

## A few additional examples

The purpose of this handout is to give a few more examples of the guiding principle mentioned in the talk.

① Let  $\mathcal{M}_{3,3} := \{ \text{smooth cubic surfaces } S \subseteq \mathbb{P}^3 \} \cong \mathbb{P}^{19} - \Sigma_{\text{sing}}$

Let  $\tilde{\mathcal{M}}_{3,3} := \{ (S, L) : S \in \mathcal{M}_{3,3}, L \subset S \text{ line} \}$   
 $\uparrow$   
 $L \cong \mathbb{P}^1$

Cayley, C. Jordan:  $\tilde{\mathcal{M}}_{3,3}$   
 $\downarrow \pi$   $\pi(S, L) := S$   
 $\mathcal{M}_{3,3}$   
 "∃ 27 lines on any smooth cubic surface"

is a 27-sheeted cover, with monodromy

$$\mathcal{P}: \pi_1 \mathcal{M}_{3,3} \twoheadrightarrow W(E_6) \subset S_{27}$$

Thm (Huxford 2021, Conj. by Farb): The

homomorphism  $\mathcal{P}$  is unique up to conjugacy.

Let  $G(1,3) :=$  Projective Grassmannian of lines in  $\mathbb{P}^3$

$U\text{Conf}_m(X) := \{ \text{unordered } m\text{-tuples of distinct points in } X \}$

The above gives a morphism

$$\Psi: \mathcal{M}_{3,3} \longrightarrow U\text{Conf}_{27}(G(1,3))$$

Guiding Principle gives:  $\forall m \geq 1$

If  $F: \mathcal{M}_{3,3} \longrightarrow U\text{Conf}_m(G(1,3))$

is "nontrivial" then  $m=27$  and  $F \sim \Psi$ .

If in addition  $F$  is holomorphic then  $F = \Psi$ .

For generic  $S \subset \mathcal{M}_{3,3}$ ,

$$|\{L_i \cap L_j : L_i \neq L_j \text{ lines in } S\}| = 135$$

(only generic since  $\exists$  "Eckhardt Points")



→ 3 rational section  $\sigma$  of

$$\begin{array}{ccc} U_{\text{conf}_{135}}(S) & \rightarrow & E_{135} \\ & & \pi \downarrow \uparrow \sigma \\ & & M_{3,3} \end{array}$$

Guiding Principle:  $\sigma$  should be only holomorphic  $n$ -multisection.

② Guiding Principle #2: Any "exceptional homomorphism" of finite groups is modular: it can be explained by a map of moduli spaces.

Example #1:  $S_4 \twoheadrightarrow S_3$  explained by

$$\begin{array}{ccc} \widetilde{\text{Poly}}_4 & \xrightarrow{\text{Fer}} & \widetilde{\text{Poly}}_3 \\ S_4 \downarrow & & \downarrow S_3 \\ \text{poly}_4 & \xrightarrow{\text{Fer}} & \text{poly}_3 \end{array}$$

Example #2:  $S_6 \cong \text{SP}_4 \mathbb{F}_2$  explained by

$$\begin{array}{ccccc}
 \mathcal{M}_{0,6} & \xrightarrow{\cong} & \mathcal{M}_2[\mathbb{Z}] & \xrightarrow{\cong} & \mathbb{A}^2[\mathbb{Z}] \\
 S_6 \downarrow & & \downarrow & & \downarrow \text{SP}_4 \mathbb{F}_2 \\
 \mathcal{M}_{0,6} / S_6 & \xrightarrow[\cong]{R} & \mathcal{M}_2 & \xrightarrow[\cong]{\text{---}} & \mathbb{A}^2 \\
 & & & & \text{J period mapping}
 \end{array}$$

$R(\{P_1, \dots, P_6\}) := 2$ -sheeted cover  $X \rightarrow \mathbb{P}^1$  branched over  $\{P_1, \dots, P_6\}$

$$\mathcal{M}_2[\mathbb{Z}] = \{ (X, \mathcal{B}) : X \in \mathcal{M}_2, \mathcal{B} \text{ basis for } H_1(X; \mathbb{F}_2) \}$$

$\exists$  many more such examples,

e.g.

$$W(E_6)^+ \cong \text{PSP}_4(\mathbb{F}_3)$$

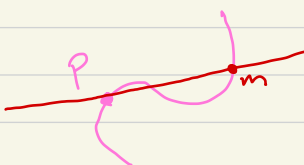
See Farb-Kisinn-Wolfson, "Modular Functions and Resolvent Problems"

for more examples.

③ A remarkable map:

Given  $\bullet S :=$  smooth cubic surface in  $\mathbb{P}^3$   
 $\bullet P \in S \setminus \{L_1, \dots, L_27\}$

$\rightsquigarrow \exists$  2-sheeted cover  $B\ell_P(S)$  exc. divisor  $e$   
 branched over a smooth quartic  $\pi \downarrow \mathbb{P}^2 \supset Q$

$\pi$  is:   $\pi(m) = \begin{cases} \text{line } \overline{pm} & m \notin e \\ [m] & m \in e \end{cases}$

Fact:  $\{28 \text{ bitangents to } Q\} = \pi(\{L_1, \dots, L_{27}\} \text{ on } S) \cup \pi(e)$

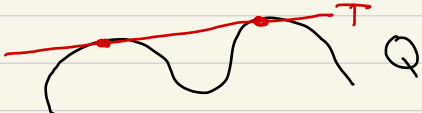
Let  $S \rightarrow E_{3,3} := \{(S, P): S \in \mathcal{M}_{3,3}, P \in S\}$   
 $\downarrow$   
 $\mathcal{M}_{3,3}$

Let  $\mathcal{M}_{4,2} = \{\text{smooth quartic curves in } \mathbb{P}^2\}$

The above gives a dominant rational map

$$\Psi: U_{3,3} \dashrightarrow \{ (Q, T) : Q \in \mathcal{M}_{4,2}, T \text{ is bitangent to } Q \}$$

$\downarrow 28$   
 $\mathcal{M}_{4,2}$



Exercise: Apply Guiding Principle to  $\Psi$  to give conjectures on holomorphic and topological rigidity.

(4) Many classical moduli spaces  $\mathcal{M}$  locally symmetric varieties, i.e.

$$\mathcal{M} = \Gamma \backslash G / K$$

$G$  = real semisimple Lie group of Hermitian type

$K$  = max compact subgroup of  $G$

$\Gamma$  = arithmetic lattice in  $G$

and the type of rigidity conjectures I've been giving can be proved using Margulis Super-rigidity, Congruence Subgroup Property, etc.