The Geometric Manin's Conjectures

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Manin's conjecture

Let K be a number field and X be a smooth projective Fano variety over K. Let r be the rank of the Picard group of X.

The number of $x \in X(K)$, outside a thin set, with anticanonical height $< t$, is asymptotic to $c \cdot e^t \cdot t^{r-1}$,

- where the anticanonical height is defined by fixing a proper model $\mathcal{X}$ of X over $\mathcal{O}_K$, a line bundle $\mathcal{L}$ on $\mathcal{X}$ isomorphic to the anticanonical bundle on X, and a metric on $\mathcal{L}$ at each infinite place, and taking $h(x) = \sum_v \log |s(x)|_v$ for an arbitrary nonvanishing section s of $\mathcal{L}$,
- c is an explicit constant due to Peyre (depending on $\mathcal{X}, \mathcal{L}$ and the metric)
- and a thin set is the union of the images of $Y_1(K), \ldots, Y_n(K)$ for Y_i either a proper subvariety or a degree > 1 cover of X.

Function Field Manin's conjecture

Let $\mathbb{F}_q$ be a finite field, C a curve over $\mathbb{F}_q$, and X a smooth projective simply-connected Fano variety over $\mathbb{F}_q(C)$. Let r be the rank of the Picard group of X.

The number of $x \in X(\mathbb{F}_q(C))$, outside a thin set, with anticanonical height $< t$, is asymptotic to $c \cdot q^t \cdot t^{r-1}$,

- where the anticanonical height is defined by fixing a proper model $\mathcal{X}$ of X over C and a line bundle $\mathcal{L}$ on $\mathcal{X}$ isomorphic to the anticanonical bundle on X, and taking the degree of $f_x^*\mathcal{L}$, where $f_x \colon C \to \mathcal{X}$ is defined by spreading out x.

Interpretation via Lang-Weil

Let $\mathrm{Sec}_e(\mathcal{X})$ be the moduli space of sections $f \colon C \to \mathcal{X}$ with $f^*\mathcal{L}$ of degree e.

Let $\mathrm{Sec}'_e(\mathcal{X})$ be the open subset of sections whose generic point does not lie in the relevant thin set.

Then the number of $\mathbb{F}_q$-points in $\mathrm{Sec}'_e(\mathcal{X})(\mathbb{F}_q)$ is equal to the number of $x \in X(\mathbb{F}_q(C))$, outside a thin set with anticanonical height e.

Lang-Weil: irreducible varieties of dimension N have $\approx q^N$ $\mathbb{F}_q$-points.

Conclusion: $\mathrm{Sec}'_e(\mathcal{X})$ should have $\approx e^{r-1}$ irreducible components, of dimension $e + O(1)$.

The simplest case: Constant varieties

Suppose X is defined over $\mathbb{F}_q$, and $\mathcal{X}$ is simply $X \times C$. Then $\mathrm{Sec}_e(\mathcal{X}) = \mathrm{Mor}_e(C, X)$, the space of morphisms $f \colon C \to X$ with $f^* K_X^{-1}$ of degree e.

The tangent space of $\mathrm{Mor}_e(C, X)$ at f is $H^0(C, f^*\mathcal{T}_X)$. By Riemann-Roch, this has expected dimension

$$n(1-g) + \deg \det f^* \mathcal{T}_X = n(1-g) + e.$$

We can split this into components $\mathrm{Mor}_\beta(C, X)$, the moduli space of maps $C \to X$ of class β, for each numerical-Brauer class β of curves on X satisfying $\beta \cdot K_X^{-1} = e$. Only nef classes β will contain curves not lying in a thin set. The number of such β is proportional to e^{r-1}.

Question 1 (Geometric Manin's Conjecture, Batyrev-Lehmann-Tanimoto): Is it true that, for all but $o(e^{r-1})$ nef curve classes of β satisfying $\beta \cdot K_X^{-1} = e$, the moduli space $\mathrm{Mor}_\beta(C, X)$ is geometrically irreducible of dimension $n(1-g) + e$?

Another test case: Blow-ups

Even when X is defined over $\mathbb{F}_q$, there is no reason to take $\mathcal{X} = X \times C$. Instead, we can take it to be the blow-up of $X \times C$ at m points.

One component of $\mathrm{Sec}_\beta(\mathcal{X})$ is then the moduli space of maps $f : C \to X$ of curve class β with $f(x_i) = y_i$ for some fixed points $x_1, \ldots, x_m \in C$, $y_1, \ldots, y_m \in X$. (Can express as a fiber of the natural map $\mathcal{M}_{g,m}(X, \beta) \to \mathcal{M}_{g,m} \times X^m$). Expected dimension is $n(1 - g - m) + e$.

When is this irreducible of the expected dimension? Not always! When X is a product like $\mathbb{P}^1 \times \mathbb{P}^1$, β is a bidegree (e_1, e_2). Moduli space splits as a product of two spaces, with expected dimensions $(1 - g - m) + 2e_1$ and $1 - g - m + 2e_2$ respectively. If either e_1 or e_2 is small then the expected dimension of one of the moduli spaces is negative, hence it never is nonempty of the right dimension.

A question in the blow-up case

Question 2 (Geometric Manin's Conjecture: For X a smooth projective Fano variety, m a natural number, β a nef curve class on X sufficiently far, depending on X, m, from the boundary of the nef cone, are all fibers of $\mathcal{M}_{g,m}(X,\beta) \to \mathcal{M}_{g,m} \times X^m$ geometrically irreducible of dimension $n(1-g-m) + \beta \cdot K_X^{-1}$?

- It is easy to see that the answer is "yes" for easy varieties like projective spaces.
- The "geometry via arithmetic" method of Browning-Vishe or Browning-S should give a positive answer for any X where it works for all, but no one has checked this. (Arithmetically, congruence conditions are usually only a small increase in difficulty.)
- It seems hard to apply sophisticated geometric methods like bend-and-break to this problem, in part because the fibers of $\overline{\mathcal{M}}_{g,m}(X,\beta) \to \overline{\mathcal{M}}_{g,m} \times X^m$ often have the wrong dimension.

What is the maximum class of varieties for which purely geometric methods can answer this question?

Preparations for the general case

If $\mathcal{X}$ is regular, then sections of $\mathcal{X}$ lie in the smooth locus of $\mathcal{X} \to C$. May as well assume that $\mathcal{X}$ is smooth (but drop proper assumption on special fibers). Each section goes through exactly one connected component of each fiber. It's convenient to drop all but one component.

For β a curve class on $\mathcal{X}$, can form the moduli space $\mathrm{Sec}_\beta(\mathcal{X})$ of sections of $\mathcal{X}$ whose image has class β.

This will be empty unless $\beta \cdot F = 1$ for F the class of a fiber. We work with the cone of classes that are both nef and satisfy $\beta \cdot F = 1$.

The relative tangent bundle is relevant in deformations, so we use the relative anticanonical bundle.

A question in the general case

Question 3 (Geometric Manin's Conjecture): Let $\mathcal{X} \to C$ be a smooth morphism, with connected geometric fibers, with projective Fano generic fiber. For β a nef curve class on X, with $\beta \cdot F = 1$, if β is sufficiently far from the boundary of the cone of such classes, is $\mathrm{Sec}'_\beta(\mathcal{X})$ geometrically irreducible of dimension $n(1-g) + \beta \cdot K_{\mathcal{X}/C}^{-1}$?

For arbitrary β, is the dimension at most $n(1-g) + \beta \cdot K_{\mathcal{X}/C}^{-1} + O(1)$? Is the sum over β within $O(1)$ of the boundary of the cone, satisfying $\beta \cdot K_{\mathcal{X}/C}^{-1} = e$, of the number of irreducible components of $\mathrm{Sec}'_\beta(\mathcal{X})$, at most $o(e^{r-1})$?

Last two questions are there to ensure we don't lose information in passing from e to β.

More precise information: Cohomology

Lang-Weil, precisely, says that an N-dimensional geometrically irreducible variety over $\mathbb{F}_q$ has $(1 + O(q^{-1/2}))q^N$ points over $\mathbb{F}_q$. But Manin's conjecture, with Peyre's explicit constant, should give a much more precise formula for the number of points. How to explain this geometrically?

Ellenberg and Venkatesh suggested the answer was cohomology. The Lefschetz fixed point formula gives an expact formula for the number of $\mathbb{F}_q$ points of a variety in terms of its compactly-supported cohomology groups.

Cohen-Jones-Segal: The low-degree cohomology of the moduli space of maps $\mathbb{P}^1 \to X$ should look like the low-degree cohomology of the moduli space of continuous maps $S^2 \to X(\mathbb{C})$, aka $\mathbb{P}^1(\mathbb{C}) \to X(\mathbb{C})$.

Ellenberg-Venkatesh made plausible guesses for how Frobenius would act on the low-degree cohomology of the moduli space of continuous maps $S^2 \to X(\mathbb{C})$, and noted that they match, via the Lefschetz formula, Peyre's explicit constant in Manin's conjecture.

Preparations for a cohomological question

For $\mathcal{X} \to C$ a smooth morphism of varieties over $\mathbb{C}$ and β a nef curve class on $\mathcal{X}$, let $\mathrm{Sec}^{\mathrm{cont}}_\beta(\mathcal{X}(\mathbb{C}))$ be the moduli space of *continuous* sections $C(\mathbb{C}) \to \mathcal{X}(\mathbb{C})$ whose image has *homology* class β.

There is a map $\mathrm{Sec}'_\beta(\mathcal{X}) \to \mathrm{Sec}^{\mathrm{cont}}_\beta(\mathcal{X}(\mathbb{C}))$ inducing a map on cohomology groups the other way.

We might want to ask if this is an isomorphism in low degree, but this isn't quite the right question to ask. For Lefschetz, we need the compactly-supported cohomology, and high-degree gives the main term.

For smooth space, Poincaré duality relates high-degree compactly-supported cohomology to low-degree usual cohomology, but $\mathrm{Sec}'_\beta(\mathcal{X})$ is not necessarily smooth. (Alternate approach: Pass to the smooth locus.) In general, get a trace pairing between them.

A cohomological question

Question 4 (Geometric Manin's Conjecture): Let $\mathcal{X} \to C$ be a smooth morphism ove $\mathbb{C}$, with connected geometric fibers, with projective Fano generic fiber. For β a nef curve class on X, with $\beta \cdot F = 1$, is the trace pairing

$$H^i(\mathrm{Sec}^{\mathrm{cont}}_{\beta}(\mathcal{X}(\mathbb{C}))) \times H_c^{2(n(1-g)+\beta\cdot K^{-1}_{\mathcal{X}/C})-i}(\mathrm{Sec}'_{\beta}(\mathcal{X}))$$

a perfect pairing for i less than c_1 times the distance of β from the boundary of the nef cone minus c_2, for some constants c_1, c_2?

For any β, is

$$\dim H_c^{2(n(1-g)+\beta\cdot K^{-1}_{\mathcal{X}/C})-i}(\mathrm{Sec}'_{\beta}(\mathcal{X})) \leq c_3^i c_4$$

for some constants c_3, c_4?

Positive answer to the second question implies the terms with unusual β do not contribute much to the Lefschetz fixed point formula for the number of rational points, at least as long as $q > c_3^2$, by Deligne's RH.

What did I actually do?

In joint work with Tim Browning, we gave a positive answer to something like question 4 for low-degree hypersurfaces.

Why only "something like"?

- We worked with an affine variety, not a projective variety. (We chose X a smooth affine hypersurface over $\mathbb{F}_q$, and $\mathcal{X} \to \mathbb{P}^1$ a model chosen so that sections are polynomial maps $\mathbb{A}^1 \to X$ of fixed degree with fixed leading term).
- We showed the high-degree compactly-supported cohomology was equal to something that quacks like the cohomology of the moduli space of continuous sections of $\mathcal{X}$, but we didn't show it is the cohomology of this moduli space. (We showed it is computed by a spectral sequence that, if it degenerates, has the same Betti numbers as this moduli space.)

Browning-S method

Step 1: count points on the moduli space $\mathrm{Sec}_e(\mathcal{X})$.

- This is counting tuples of polynomials in one variable over $\mathbb{F}_q$ satisfying a polynomial equation.
- We solve this via a classical strategy of Birch, using the circle method.

Potential step 2: deduce the cohomological statement from the point-counting statement.

- This seems impossible!

Actual step 2: translate the strategy step-by-step into a geometric strategy.

- Every function appearing in the original argument becomes a sheaf.
- Sums and integrals become sheaf cohomology groups.
- Replace each analytic statement with a geometric statement that implies it (using the Lefschetz fixed point formula and Deligne's RH)

Construction of sheaves

Let $\mathcal{F}$ be an ℓ-adic sheaf on a variety Y over a field $\mathbb{F}_q$. For each point $x \in Y$, the stalk $\mathcal{F}_x$ admits an action of Frobenius Frob_q.

- From the sheaf $\mathcal{F}$, we construct the trace function $\mathrm{tr}(\mathrm{Frob}_q, \mathcal{F}_x)$.
- Given a function $G(x)$, we can often find a sheaf $\mathcal{F}$ whose trace function $\mathrm{tr}(\mathrm{Frob}_q, \mathcal{F}_x) = G(x)$.

Example: In the usual circle method, to count solutions to $F(x_1, \dots, x_n) = 0$, we write down the function $e(\theta \cdot F(x_1, \dots, x_n))$ where $\theta \in \mathbb{R}/\mathbb{Z}$.

In the function field circle method, we instead write down $\psi(\alpha \cdot F(f_1, \dots, f_n))$ where $F(f_1, \dots, f_n)$ is a polynomial of degree de over $\mathbb{F}_q$, α is a linear form on polynomials of degree de, and $\psi \colon \mathbb{F}_q \to \mathbb{C}^\times$ is an additive character.

To construct a sheaf with trace function $\psi(\alpha \cdot F(f_1, \dots, f_n))$, we consider the finite étale covering with equation $y^q - y = \alpha \cdot F(f_1, \dots, f_n)$, with automorphism group $\mathbb{F}_q$, giving a homomorphism $\pi_1 \to \mathbb{F}_q$, and compose with ψ, obtaining a representation of π_1, giving a locally constant sheaf.

The major arcs stratification

The space of linear forms on polynomials of degree *de* is the analogue of the circle $\mathbb{R}/\mathbb{Z}$ in the classical circle method.

Classically, we divide this circle into major arcs and minor arcs.

Geometrically, we define a stratification by a series of closed subsets, where the m'th step is the closure of the set of linear forms that only depend on the value of the polynomial at m points.

Strata for $m \leq e$ are the major arcs. For these linear forms, we can calculate the sums and cohomology groups exactly. The stratification defines a spectral sequence.

Strata for $m > e$ are the minor arcs. Here we can only bound the sums using Weyl differencing, or bound the highest degree of nonvanishing compactly supported cohomology with geometric Weyl differencing.