Pseudoholomorphic curves on lcs manifolds

IBS-Center for Geometry and Physics & POSTECH

yongoh1@postech.ac.kr

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Outline

1. lcs manifolds and motivating examples
2. Contact instantons and lcs instantons
3. Geometry and analysis of lcs-instantons

Motivating problems

There are two natural motivating questions that prompted us to search for the theory of pseudoholomorphic curves on lcs manifolds.

1. Study of topology of the group $\mathrm{Cont}(M, \xi)$ of contactomorphisms by considering the mapping torus of $\psi \in \mathrm{Cont}(M, \xi)$. For example the study of π_1 similarly as done by Seidel for the group of Hamiltonian diffeomorphisms. The resulting mapping tous M_ψ carries a natural lcs structure.
2. (Savelyev's question) Is there any non-squeezing type result on exact lcs manifolds, i.e., when $\omega = d^{\mathfrak{b}}\lambda = d\lambda - \mathfrak{b} \wedge \lambda$ for closed one-form $\mathfrak{b}$.?

Motivated by these problems, we investigate the pseudoholomorphic curve theory on these classes of lcs manifolds, especially on exact lcs manifolds.

lcs manifolds

Definition

An lcs *manifold is a triple* $(M, \omega, \mathfrak{b})$ *where* $\mathfrak{b}$ *is a closed one-form and* ω *is a nondegenerate 2-form satisfying the relation*

$$d\omega + \mathfrak{b} \wedge \omega = 0. \tag{1.1}$$

Definition

Let $(M, \omega, \mathfrak{b})$ *and* $(M', \omega', \mathfrak{b}')$ *be two lcs manifolds.*

1. *A diffeomorphism* $\phi : M \to M'$ *is called* lcs *if there exists* $a \in C^\infty(M, \mathbb{R} \setminus \{0\})$ *such that*

$$\phi^*\omega' = (1/a)\omega, \quad \phi^*\mathfrak{b}' = \mathfrak{b} + d(\ln|a|).$$

2. *An lcs diffeomorphism is positive (resp. negative) if the function* a *is positive (resp. negative).*

- For a positive lcs diffeomorphism, we write the defining condition as

$$\phi^*\omega' = e^f\omega, \quad \phi^*\mathfrak{b}' = \mathfrak{b} - df. \tag{1.2}$$

  This manifests its similarity to the defining condition $\psi^*\lambda' = e^g\lambda$ of contactormosphism ψ with *conformal exponent* $g \in C^\infty(M)$.
- Locally by choosing $\mathfrak{b} = d\ell$ for a local function $\ell : U \to \mathbb{R}$ on an open neighborhood U, (1.2) is equivalent to

$$d(e^{-\ell}\omega) = 0 \tag{1.3}$$

  and so the local geometry of l.c.s manifold is exactly the same as that of symplectic manifolds.
- In particular one can define the notion of Lagrangian submanifolds in the same way as in the symplectic case since the definitions require only nondegeneracy of the two-form ω.

$\mathfrak{lcs}$-fication of contact manifolds

- A natural class of lcs manifolds arise as the mapping tori of contactomrophisms ψ which becomes Banyaga's lcs manifold on $Q \times S^1$ when $\psi = id$.

Definition ($\mathfrak{lcs}$-fication)

We call the pair $(Q \times S^1, \omega_\lambda)$ an $\mathfrak{lcs}$-fication of the contact manifold (Q, λ) where $S^1 = \mathbb{R}/\mathbb{Z}$, where

$$\omega_\lambda := d\lambda + \mathfrak{b} \wedge \lambda = d^{\mathfrak{b}}\lambda, \quad \mathfrak{b} = \frac{1}{2\pi}\pi^* d\theta. \tag{1.4}$$

- By varying the size of the circle $(S^1, d\theta) := (\mathbb{R}/\mathbb{Z}, [dt])$ to $(S^1_R, d\theta) := (\mathbb{R}/R\mathbb{Z}, [dt])$, or by taking the one-form $\mathfrak{b}_R := \frac{1}{2\pi R}\pi^*\theta$ on $S^1 = R/\mathbb{Z}$, we may regard the lcs-manifold $(Q \times \mathbb{R}, d\lambda + ds \wedge \lambda)$ as the limiting case of

$$(Q \times S^1, d\lambda + \mathfrak{b}_R \wedge \lambda), \quad \mathfrak{b}_R = \frac{1}{2\pi R} d\theta$$

as $R \to \infty$.

Mapping tori of contactomorphisms

Let (Q, ξ) be a contact manifold and $\psi \in \mathrm{Cont}(Q, \xi)$. By definition,

$$d\psi(\xi) \subset \xi.$$

- We consider the product $Q \times \mathbb{R}$ and the distribution given by

$$\xi \oplus \{0\} \subset \xi \oplus \mathbb{R}\langle \frac{\partial}{\partial s} \rangle \subset T(Q \times \mathbb{R}).$$

  Since $d\psi(\xi) \subset \xi$, the distribution descends to the mapping torus

$$M_\psi := \frac{Q \times \mathbb{R}}{(t, \psi(x)) \sim (t+1, x)} \longrightarrow S^1.$$

- We denote by $\widetilde{\xi} = [\xi \oplus \{0\}]$ the resulting distribution. Then we have

$$\widetilde{\xi} \subset \left[\xi \oplus \mathbb{R}\langle \frac{\partial}{\partial s} \rangle \right] \subset TM_\psi$$

  on M_ψ.

- The above mapping torus is well-defined for the contact manifold (M, ξ) not for the one (M, λ) with a given contact form: Unless ϕ is strict, i.e., $\phi^*\lambda = \lambda$, the obvious pull-back form $\pi^*\lambda$ on $Q \times \mathbb{R}$ does not descend but the contact distribution does.
- Since $\left[\xi \oplus \mathbb{R}\langle \frac{\partial}{\partial s}\rangle\right]$ is cooriented if ξ is, we can take a one-form κ on M_ϕ such that

$$\ker \kappa = \left[\xi \oplus \mathbb{R}\langle \frac{\partial}{\partial s}\rangle\right]. \tag{1.5}$$

Proposition

For each contactomorphism $\psi \in \mathrm{Cont}(Q, \xi)$, the two form $d\kappa + d\theta \wedge \kappa =: \omega_\kappa$ is an lcs form. Furthermore if κ' is another such form satisfying (1.5), the two lcs forms ω_κ and $\omega_{\kappa'}$ are (positively) conformally diffeomorphic.

Definition (lcs-mapping torus of ψ)

*We call the isomorphism class of the above constructed lcs-manifold (M_ψ, ω_κ) the **lcs-mapping torus of ψ**.*

- This class of lcs manifolds is special in the following sense: When the one-form κ is given, heir tangent bundle *TM* has the splitting

$$TM = \widetilde{\xi} \oplus \mathcal{V}_\kappa, \quad \mathcal{V}_\kappa := (\widetilde{\xi})^{\omega_\kappa}$$

  with $\mathcal{V}_\kappa$ of constant rank 2. I.e., (M_ψ, ω_κ) is an exact lcs manifold of the first kind.
- This decomposition will provide some particular class of almost complex structures *preserving the decomposition*, and provide a nice theory of pseudoholomorphic curves which we call lcs instantons.
- They are closely tied to contact instantons on general contact manifolds (Q, ξ).

Contact instantons

Definition (Contact triad)

Let (M, ξ) be a contact manifold, and λ be a contact form of ξ. An endomorphism $J : TM \to TM$ is called a λ-adapted CR-almost complex structure if it satisfies

1. $J(\xi) \subset \xi$, $JR_\lambda = 0$ *and* $J|_\xi^2 = -id|_\xi$,
2. $g_\xi := d\lambda(\cdot, J|_\xi \cdot)|_\xi$ *defines a Hermitian vector bundle* $(\xi, J|_\xi, g_\xi)$.

We call the triple (M, λ, J) a contact triad.

- For given such a triad, we first decompose any TM-valued one-form Ξ on a Riemann surface (Σ, j) into

$$\Xi = \Xi^\pi + \lambda(\Xi)\, R_\lambda$$

and then we further decompose

$$\Xi^\pi = \Xi^{\pi(1,0)} + \Xi^{\pi(0,1)}.$$

into J linear and J anti-linear parts of Ξ^π.

A map $w : (\Sigma, j) \to (Q, J)$ is called a contact instanton if it satisfies

$$\overline{\partial}^\pi w = 0, \; d(w^*\lambda \circ j) = 0, \quad \overline{\partial}^\pi := (du)^{\pi(0,1)}. \tag{2.1}$$

Theorem (–, 2021)

The associated boundary value problem with Legendrian boundaries is a nonlinear elliptic boundary value problem.

- Contact instanton equation itself is first introduced by Hofer.
- When the one-form $w^*\lambda \circ j$ is exact, the equation can be lifted to a pseudoholomorphic curve equation on the symplectization $SQ = Q \times \mathbb{R}$.
- We used this equation to develop the contact analog to the Lagrangian intersection theory in symplectic geometry to investigate entanglement problem of Legendrian links in general. In particular, we have proved a conjecture by Sandon-Shelukhin on the existence question of *translated points* of a contactomorphism ψ when its Hofer-type norm is smaller than period gap of the contact manifold (M, λ). – arXiv:2111.02597–

Asymptotic invariants of contact instantons

We start with the following subsequence convergence result of contact instantons.

Hypothesis

Let $w : \dot\Sigma \to Q$ be a contact instanton satisfying

1. $E^\pi(w) < \infty$ *(finite π-energy);*
2. $\|dw\|_{C^0(\dot\Sigma)} < \infty$.

Then we associate two natural asymptotic invariants at each puncture defined as

$$T := \frac{1}{2}\int_{[0,\infty)\times S^1} |d^\pi w|^2 \, dA + \int_{\{0\}\times S^1} (w|_{\{0\}\times S^1})^* \lambda$$

$$Q := \int_{\{0\}\times S^1} ((w|_{\{0\}\times S^1})^* \lambda \circ j).$$

We call T the *asymptotic contact action* and Q the *asymptotic contact charge* of the contact instanton w at the given puncture.

Subsequence convergence theorem

Let $\dot\Sigma$ be a punctured Riemann surface with a finite number of marked points $\{p_1, \cdots, p_k\}$. Let w be a contact instanton satisfying the conditions laid out above.

Theorem (– and R. Wang, 2018)

For any given sequence $l = \{\tau_k\}$ with $\tau_k \to \infty$, there exists a subsequence $l' \subset l$ and a closed parameterized Reeb orbit $\gamma = \gamma_{l'}$ of period $|T|$ and some $(\tau_0, t_0) \in \mathbb{R} \times S^1$ and a massless instanton $w_\infty(\tau, t)$ (i.e., $E^\pi(w_\infty) = 0$) on the cylinder $\mathbb{R} \times S^1$ such that

1. $\lim_{i\to\infty} w(\tau + \tau_{k_i}, t) = w_\infty$ *in compact C^∞ topology.*
2. *Furthermore w_∞ has the formula*

$$w_\infty(\tau, t) = \gamma(-Q(p)\tau + T(p)\, t)$$

   for a Reeb trajectory γ,
3. *For the case of $Q = 0$ and $T \neq 0$, the trajectory is a closed Reeb orbit of R_λ with period $|T|$.*

Spirling cusp instantons along a Reeb core

- In the above theorem, the asymptotic massless instanton

$$w_\infty(\tau, t) = \gamma(-Q(p)\tau + T(p)\, t)$$

  could be an Reeb trajectory that is a immersion of $\mathbb{R}$, when $Q(p) \neq 0 and T(p) = 0$. In this case, the asymptotic shapes of the subsequence w_j are 'cusp-like' cylinders in that its asymptotic limit is one-dimensional Reeb trajectory.
- To perform the study of compactification and Fredholm theory, we must find a way of getting rid of this bad limit.
- In the preprint (arXiv:2103.15390), we show that under the Legendrian boundary conditions, the asymptotic charge $Q(p)$ always vanishes. This enables us to carry out the compactification and the Fredholm theory for the contact instantons with Legendrian boundary conditions.

$\mathfrak{lcs}$-instantons

Definition (λ-admissible almost complex structure)

We say an almost complex structure J on $M = Q \times S^1$ is λ-admissible (resp. ω_λ-admissible) if J satisfies the following:

1. *J is tame to ω_λ in standard sense,*
2. *J preserves the splitting $TM = \widetilde{\xi} \oplus \mathcal{V}$, i.e., $J(\xi) = \xi$, $J(\mathcal{V}) = \mathcal{V}$, where $\mathcal{V} = (\widetilde{\xi})^{\omega_\lambda}$.*
3. *J satisfies $J\frac{\partial}{\partial\theta} = R_\lambda$.*

(resp. if J satisfies (1) and (2).) We denote by $\mathcal{J}(Q \times S^1, \lambda)$ (resp. $\mathcal{J}(Q \times S^1, \omega_\lambda)$ the set of λ-admissible (resp. ω_λ-sdmissible) almost complex structures.

- The associated J-holomorphic curve equation for the map $u = (w, f) : \dot{\Sigma} \to Q \times S^1$ is reduced to

$$\overline{\partial}^\pi w = 0, \quad w^*\lambda \circ j = f^* d\theta. \tag{3.1}$$

We call a solution (w, f) an $\mathfrak{lcs}$-instanton.

- In particular w is a contact instanton.
- The equation resembles the J-holomorphic curve equation in the symplectization $\mathbb{R} \times Q$ except that the *exact* one-form $f^* dr$ in the symplectization case is replaced by the *closed* one-form $f^* d\theta$.
- Indeed if we replace S^1 by $\mathbb{R}$, our study of pseudoholomorphic curves on the lcs manifold equipped with the form $d\lambda + \mathfrak{b} \wedge \lambda$ with $\mathfrak{b} = dr$ reduced to that of the symplectization of (Q, λ).
- In this sense, Hofer's treatment of energy controlling the behavior of pseudoholomrphic curves on symplectization hints the way how one should proceed with the analysis of pseudoholomorphic curves on lcs manifolds.
- To correctly perform this analysis, we need to fix the charge class $\eta = [f^* d\theta] \in H^1(\dot{\Sigma}, \mathbb{Z})$ of the lcs instanton (w, f).

Charge class and lcs instanton energy

Let (w, f) be an lcs-instanton satisfying

$$\overline{\partial}^\pi w = 0, \; w^*\lambda \circ j = f^* d\theta$$

for the canonical angular form on S^1.

Definition (Period map and the charge class)

Let $f : \dot\Sigma \to Q \times S^1$ be a smooth map.

1. *We call the map*

$$C^\infty(\dot\Sigma, Q) \to H^1(\dot\Sigma, \mathbb{Z}); \quad f \mapsto [f^* d\theta]$$

   the period map and call the cohomology class $[f^* d\theta]$ the charge class of the map f.

2. *For an lcs instanton $u = (w, f) : \dot\Sigma \to Q \times S^1$, we call the cohomology class*

$$[f^* d\theta] \in H^1(\dot\Sigma, \mathbb{Z})$$

   the charge class of u and write

Definition

Let λ be a contact form of contact manifold (Q, ξ). Denote by $\mathfrak{R}eeb(Q, \lambda)$ the set of closed Reeb orbits. We define $\mathrm{Spec}(Q, \lambda)$ to be the set

$$\mathrm{Spec}(Q, \lambda) = \left\{ \int_\gamma \lambda \mid \lambda \in \mathfrak{R}eeb(Q, \lambda) \right\}$$

and call the action spectrum of (Q, λ). We denote

$$T_\lambda := \inf \left\{ \int_\gamma \lambda \mid \lambda \in \mathfrak{R}eeb(Q, \lambda) \right\}.$$

We set $T_\lambda = \infty$ if there is no closed Reeb orbit. This constant T_λ will enter in a crucial way in the following period gap theorem of lcs instantons.

Off-shell energy for lcs instantons

Definition (π-energy)

$$E^\pi(w, f) := \frac{1}{2}\int_{\dot\Sigma} |d^\pi w|^2.$$

Let $[f^* d\theta] = \eta$ for a fixed class $\eta \in H^1(\dot\Sigma, \mathbb{Z})$, we can write the closed one-form $f^* d\theta$ into

$$f^* d\theta = \beta + d\widetilde{f}$$

where β is a harmonic one-form representing the class $\eta \in H^1(\dot\Sigma)$ and $\widetilde{f}$ is a real-valued function on $\dot\Sigma$.

Definition (Vertical energy)

Let $\eta \in H^1(\dot\Sigma, \mathbb{Z})$ be fixed. Let w satisfy $w^*\lambda \circ j = f^* d\theta$ with $[u]_{S^1} = \eta$. Then we define

$$E^\perp_\eta(j, u) = \sup_{\varphi \in \mathbb{C}} \int_{\dot\Sigma} df \circ j \wedge d(\psi(\widetilde{f})) = \sup_{\varphi \in \mathbb{C}} \int_{\dot\Sigma} d(\psi(\widetilde{f})) \wedge w^*\lambda.$$

Theorem (ϵ-regularity theorem)

Denote by $D^2(1)$ the closed unit disc and let $u = (w, f)$ be an lcs instanton defined on $D^2(1)$ so that $w : D^2(1) \to Q$ satisfies

$$\overline{\partial}^\pi w = 0, \; w^*\lambda \circ j = f^* d\theta.$$

Assume the vertical energy bound $E^\perp(w) < K_0$. Then for any given $0 < \epsilon < T_\lambda$ and w satisfying $E^\pi(w) < T_\lambda - \epsilon$, and for a smaller disc $D' \subset \overline{D}' \subset D$, there exists some $K_1 = K_1(D', \epsilon, K_0) > 0$

$$\|dw\|_{C^0;D'} \leq K_1 \tag{3.2}$$

where K_1 depends only on (Q, λ, J), ϵ, $D' \subset D$.

Linearization and the Fredholm theory

- We consider the map

$$\Upsilon(w, f) = \left(\overline{\partial}^\pi w, w^*\lambda \circ j - f^* d\theta\right)$$

whose zero set is the set of lcs instantons by definition.

- Let $u = (w, f) : \dot{\Sigma} \to Q \times S^1$ be an lcs instanton of a given charge class $[u] = \eta$. We decompose

$$d\pi = d^\pi w + w^*\lambda \otimes R_\lambda$$

and

$$X = (Y, v) \in \Omega^0(u^* T(Q \times S^1))$$

and $Y = Y^\pi + \lambda(Y) R_\lambda$. Denote

$$\kappa = \lambda(Y), \quad \upsilon = d\theta(v).$$

Theorem

Decompose the linearized operator into $D\Upsilon(u) = D\Upsilon_1(u) + D\Upsilon_2(u)$ *with*

$$\begin{aligned} D\Upsilon_1(u)(Y, v) &= \overline{\partial}^{\nabla^\pi} Y^\pi + B^{(0,1)}(Y^\pi) + T_{dw}^{\pi,(0,1)}(Y^\pi) \\ &\quad + \frac{1}{2}\kappa \cdot ((\mathcal{L}_{R_\lambda} J)J(\partial^\pi w)) \\ D\Upsilon_2(u)(Y, v) &= (\mathcal{L}_Y \lambda) \circ j - \mathcal{L}_v d\theta = d\kappa \circ j - d\upsilon + Y \rfloor d\lambda \circ j \end{aligned}$$

where $B^{(0,1)}$ *and* $T_{dw}^{\pi,(0,1)}$ *are the* $(0,1)$*-components of* B *and* $T_{dw}^{\pi,(0,1)}$ *respectively, where* B, $T_{dw}^\pi : \Omega^0(w^*TQ) \to \Omega^1(w^*\xi)$ *are the zero-order differential operators given by*

$$B(Y) = -\frac{1}{2} w^*\lambda((\mathcal{L}_{R_\lambda} J)JY)$$

and

$$T_{dw}^\pi(Y) = \pi T(Y, dw).$$

- More succinctly, we can express the operator $D\Upsilon(u)$ in a matrix form

$$\begin{pmatrix} \overline{\partial}^{\nabla^\pi} + B^{(0,1)} + T^{\pi,(0,1)}_{dw} & , & \frac{1}{2}(\cdot)\cdot((\mathcal{L}_{R_\lambda}J)J(\partial^\pi w)) \\ ((\cdot)^\pi \rfloor d\lambda)\circ j & , & \overline{\partial} \end{pmatrix}. \tag{3.3}$$

- Noting that the off-diagonal terms of (3.3) are zero-order operators, by the continuous invariance of the Fredholm index, we obtain

$$\operatorname{Index} D\Upsilon_{(\lambda,T)}(w) = \operatorname{Index}\left(\overline{\partial}^{\nabla^\pi} + T^{\pi,(0,1)}_{dw} + B^{(0,1)}\right) + \operatorname{Index}(\overline{\partial}). \tag{3.4}$$

Index formula

Theorem

We fix a trivialization $\Phi : E \to \overline{\Sigma}$ *and denote by* Ψ_i^+ *(resp.* Ψ_j^-*) the induced symplectic paths associated to the trivializations* Φ_i^+ *(resp.* Φ_j^-*) along the Reeb orbits* γ_i^+ *(resp.* γ_j^-*) at the punctures* p_i *(resp.* q_j*) respectively. Then we have*

$$
\begin{aligned}
& \operatorname{Index} D\Upsilon_{(\lambda,T)}(u) \\
= \; & n(2 - 2g - s^+ - s^-) + 2c_1(w^*\xi) \\
& + \sum_{i=1}^{s^+} \mu_{CZ}(\Psi_i^+) - \sum_{j=1}^{s^-} \mu_{CZ}(\Psi_j^-) \\
& + \sum_{i=1}^{s^+} (2m(\gamma_i^+) + 1) + \sum_{j=1}^{s^-} (2m(\gamma_j^-) + 1) - 2g.
\end{aligned}
$$

Thank you for your attention!