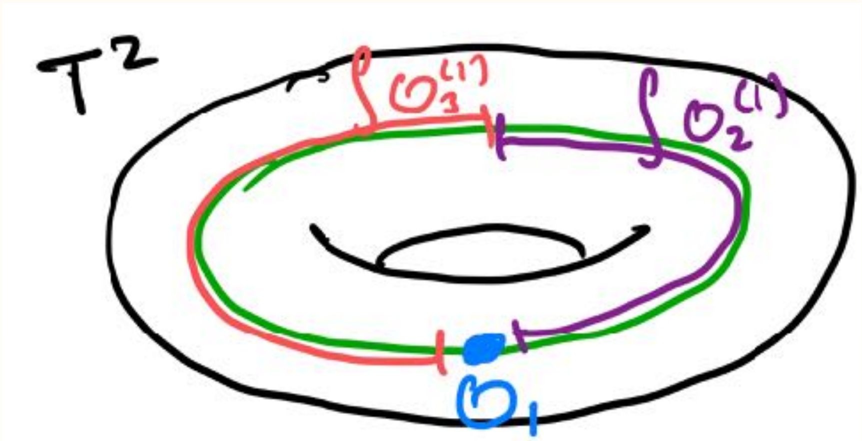

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s.t. $\chi\left[\mathcal{H}(T^2, 0)\right] = 2k = \dim \mathcal{H}(T^2, \mathcal{A}_{abel})$

$$K_0(\mathcal{C}_1) \simeq \mathbb{C}[X] \Big/ (X^2 - 4) \prod_{j=1}^{k-1} \left(X - 2\cos\frac{\pi j}{k}\right)^2 \qquad X = [V_2^+]$$

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We're looking for a topological QFT that...

- is labelled by $G,\ k \in \mathbb{Z}$
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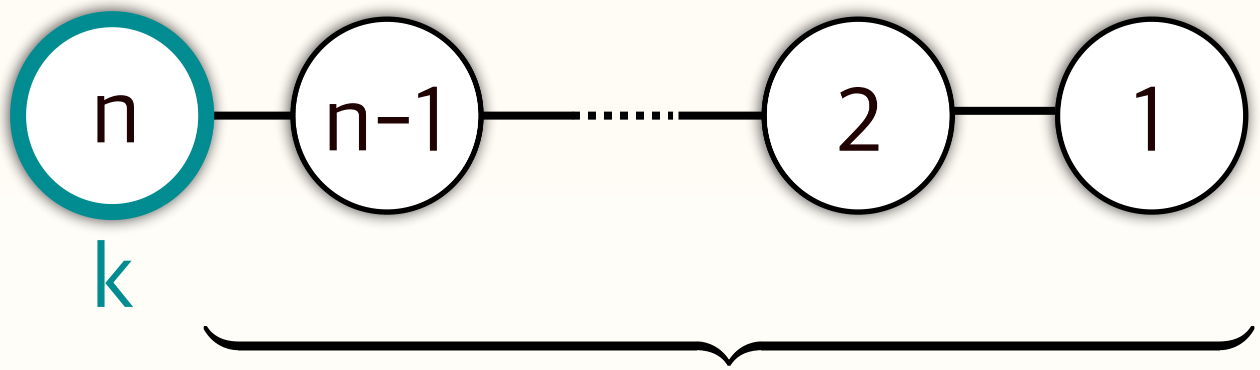

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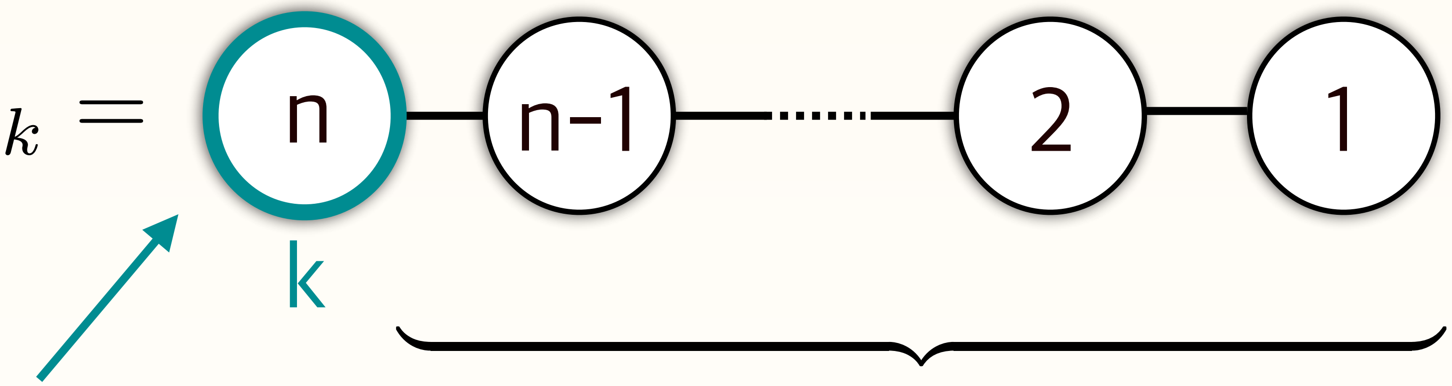

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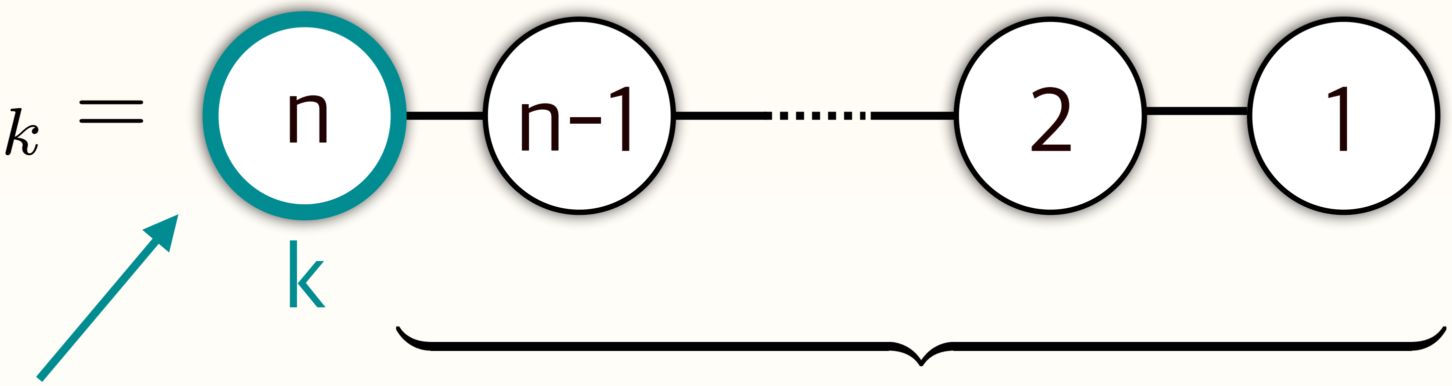

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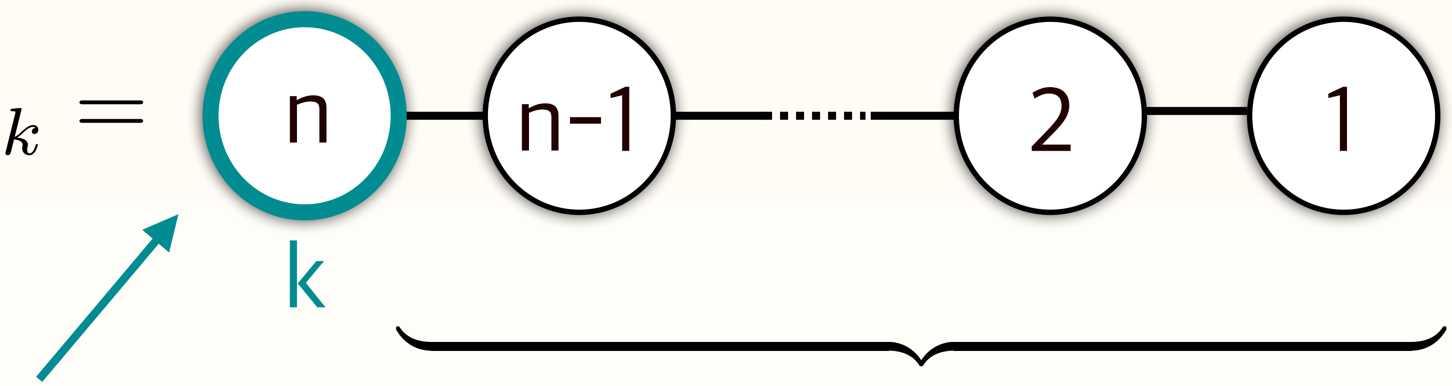

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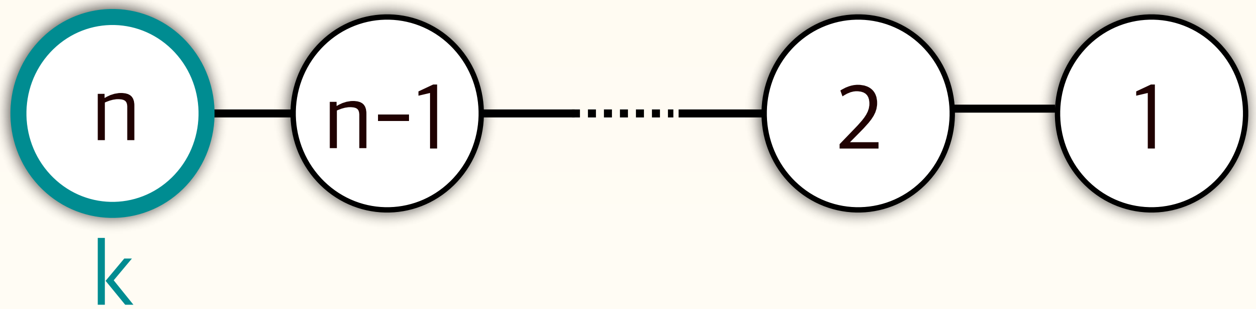

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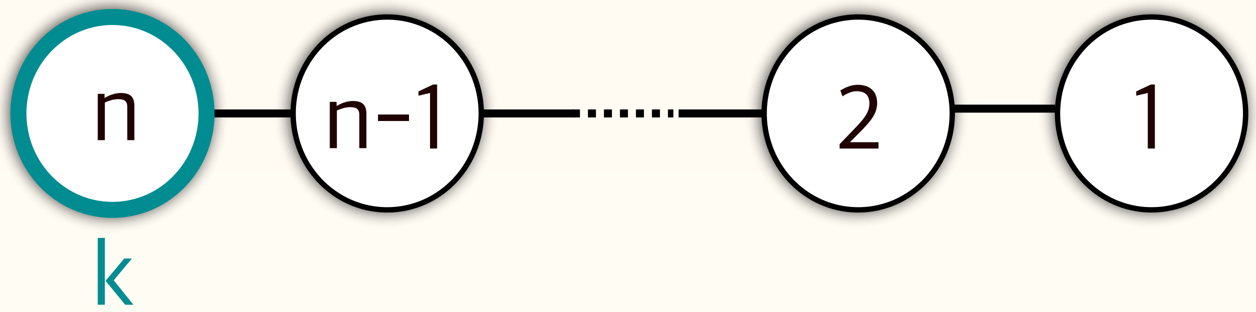
$\mathcal{T}_{n,k} =$ 

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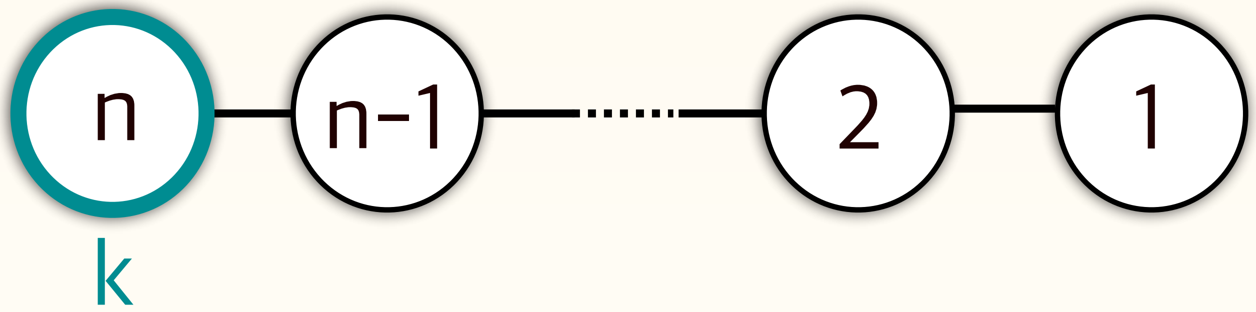

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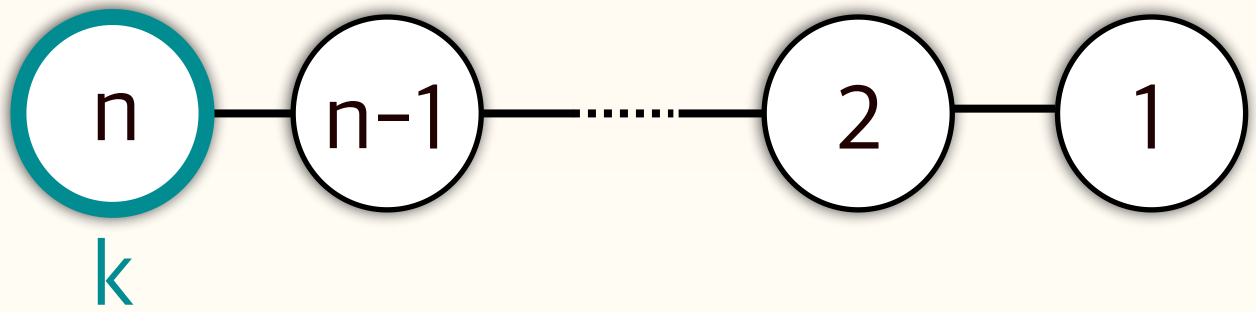

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extra structure: $\chi\big[\mathcal{H}(\Sigma_g)\big]$ as $G^\vee_{\mathbb{C}}$ characters

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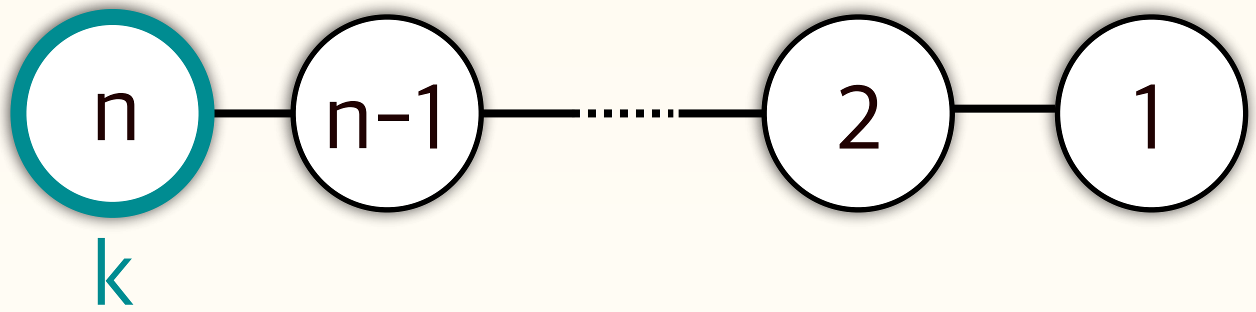

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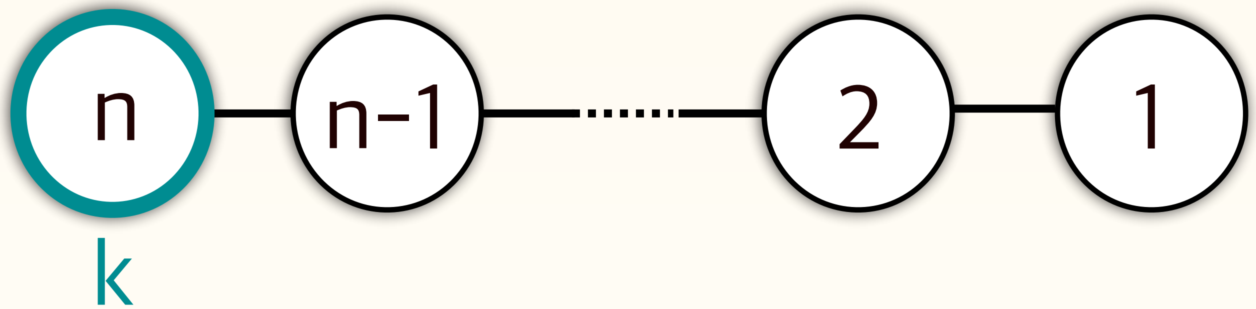

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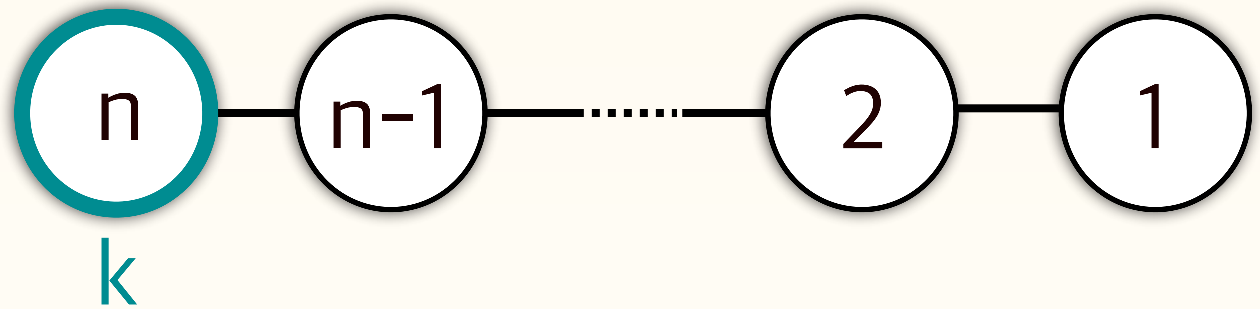

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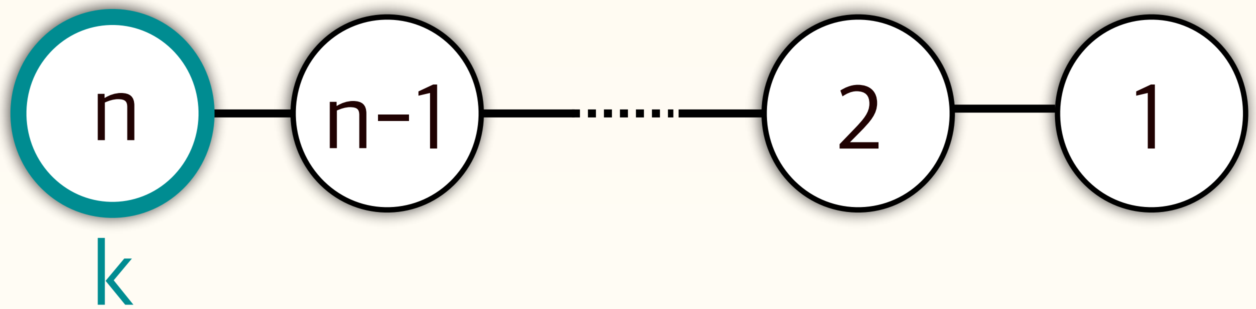


$$V = \prod_{i=1}^{n-1} \mathrm{Hom}(\mathbb{C}^i, \mathbb{C}^{i+1})$$

$$\Gamma = \prod_{i=1}^{n-1} GL(i)$$

More interesting: spaces, algebras, categories

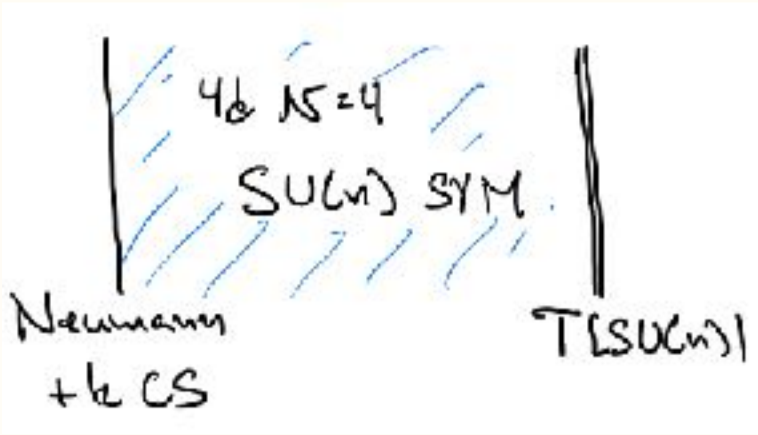
via (generalizations of) modern methods

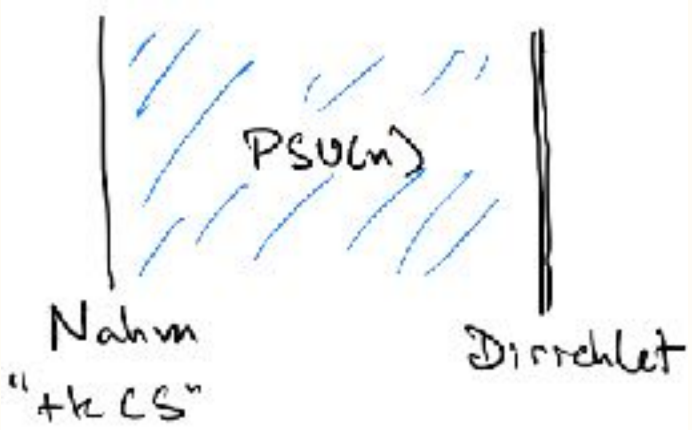
	$\mathrm{CS}_{n,k}$	$\mathcal{T}^A_{n,k}$ $\quad G = SL_n(\mathbb{C})$
$\mathcal{C}$	$\mathrm{Coh}_{LG_k}(pt)$	$\mathrm{Coh}_{LG_k}(T^*\widehat{L(V/\Gamma)})$
$\mathcal{H}(\Sigma)$	$H^{0,\bullet}_{\bar\partial}(\mathrm{Bun}_G(\Sigma), \mathcal{L}^k)$	$H^{0,\bullet}_{\bar\partial}(\mathrm{Bun}_G(\Sigma), \mathcal{L}^k \otimes \mathcal{E}_{T[G]})$
$\mathcal{H}(S^2)$	$\mathbb{C}$	$\mathbb{C}\left[T^*[2](G/B)\right]$
$\mathcal{H}(T^2)$	$\mathbb{C}^{d(n,k)}$	$\approx \mathbb{C}^{d(n,k)} \otimes H^{\bullet,\bullet}_{\bar\partial}(T^*[2](G/B))$

$d(n,k) \sim \frac{1}{n} k^n$

QFT $\mathcal{T}^A_{n,k} := \left(T[SU(n)]/SU(n)_k\right)^A$

4d sandwiches & brane webs [Hanany-Witten '96] [Aharony-Hanany-Kol '97] [Gaiotto-Witten '08] [Gaiotto-Rapcak '17]

$\mathcal{T}_{n,k} \simeq$ 

S-duality $\simeq$ 

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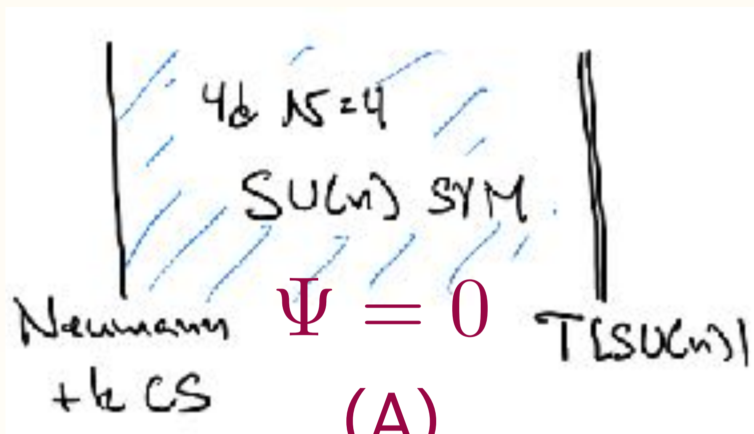
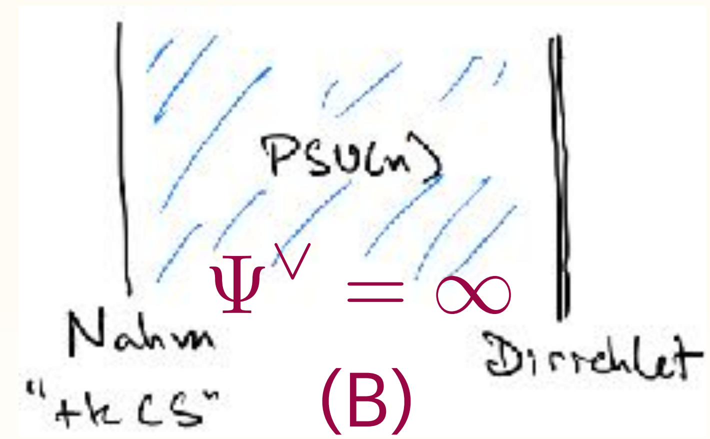


(A)

S-duality $\simeq$



(B)

[Kapustin-Witten '06]

- 4d A (resp. B) twist induces boundary A (resp. B) twist

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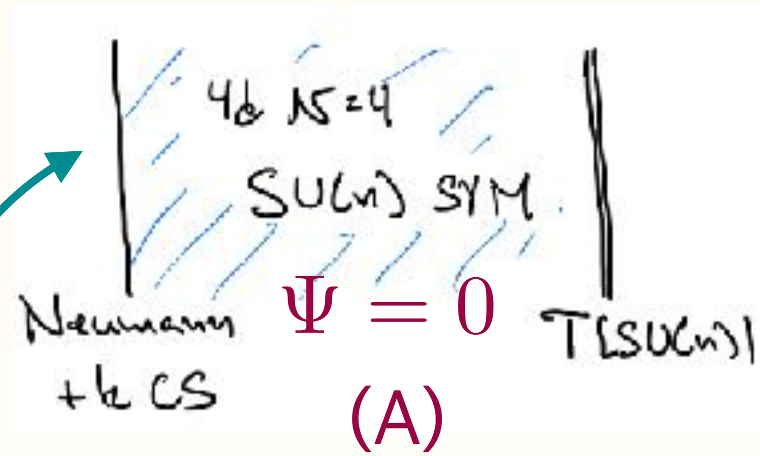

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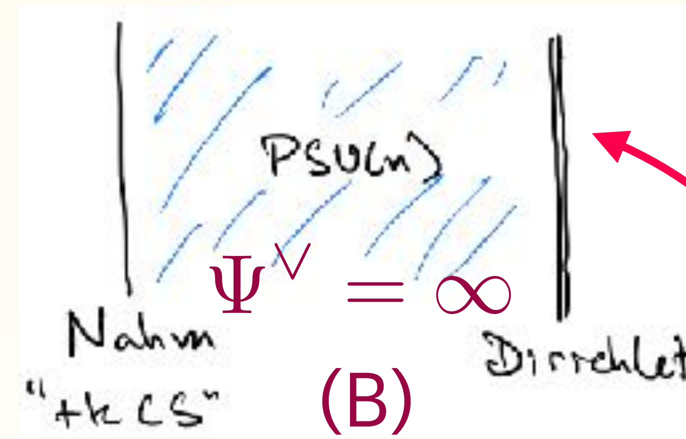

Wilson lines

flat $PGL_n(\mathbb{C})$ connection

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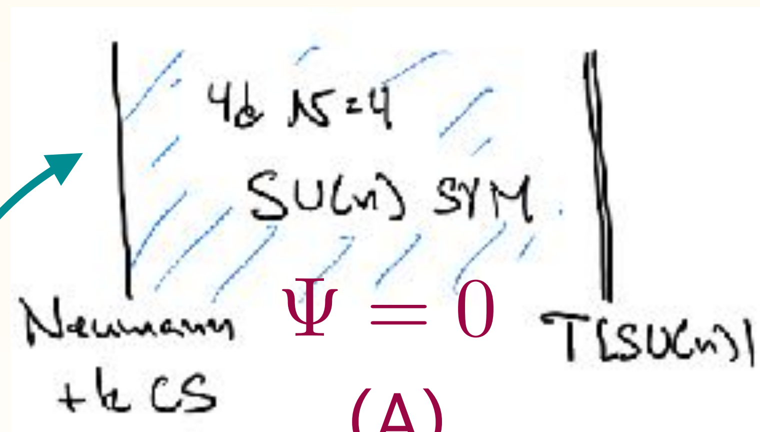
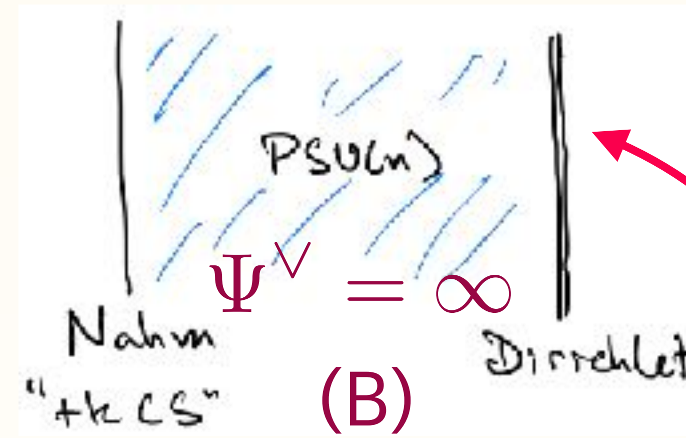


$\Psi = 0$

(A)

S-duality $\simeq$



$\Psi^{\vee} = \infty$

(B)

Wilson lines

flat $PGL_n(\mathbb{C})$ connection

[Kapustin-Witten '06]

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- expect S-dual (B) setup to reproduce RW theory of [GHNPPK '20] generic Ψ : B-twisted sigma-model to $T^*\mathrm{Gr}_G$

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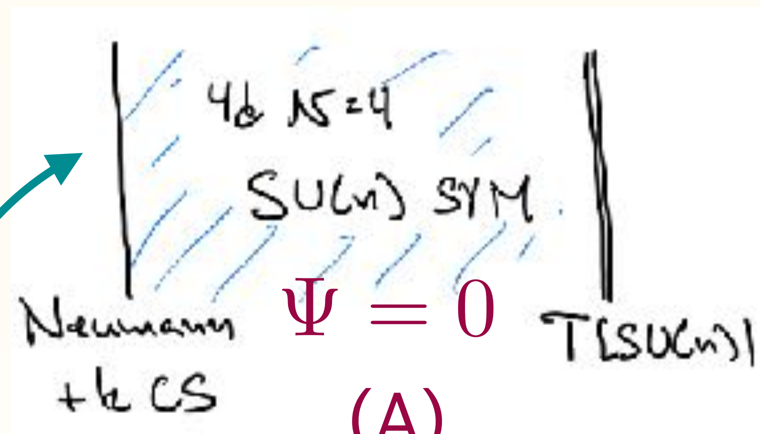

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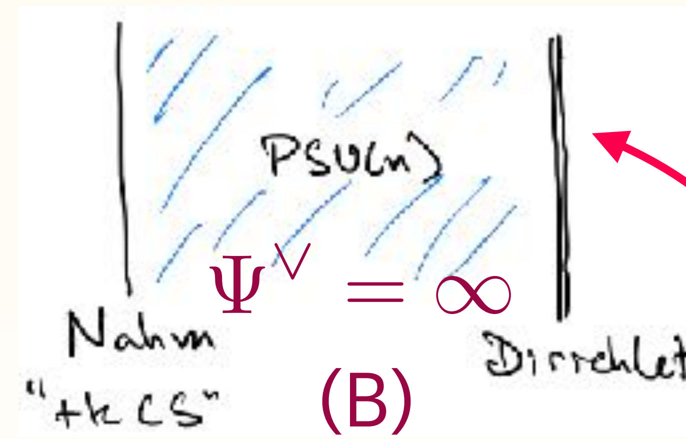

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- 4d A (resp. B) twist induces boundary A (resp. B) twist
- expect S-dual (B) setup to reproduce RW theory of [GHNPPK '20]
  generic Ψ : B-twisted sigma-model to $T^*\mathrm{Gr}_G$
  originates in 6d theory on $\mathbb{R}^3 \times D^2 \times S^1$ generic Ψ
  hopefully: categorification

  cf. [Witten '11], [Gukov-Pei-Putrov-Vafa '17]

VOA's, two ways

$3\text{d}\ \mathcal{N}=2$ realization of $\text{CS}_{n,k}$:

$\exists$ a pair of holomorphic boundary conditions supporting VOA's

$$\begin{array}{ccc} V^{k+n}(\mathfrak{sl}_n) & & V^{k+n}(\mathfrak{sl}_k) \\ & \searrow \quad \swarrow & \\ & \text{FF}^{\otimes nk} & \end{array}$$

mutual commutants (cosets) $\rightsquigarrow$ level-rank duality [Nakanishi-Tsuchiya '92]

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\mathcal{V}_{n,k} & & \mathcal{FT}_k(\mathfrak{sl}_n) \\
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$$

[Feigin-Tipunin '10]

Conjecture:

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Conjecture:

$$\mathcal{V}_{n,k} \searrow \quad \swarrow \mathcal{FT}_k(\mathfrak{sl}_n)$$
$$\mathrm{FF}^{\otimes nk}$$

[Feigin-Tipunin '10]

reminder: $\mathcal{FT}_k(\mathfrak{sl}_n)\text{-mod} \simeq u_q(\mathfrak{sl}_\mathfrak{n})\text{-mod} \overset{\text{want}}{\simeq} \mathcal{C}_{g=1}$

[Feigin-Gainutdinov-Semikhatov-Tipunin '05, '06] [Nagatomo-Tsuchiya '09]

... [Creutzig-Yang-McRae '20]

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$\mathcal{T}^A_{n,k}$:

$$\mathcal{V}_{n,k} \searrow \mathrm{FF}^{\otimes nk} \swarrow \mathcal{FT}_k(\mathfrak{sl}_n)$$

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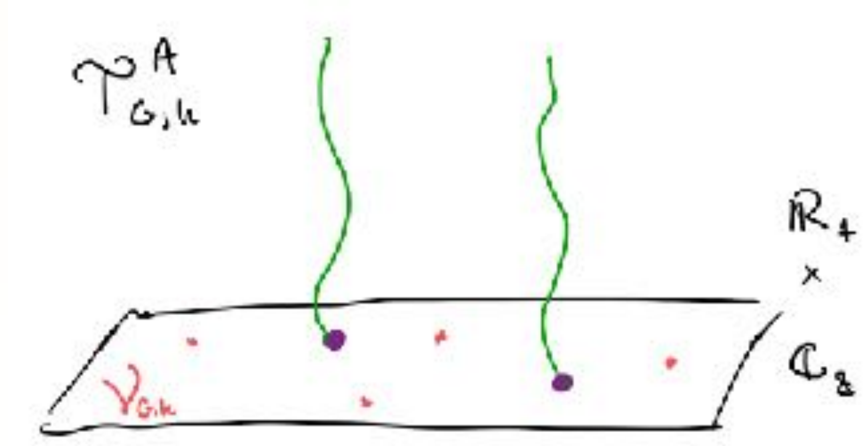
[Costello-Gaiotto '18] [TD-Costello-Gaiotto '20]



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$\mathcal{T}^A_{n,k}$:

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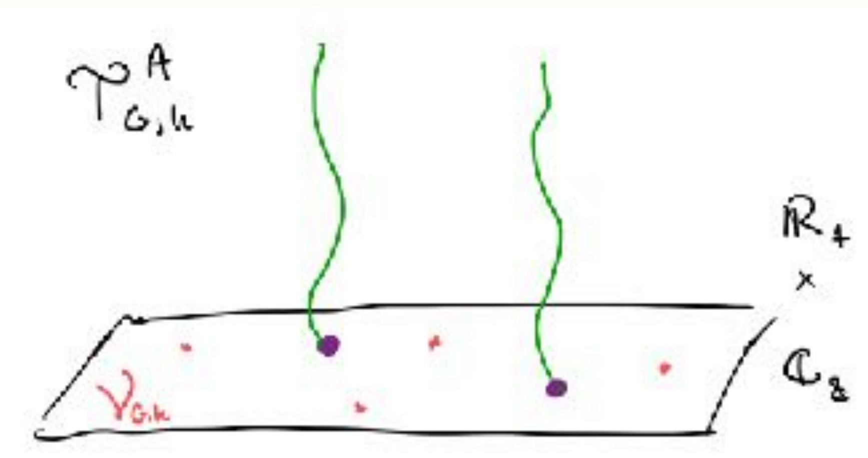
$$\mathcal{V}_{n,k} := \left(A[\mathfrak{sl}_n]\otimes \mathrm{FF}^{\otimes n(k-1)}\right)\Big/ V^k(\mathfrak{sl}_n)$$

$$A[\mathfrak{sl}_n] \simeq (\beta\gamma\otimes \mathrm{FF})[V] \underset{\mathrm{BRST}}{/\!/} \mathfrak{gl}_1\times\cdots\times\mathfrak{gl}_{n-1}$$

limit of "Langlands kernel" VOA [Creutzig-Gaiotto '17]

$A[\mathfrak{sl}_2] \simeq psu(2|2)_1$



VOA's, two ways

$\mathcal{T}^A_{n,k}$:

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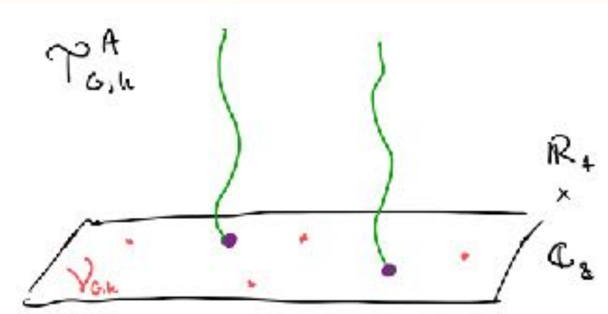
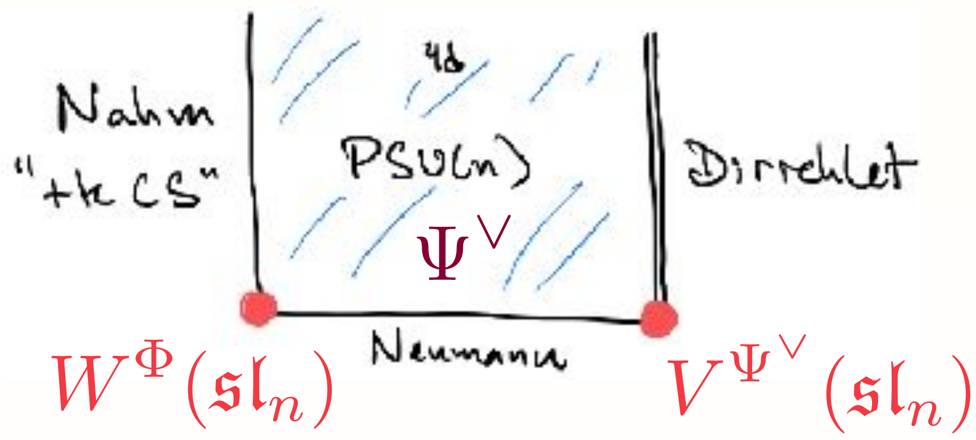


- $\mathcal{FT}_k(\mathfrak{sl}_n)$ is derived more readily from S-dual 4d sandwich

  collision of corner VOA's [Gaiotto-Rapcak '17] [Creutzig-Gaiotto'17]

  cf. [Frenkel-Gaiotto '18] [Feigin-Gukov '18]



VOA's, two ways

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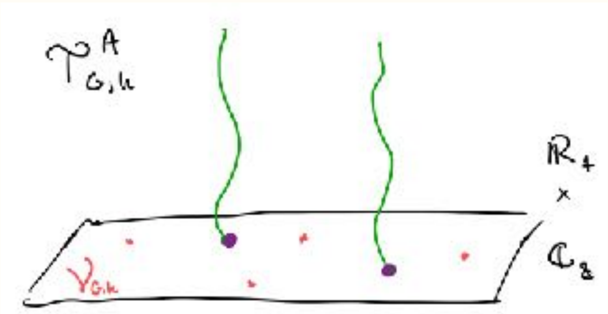

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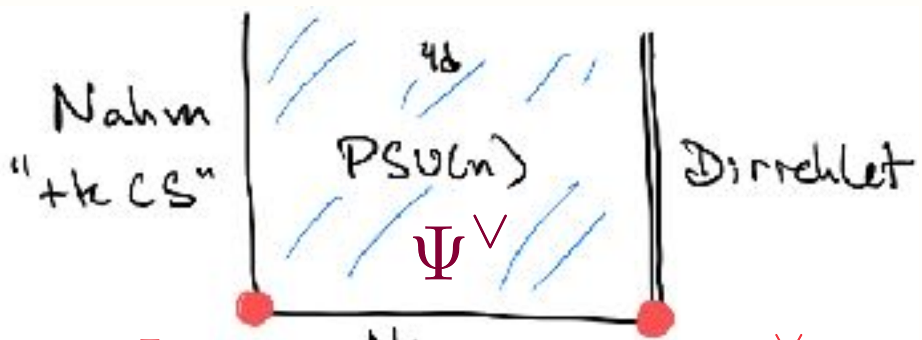

$W^\Phi(\mathfrak{sl}_n)$ $V^{\Psi^\vee}(\mathfrak{sl}_n)$

$$\frac{1}{\Phi} + \frac{1}{\Psi^\vee} = k$$

$$\mathrm{extension}\left[W^\Phi(\mathfrak{sl}_n) \otimes V^{\Psi^\vee}(\mathfrak{sl}_n)\right] \overset{\Psi^\vee \to \infty}{\rightsquigarrow} \mathcal{FT}_k(\mathfrak{sl}_n)$$

[Sugimoto '20]

VOA's, two ways

$\mathcal{T}^{A}_{n,k}$:

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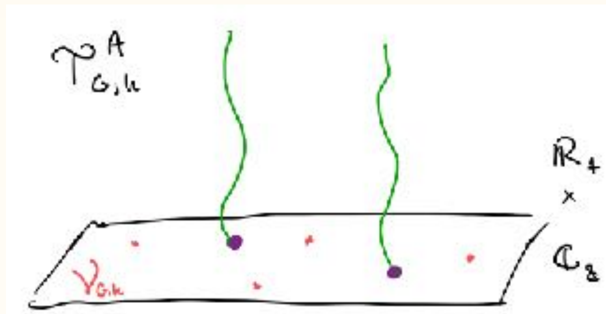

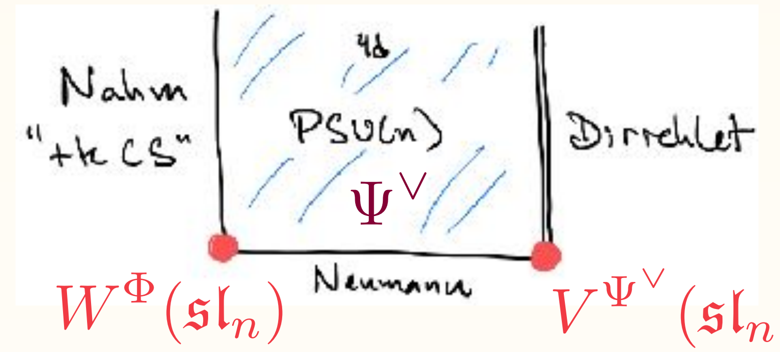

mutual commutants (conj.) => equivalent module cat's

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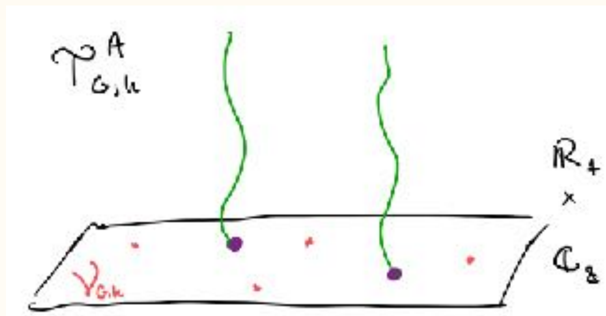

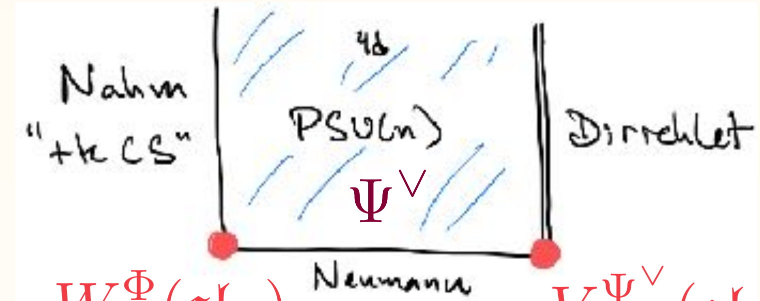

$W^{\Phi}(\mathfrak{sl}_n)$ $V^{\Psi^\vee}(\mathfrak{sl}_n)$

mutual commutants (conj.) => equivalent module cat's

Lots left to do

- S & T matrices (from QFT), general 3-manifold invariants
- derived TQFT cf. Kontsevich, Lurie,...

[Schweigert-Woike '19], [Kapustin-Rozansky-Saulina '08],...

Thank you!