Lipschitz bounds and non-uniform ellipticity

Lisa Beck

University of Augsburg

Workshop

“Nonlinear Potential Theoretic Methods in Partial Differential Equations”

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Plan of the talk

1. Introduction to the general topic
   - non-uniform ellipticity and relevance of Lipschitz regularity
   - model examples: (p, q)-growth and exponential growth functionals
2. Lipschitz regularity criteria for uniformly elliptic problems
   - Lipschitz regularity criteria for the Poisson equation
   - Extension to more general equations and minimization problems
3. Lipschitz regularity criteria for (p, q)-growth problems
   - Regularity results under smallness of the gap $q - p$
   - Strategy of proof – fake it till you make it

based on the joint paper

L. B., G. Mingione:
Lipschitz bounds and non-uniform ellipticity
Comm. Pure Appl. Math. 73 (2020)

Introduction
to the general topic

The variational setting

We study the minimization of variational integrals of the form

$$w \mapsto \int_\Omega \left[F(\mathrm{D}w) - f \cdot w\right] \mathrm{d}x$$

among all functions $w \in \mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$, with $\Omega \subset \mathbb{R}^n$ open and $n \geq 2$, $N \geq 1$.

Aim: Identify sharp conditions on the datum f which guarantee local Lipschitz continuity of minimizers for integrands $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfying suitable convexity and growth assumptions, in particular in non-uniformly elliptic settings.

- The Euler–Lagrange equation reads formally as $-\operatorname{div} \partial F(\mathrm{D}u) = f$ and non-uniform ellipticity refers to the situation when the ellipticity ratio

$$\mathcal{R}(z) := \frac{\text{largest eigenvalue of } \partial^2 F(z)}{\text{smallest eigenvalue of } \partial^2 F(z)}$$

  is unbounded for $|z| \to \infty$.
- Once Lipschitz continuity of minimizers is known, then the equation becomes uniformly elliptic at infinity and classical methods apply.

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- In the vectorial case $N > 1$ there are many examples of elliptic systems of variational and non-variational type, which admit irregular (discontinuous) solutions.

  [De Giorgi 1968, Giusti & Miranda 1968, Frehse 1973, Nečas 1975, Šverák & Yan 2000, . . .]

  and here one usually supposes radial structure $F(z) = \widetilde{F}(|z|)$ which is known to rule out singularities, e.g. for the p-Laplacean system.

  [Uhlenbeck 1977]

Model examples of non-uniform ellipticity

Polynomial, but unbalanced growth conditions:

This is the case for anisotropic variational functionals (used e.g. for the modeling of electrorheological fluids), where

$$F(z) = |z|^p + \sum_{i=1}^{n} |z_i|^{q_i} \quad \text{with } p, q_1, \dots, q_n > 1\,,$$

or for perturbations of standard p-growth variational functionals, where

$$F(z) = |z|^p \log(\mathrm{e} + |z|) \quad \text{with } p > 1\,.$$

These are special examples of integrands satisfying a (p, q)-growth condition

$$\begin{cases} |z|^p \lesssim F(z) \lesssim |z|^q + 1 \\ |z|^{p-2}|\xi|^2 \lesssim \partial^2 F(z)\xi \cdot \xi \quad \text{and} \quad |\partial^2 F(z)| \lesssim |z|^{q-2} + 1 \end{cases}$$

meaning that the ellipticity ratio $\mathcal{R}(z) \lesssim |z|^{q-p} + 1$ becomes unbounded as $|z| \to \infty$ with a speed which is proportional to the gap $q - p$.

(investigated extensively starting from the papers [Marcellini 1989 & 1991] for $f \equiv 0$)

Model examples of non-uniform ellipticity

Very fast growth conditions:

This happens for instance for variational functionals of exponential growth, e.g.

$$F(z) = \exp(|z|^p) \quad \text{with } p > 1$$

or the iterated versions

$$F(z) = \exp(\exp(\dots \exp(|z|^p) \dots)) \quad \text{with } p > 1\,.$$

These can be formulated as growth conditions of the form

$$g_1(|z|)|\xi|^2 \lesssim \partial^2 F(z)\xi \cdot \xi \quad \text{and} \quad |\partial^2 F(z)| \lesssim g_2(|z|)$$

with functions g_1, g_2, and in the above examples, the ellipticity ratio $\mathcal{R}(z) = g_2(|z|)/g_1(|z|)$ is bounded by a polynomial function or a lower order exponential function.

(research initiated starting from the papers [Duc & Eells 1991, Lieberman 1992, Marcellini 1996] for $f \equiv 0$)

Lipschitz regularity criteria
for uniformly elliptic problems

Poisson equation – classical regularity criteria

A prototypic example of a uniformly elliptic equation is the Poisson equation

$$-\Delta u = f \qquad \text{in } \Omega \subset \mathbb{R}^n \text{ for } n \geq 3\,,$$

with a source term $f\colon \Omega \to \mathbb{R}$, which is the Euler–Lagrange equation of the variational integral

$$w \mapsto \int_\Omega \big[|\mathrm{D}w|^2 - fw\big]\,\mathrm{d}x\,.$$

Regularity criteria for weak solutions $u \in \mathrm{W}^{1,2}(\Omega)$:

- Schauder theory:

  Natural candidate for a solution is the Newton-potential

  $$N_f(x_0) := \int_\Omega \Phi(x_0 - y) f(y) \mathrm{d}y$$

  with $\Phi(x) \sim |x|^{2-n}$ the fundamental solution to the Laplace equation, and we have

  - $f \in \mathrm{L}^\infty \Rightarrow N_f \in C^{1,\beta}$ for all $\beta \in [0,1)$
  - $f \in \mathrm{C}^{0,\alpha} \Rightarrow N_f \in \mathrm{C}^{2,\alpha}$ for $\alpha \in (0,1)$ and solution property (but $N_f \notin \mathrm{C}^2$ is possible for $f \in \mathrm{C}^0$ only!)

[Hölder 1882]

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- L^p theory:
  - For solutions $v \in \mathrm{W}^{1,2}_0(\Omega)$ to the Poisson equation $\Delta v = \operatorname{div} g$ with data in divergence form, we have

    $$\|\mathrm{D}v\|_{L^2} \lesssim \|g\|_{L^2} \quad \text{and} \quad \|\mathrm{D}v\|_{\mathrm{BMO}} \lesssim \|g\|_{L^\infty}\,,$$

    and an interpolation argument shows

    $$\|\mathrm{D}v\|_{L^p} \lesssim \|g\|_{L^p}\,.$$

  - Passage to the differentiated equation

    $$\Delta \mathrm{D}_k u = \mathrm{D}_k f$$

    and a localization argument then show $f \in \mathrm{L}^p \Rightarrow u \in \mathrm{W}^{2,p}_{\mathrm{loc}}$.

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  in particular: $f \in \mathrm{L}^{n+\varepsilon} \Rightarrow u \in \mathrm{C}^1$ for each $\varepsilon > 0$

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  in particular: $f \in \mathrm{L}^{n+\varepsilon} \Rightarrow u \in \mathrm{C}^1$ for each $\varepsilon > 0$
- Riesz potential theory:
  - Again by representation via the fundamental solution

$$|N_f(x_0)| \lesssim \int_{\mathbb{R}^n} |x_0 - y|^{2-n} |f(y)|\,\mathrm{d}y =: \mathrm{I}_2^f(x_0)$$

  - and after differentiation

$$|\mathrm{D}N_f(x_0)| \lesssim \int_{\mathbb{R}^n} |x_0 - y|^{1-n} |f(y)|\,\mathrm{d}y =: \mathrm{I}_1^f(x_0)$$

[Riesz 1949]

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- Riesz potential theory: $\mathbb{I}_1^f \in \mathrm{L}^\infty \Rightarrow u \in \mathrm{C}^1$

  By considering truncated Riesz potentials, we also obtain local Lipschitz estimates

$$|\mathrm{D}u(x_0)| \lesssim \fint_{B_R(x_0)} |\mathrm{D}u(x)|\,\mathrm{d}x + \underbrace{\int_0^R \rho^{1-n} \int_{B_\rho(x_0)} |f(y)|\,\mathrm{d}y \,\frac{\mathrm{d}\rho}{\rho}}_{=:\mathbb{I}_1^f(x_0,R)}$$

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  in particular: $f \in \mathrm{L}^{n+\varepsilon} \Rightarrow u \in \mathrm{C}^1$ for each $\varepsilon > 0$
- Riesz potential theory: $\mathbb{I}_1^f \in \mathrm{L}^\infty \Rightarrow u \in \mathrm{C}^1$
- Sharp Lipschitz regularity criterion: $f \in \mathrm{L}^{n,1} \Rightarrow u \in \mathrm{C}^1$

[Stein 1981, Cianchi 1992]

Intermezzo on Lorentz spaces

The Lorentz spaces $\mathrm{L}^{p,\gamma}$ are refinements of the classical Lebesgue spaces, with quasi-norm

$$\|f\|_{\mathrm{L}^{p,\gamma}} = \left(p \int_0^\infty \left(\lambda^p |\{y \in \Omega \colon |f(y)| > \lambda\}|\right)^{\frac{\gamma}{p}} \frac{\mathrm{d}\lambda}{\lambda} \right)^{\frac{1}{\gamma}},$$

with the properties

- $\mathrm{L}^{p,p} = \mathrm{L}^p$ (Cavalieri principle)
- $\mathrm{L}^{p,\gamma} \subset \mathrm{L}^{p,\gamma'}$ for $\gamma \leq \gamma'$
- $\mathrm{L}^q \subset \mathrm{L}^{p,1} \subset \mathrm{L}^p$ for all $q > p$

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For the critical space $\mathrm{L}^{n,1}$ in our setting we have $\mathrm{L}^{n+\varepsilon} \subset \mathrm{L}^{n,1} \subset \mathrm{L}^n$ for each $\varepsilon > 0$ and a connection to the truncated Riesz potential and a nonlinear Riesz potential via the estimate

$$\mathbb{I}_1^f(x_0, R) \lesssim \mathbb{P}^f(x_0, R) := \int_0^R \left(\rho^2 \fint_{B_\rho(x_0)} |f(y)|^2 \,\mathrm{d}y\right)^{1/2} \frac{\mathrm{d}\rho}{\rho} \lesssim \|f\|_{\mathrm{L}^{n,1}}$$

An example of a nontrivial function $f \in \mathrm{L}^{n,1} \setminus \mathrm{L}^{n+\varepsilon}$ is given by

$$f(x) = \frac{1}{|x| \log^\beta(1/|x|)} \quad \text{for } \beta > 1\,.$$

Extension of the Lipschitz regularity criterion

Uniformly elliptic problems of p-growth:

We now pass to quasi-linear equations

$$-\operatorname{div}(a(\mathrm{D}u)) = f$$

or the minimization of the variational integral

$$w \mapsto \int_\Omega [F(\mathrm{D}w) - fw]\,\mathrm{d}x$$

among all functions $w \in \mathrm{W}^{1,p}(\Omega)$, under suitable uniform p-growth and ellipticity / convexity assumptions with $p \geq 2$. The prototypic example here is the p-Laplace equation with $a(z) = |z|^{p-2}z$ and $F(z) = |z|^p$.

Surprisingly, for solutions $u \in \mathrm{W}^{1,p}(\Omega)$ very similar estimates (in Lebesgue points of $\mathrm{D}u$) are available as in the linear case:

$$|\mathrm{D}u(x_0)|^{p-1} \lesssim \left(\fint_{B_R(x_0)} |\mathrm{D}u(x)|\,\mathrm{d}x \right)^{p-1} + \mathbb{I}_1^f(x_0, R) + 1$$

[Kuusi & Mingione 2013]

Extension of the Lipschitz regularity criterion

Uniformly elliptic problems of p-growth:

For solutions $u \in \mathrm{W}^{1,p}(\Omega)$ to $-\operatorname{div}(|\mathrm{D}u|^{p-2}\mathrm{D}u) = f$ very similar estimates are available as in the linear case:

$$|\mathrm{D}u(x_0)|^{p-1} \lesssim \left(\fint_{B_R(x_0)} |\mathrm{D}u(x)| \,\mathrm{d}x \right)^{p-1} + \mathbb{I}_1^f(x_0, R)$$

[Kuusi & Mingione 2013]

Notice:

- We still have $f \in \mathrm{L}^{n,1} \Rightarrow \mathbb{I}_1^f(\cdot, R) \in \mathrm{L}^\infty_{\mathrm{loc}} \Rightarrow u \in \mathrm{W}^{1,\infty}_{\mathrm{loc}}$, independently from $p \in (1, \infty)$ (or more generally, of the equation or variational functional considered).
- For $f \equiv 0$, this is the classical L^∞-L^p-estimate for p-harmonic functions.

  [Manfredi 1988]
- From a pointwise version, local higher integrability or gradient estimates are recovered.

  [Iwaniec 1983, Di Benedetto & Manfredi 1993, . . .]
- The result extends to the vectorial case.

  [Kuusi & Mingione 2018]

Extension of the Lipschitz regularity criterion

Uniformly elliptic problems of p-growth:

For solutions $u \in \mathrm{W}^{1,p}(\Omega)$ to $-\operatorname{div}(|\mathrm{D}u|^{p-2}\mathrm{D}u) = f$ very similar estimates are available as in the linear case:

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[Kuusi & Mingione 2013]

Some heuristic explanation of the linearization phenomenon:

- Though the equation is non-linear in the solution u, it is linear in some intrinsic quantity

$$\operatorname{div} v = f \qquad \text{for} \qquad v = |\mathrm{D}u|^{p-2}\mathrm{D}u\,.$$

- Thus, we might expect $\mathbb{I}_1^f(\cdot,R) \in \mathrm{L}^\infty_{\mathrm{loc}} \Rightarrow v \in \mathrm{L}^\infty_{\mathrm{loc}}$ with pointwise estimate

$$|v(x_0)| \lesssim \fint_{B_R(x_0)} |v(x)|\,\mathrm{d}x + \mathbb{I}_1^f(x_0,R)$$

which leads to the above estimate when setting $|v| = |\mathrm{D}u|^{p-1}$ (and using a reverse Hölder inequality).

Extension of the Lipschitz regularity criterion

Uniformly elliptic problems of p-growth:

For solutions $u \in \mathrm{W}^{1,p}(\Omega)$ to $-\operatorname{div}(|\mathrm{D}u|^{p-2}\mathrm{D}u) = f$ very similar estimates are available as in the linear case:

$$|\mathrm{D}u(x_0)|^{p-1} \lesssim \left(\fint_{B_R(x_0)} |\mathrm{D}u(x)| \,\mathrm{d}x \right)^{p-1} + \mathbb{I}_1^f(x_0, R)$$

[Kuusi & Mingione 2013]

More general uniformly elliptic problems of general growth:

Similarly, for solutions $u \in \mathrm{W}^{1,1}(\Omega)$ to $-\operatorname{div}(g(|\mathrm{D}u|)\mathrm{D}u/|\mathrm{D}u|) = f$ where g is a positive function of class C^1 satisfying the uniform ellipticity condition

$$1 \le \nu \le \frac{g'(t)t}{g(t)} \le L$$

plus non-degeneracy-conditions $\lim_{t \searrow 0} g(t)/t = 0$ and $\lim_{t \to \infty} g(t)/t = \infty$, one has (in Lebesgue points of $\mathrm{D}u$) the intrinsic estimate

$$g(|\mathrm{D}u(x_0)|) \lesssim g\left(\fint_{B_R(x_0)} |\mathrm{D}u(x)| \,\mathrm{d}x \right) + \mathbb{I}_1^f(x_0, R)\,.$$

[Baroni 2015]

Lipschitz regularity criteria for (p, q)-growth problems

Classical existence and regularity results

We first consider the minimization of the variational integral

$$w \mapsto \int_\Omega F(\mathrm{D}w)\,\mathrm{d}x$$

among all functions $w \in \mathrm{W}^{1,1}(\Omega)$ (with given boundary values) without source term f, for an integrand F satisfying a (p,q)-growth condition (for $1 < p \le q$)

$$\begin{cases} |z|^p \lesssim F(z) \lesssim |z|^q + 1 \\ |z|^{p-2}|\xi|^2 \lesssim \partial^2 F(z)\xi\cdot\xi \quad \text{and} \quad |\partial^2 F(z)| \lesssim |z|^{q-2} + 1 \end{cases}$$

- Existence of minimizers: follows by the direct method (minimizing sequences are bounded in $\mathrm{W}^{1,p}(\Omega)$).

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- Existence of minimizers: follows by the direct method (minimizing sequences are bounded in $\mathrm{W}^{1,p}(\Omega)$).
- Existence of irregular minimizers: a condition on the gap $q/p \leq 1 + o(n)$ is necessary for boundedness/regularity.

[Marcellini 1987 & 1991, Giaquinta 1987, Hong 1992]

For the variational integral with integrand

$$F(z) = \frac{1}{2}\sum_{i=1}^{n-1} |z_i|^2 + \frac{1}{q}|z_n|^q \qquad \text{with } q > 2\,\frac{n-1}{n-3}\,,$$

there exists an unbounded minimizer with a discontinuity along a line!

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- Regularity of minimizers: the condition $q/p \leq 1 + o(n)$ is also sufficient for $\mathrm{W}^{1,\infty}$-regularity of solutions.

A lot of results, starting from

- $u \in \mathrm{W}^{1,q} \Rightarrow u \in \mathrm{W}^{1,\infty}$ if $q/p < n/(n-2)$ [Marcellini 1991]
- $u \in \mathrm{W}^{1,p} \Rightarrow u \in \mathrm{W}^{1,q}$ if $q/p < 1 + 2/n < n/(n-2)$ [Esposito, Leonetti & Mingione 1999]

with local Lipschitz estimate reflecting the unbalanced polynomial growth

$$\|\mathrm{D}u\|_{\mathrm{L}^\infty(B_R(x_0))} \lesssim \left(\fint_{B_{2R}(x_0)} F(\mathrm{D}u)\,\mathrm{d}x \right)^{\frac{2}{(n+2)p-nq}} + 1$$

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$$\begin{cases} |z|^p \lesssim F(z) \lesssim |z|^q + 1 \\ |z|^{p-2}|\xi|^2 \lesssim \partial^2 F(z)\xi \cdot \xi \quad \text{and} \quad |\partial^2 F(z)| \lesssim |z|^{q-2} + 1 \end{cases}$$

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    [Marcellini 1991]
  - $u \in \mathrm{W}^{1,p} \Rightarrow u \in \mathrm{W}^{1,q}$ if $q/p < 1 + 2/n < n/(n-2)$

    [Esposito, Leonetti & Mingione 1999]

  This bound was recently improved in this general setting to $q/p < 1 + 2/(n-1)$.

  [Bella & Schäffner 2019]

A Caccioppoli inequality for an intrinsic quantity

We next include a source term and consider minimizers of the variational integral $w \mapsto \int_\Omega \big[F(\mathrm{D}w) - fw\big]\,\mathrm{d}x$ under (p,q)-growth assumptions (for $2 \le p \le q$)

$$\begin{cases} |z|^p \lesssim F(z) \lesssim |z|^q + 1 \\ |z|^{p-2}|\xi|^2 \lesssim \partial^2 F(z)\xi\cdot\xi \quad \text{and} \quad |\partial^2 F(z)| \lesssim |z|^{q-2} + 1 \end{cases}$$

Theorem (Caccioppoli inequality)

Consider a ball B_r in $\mathbb{R}^n$, $f \in \mathrm{L}^2(B_r)$, and a minimizer $u \in \mathrm{W}^{1,\infty}(B_r)$. Then, for each $k \in \mathbb{R}^+$, we have

$$\int_{B_{r/2}} \big|\mathrm{D}(|\mathrm{D}u|^p - k)_+\big|^2\,\mathrm{d}x \lesssim r^{-2}\int_{B_r} |\mathrm{D}u|^{q-p}(|\mathrm{D}u|^p - k)_+^2\,\mathrm{d}x + \int_{B_r} |\mathrm{D}u|^2|f|^2\,\mathrm{d}x$$

(*independently of r*).

($\mathrm{W}^{1,\infty}$-regularity to be justified in the end by approximation)

Strategy of proof:

1. Pass to the Euler–Lagrange equation

$$\int_{B_r} \partial F(\mathrm{D}u) \cdot \mathrm{D}\varphi \,\mathrm{d}x = \int_{B_r} f\varphi \,\mathrm{d}x \quad \text{for all } \varphi \in \mathrm{C}^1_0(B_r)\,.$$

2. Due to the $\mathrm{W}^{1,\infty}$ assumption, standard regularity theory applies and we have

$$u \in \mathrm{W}^{2,2}_{\mathrm{loc}}(B_r)\,.$$

3. Pass to the differentiated Euler–Lagrange equation ($\varphi \to \mathrm{D}_s\varphi$)

$$\int_{B_r} \partial^2 F(\mathrm{D}u)\mathrm{D}\mathrm{D}_s u \cdot \mathrm{D}\varphi \,\mathrm{d}x = -\int_{B_r} f\mathrm{D}_s\varphi \,\mathrm{d}x \quad \text{for all } \varphi \in \mathrm{C}^1_0(B_r)\,.$$

4. Use $\varphi = \mathrm{D}_s u(|\mathrm{D}u|^p - k)_+\eta^2$, with a localization function $\eta \in \mathrm{C}^1_0(B_r;[0,1])$ (which is then chosen with $\eta \equiv 1$ in $B_{r/2}$ and $|\mathrm{D}\eta| \lesssim r^{-1}$) and employ the growth assumptions on $\partial^2 F$.

The Caccioppoli inequality implies local Lipschitz regularity

So far ... a minimizer $u \in \mathrm{W}^{1,\infty}(B_r)$ satisfies

$$\int_{B_{r/2}} |\mathrm{D}(|\mathrm{D}u|^p - k)_+|^2 \, \mathrm{d}x \lesssim r^{-2} \int_{B_r} |\mathrm{D}u|^{q-p}(|\mathrm{D}u|^p - k)_+^2 \, \mathrm{d}x + \int_{B_r} |\mathrm{D}u|^2 |f|^2 \, \mathrm{d}x$$

The Caccioppoli inequality implies local Lipschitz regularity

So far ... a minimizer $u \in \mathrm{W}^{1,\infty}(B_r)$ satisfies for all $k \geq 1$ and $r \leq R$

$$\begin{aligned} &\int_{B_{r/2}} |\mathrm{D}(|\mathrm{D}u|^p - k)_+|^2 \,\mathrm{d}x \\ &\quad\lesssim r^{-2} \int_{B_r} |\mathrm{D}u|^{q-p}(|\mathrm{D}u|^p - k)_+^2 \,\mathrm{d}x + \int_{B_r} |\mathrm{D}u|^2 |f|^2 \,\mathrm{d}x \\ &\quad\lesssim \|\mathrm{D}u\|_{\mathrm{L}^\infty(B_R)}^{q-p} r^{-2} \int_{B_r} (|\mathrm{D}u|^p - k)_+^2 \,\mathrm{d}x + \|\mathrm{D}u\|_{\mathrm{L}^\infty(B_R)}^2 \int_{B_r} |f|^2 \,\mathrm{d}x\,, \end{aligned}$$

a Caccioppoli inequality of balanced growth, but involving constants in terms of $\|\mathrm{D}u\|_{\mathrm{L}^\infty}$.

The Caccioppoli inequality implies local Lipschitz regularity

So far ... a minimizer $u \in \mathrm{W}^{1,\infty}(B_r)$ satisfies

$$\begin{aligned}\int_{B_{r/2}} |\mathrm{D}(|\mathrm{D}u|^p - k)_+|^2 \,\mathrm{d}x &\lesssim r^{-2} \int_{B_r} |\mathrm{D}u|^{q-p}(|\mathrm{D}u|^p - k)_+^2 \,\mathrm{d}x + \int_{B_r} |\mathrm{D}u|^2 |f|^2 \,\mathrm{d}x \\ &\lesssim \|\mathrm{D}u\|_{\mathrm{L}^\infty(B_R)}^{q-p} r^{-2} \int_{B_r} (|\mathrm{D}u|^p - k)_+^2 \,\mathrm{d}x + \|\mathrm{D}u\|_{\mathrm{L}^\infty(B_R)}^2 \int_{B_r} |f|^2 \,\mathrm{d}x \,,\end{aligned}$$

a Caccioppoli inequality of balanced growth, but involving constants in terms of $\|\mathrm{D}u\|_{\mathrm{L}^\infty}$.

De Giorgi iteration à la [Kilpeläinen & Malý 1994]:
Consider a ball $B_R = B_R(x_0)$ in $\mathbb{R}^n$, $f \in \mathrm{L}^2(B_R)$, and a function $v \in \mathrm{W}^{1,2}(B_R)$, which satisfies

$$\int_{B_{r/2}} |\mathrm{D}(v-k)_+|^2 \,\mathrm{d}x \lesssim M_1^2 r^{-2} \int_{B_r} (v-k)_+^2 \,\mathrm{d}x + M_2^2 \int_{B_r} |f|^2 \,\mathrm{d}x$$

for all $k \geq k_0$ and $r \leq R$. Then we have

$$v(x_0)_+ - k_0 \lesssim M_1^{\frac{n}{2}} \Big(\fint_{B_R} v_+^2 \,\mathrm{d}x \Big)^{\frac{1}{2}} + M_1^{\frac{n-2}{2}} M_2 \mathbb{P}^f(x_0, R)$$

The Caccioppoli inequality implies local Lipschitz regularity

So far ... a minimizer $u \in \mathrm{W}^{1,\infty}(B_r)$ satisfies

$$\int_{B_{r/2}} |\mathrm{D}(|\mathrm{D}u|^p - k)_+|^2 \,\mathrm{d}x$$

$$\lesssim r^{-2} \int_{B_r} |\mathrm{D}u|^{q-p}(|\mathrm{D}u|^p - k)_+^2 \,\mathrm{d}x + \int_{B_r} |\mathrm{D}u|^2 |f|^2 \,\mathrm{d}x$$

$$\lesssim \|\mathrm{D}u\|_{\mathrm{L}^\infty(B_R)}^{q-p} r^{-2} \int_{B_r} (|\mathrm{D}u|^p - k)_+^2 \,\mathrm{d}x + \|\mathrm{D}u\|_{\mathrm{L}^\infty(B_R)}^2 \int_{B_r} |f|^2 \,\mathrm{d}x\,,$$

a Caccioppoli inequality of balanced growth, but involving constants in terms of $\|\mathrm{D}u\|_{\mathrm{L}^\infty}$. This implies

$$|\mathrm{D}u(x_0)|^p \lesssim \|\mathrm{D}u\|_{\mathrm{L}^\infty(B_R)}^{\frac{q-p}{2}\frac{n}{2}} \Big(\fint_{B_R} |\mathrm{D}u|^{2p} \,\mathrm{d}x \Big)^{\frac{1}{2}} + \|\mathrm{D}u\|_{\mathrm{L}^\infty(B_R)}^{\frac{q-p}{2}\frac{n-2}{2}+1} \mathbb{P}^f(x_0, R) + 1$$

De Giorgi iteration à la [Kilpeläinen & Malý 1994]:
Consider a ball $B_R = B_R(x_0)$ in $\mathbb{R}^n$, $f \in \mathrm{L}^2(B_R)$, and a function $v \in \mathrm{W}^{1,2}(B_R)$, which satisfies

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for all $k \geq k_0$ and $r \leq R$. Then we have

$$v(x_0)_+ - k_0 \lesssim M_1^{\frac{n}{2}} \Big(\fint_{B_R} v_+^2 \,\mathrm{d}x \Big)^{\frac{1}{2}} + M_1^{\frac{n-2}{2}} M_2 \mathbb{P}^f(x_0, R)$$

The Caccioppoli inequality implies local Lipschitz regularity

So far ... a minimizer $u \in \mathrm{W}^{1,\infty}(B_r)$ satisfies

$$|\mathrm{D}u(x_0)|^p \lesssim \|\mathrm{D}u\|_{\mathrm{L}^\infty(B_R)}^{\frac{q-p}{2}\frac{n}{2}} \Big(\fint_{B_R} |\mathrm{D}u|^{2p}\,\mathrm{d}x \Big)^{\frac{1}{2}} + \|\mathrm{D}u\|_{\mathrm{L}^\infty(B_R)}^{\frac{q-p}{2}\frac{n-2}{2}+1} \mathbb{P}^f(x_0,R) + 1$$

$$\lesssim \|\mathrm{D}u\|_{\mathrm{L}^\infty(B_R)}^{\frac{q-p}{2}\frac{n}{2}+\frac{p}{2}} \Big(\fint_{B_R} |\mathrm{D}u|^{p}\,\mathrm{d}x \Big)^{\frac{1}{2}} + \|\mathrm{D}u\|_{\mathrm{L}^\infty(B_R)}^{\frac{q-p}{2}\frac{n-2}{2}+1} \mathbb{P}^f(x_0,R) + 1$$

and this gives a reasonable a priori Lipschitz bound, provided that

$$\frac{q-p}{2}\frac{n}{2} + \frac{p}{2} < p \quad \Leftrightarrow \quad \frac{q}{p} < 1 + \frac{2}{n}$$

(then also $\frac{q-p}{2}\frac{n-2}{2} + 1 \le \frac{q-p}{2}\frac{n}{2} + \frac{p}{2} < p$). By a (technical) iteration scheme, we conclude in this case with

$$\|\mathrm{D}u\|_{\mathrm{L}^\infty(B_{R/2})} \lesssim \Big(\fint_{B_R} |\mathrm{D}u|^p\,\mathrm{d}x \Big)^{\frac{2}{(n+2)p-nq}} + \|f\|_{\mathrm{L}^{n,1}(B_R)}^{\frac{4}{4(p-1)-(q-p)(n-2)}} + 1$$

Local Lipschitz regularity result

Theorem (B.-Mingione 2020)

Consider an integrand F satisfying the (p,q)-growth condition, with $2 \leq p \leq q$, and $f \in \mathrm{L}^{n,1}(\Omega)$. If $\frac{q}{p} < 1 + \frac{2}{n}$ holds, then every minimizer u of the functional $w \mapsto \int_\Omega [F(\mathrm{D}w) - fw]\,\mathrm{d}x$ is locally Lipschitz continuous with

$$\|\mathrm{D}u\|_{\mathrm{L}^\infty(B_{R/2})} \lesssim \left(\fint_{B_R} F(\mathrm{D}u)\,\mathrm{d}x \right)^{\frac{2}{(n+2)p-nq}} + \|f\|_{\mathrm{L}^{n,1}(B_R)}^{\frac{4}{4(p-1)-(q-p)(n-2)}} + 1$$

for every ball $B_R \subset \Omega$.

Remarks:

- One needs to work with approximations $u^{(k)} \in \mathrm{W}^{1,\infty}$ (e.g. via bounded slope condition, [Bousquet & Brasco 2016]) and then pass to the limit in the uniform estimates.
- This estimate reduces to the classical estimates for p-growth and for (p,q)-growth without source term.
- The results extends to the vectorial setting under radial structure $F(z) = \widetilde{F}(|z|)$.

A more general local Lipschitz regularity result

The strategy of proof can actually be performed for a very general class of non-uniformly elliptic variational functionals $w \mapsto \int_\Omega \big[F(\mathrm{D}w) - fw\big]\,\mathrm{d}x$ with a convex integrand F satisfying

$$g_1(|z|)|\xi|^2 \lesssim \partial^2 F(z)\xi\cdot\xi \quad \text{and} \quad |\partial^2 F(z)| \lesssim g_2(|z|)$$

with functions g_1, g_2. Supposing that

- $t \mapsto t g_1(t)$ is non-decreasing and grows at least as t^σ for some $\sigma > 0$,
- $\int_0^{|z|} t g_1(t)\mathrm{d}t$ provides a "lower bound" to $F(z)$ (modulo suitable power),
- $t \mapsto g_2(t)/g_1(t)$ is non-decreasing and grows "not too fast"

and that the source term satisfies

- $f \in \mathrm{L}^{n,1}(\Omega)$,

then every minimizer is locally Lipschitz continuous with quantified estimate formulated in terms of intrinsic quantities of $\mathrm{D}u$.

Specification to a simple exponential case

Theorem (B.-Mingione 2020)

Consider the integrand $F(z) = \exp(|z|^p)$ for some $p > 1$ and $f \in \mathrm{L}^{n,1}(\Omega)$. Then every minimizer u of the functional $w \mapsto \int_\Omega \big[F(\mathrm{D}w) - fw\big]\,\mathrm{d}x$ is locally Lipschitz continuous. Moreover, in the case $f \equiv 0$, there holds

$$\|\mathrm{D}u\|^p_{\mathrm{L}^\infty(B_{R/2})} \lesssim \log\left(\fint_{B_R} \exp(|\mathrm{D}u|^p)\,\mathrm{d}x \right) + 1$$

for every ball $B_R \subset \Omega$.

Remarks:

- From the heuristic viewpoint, large values of $|\mathrm{D}u|$ are very expensive and regularity theory should be easier (as opposed to linear growth problems which are excluded in our results);
- The results extends to the vectorial setting;
- Previously, local Lipschitz regularity was only known in the case $f \equiv 0$, with a bound with a loss of an exponential scale compared to the estimate above.

[Marcellini 1996]

Thank you for the attention!