Topological Aspects of Locally Conformal Symplectic Manifolds

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Symplectic: (M, ω), $\omega \in \Omega^2(M)$

- **Closed** $d\omega = 0$
- **Non-degenerate** $\iota_\bullet \omega \colon TM \xrightarrow{\sim} T^*M$

Darboux Theorem: Locally

$$\omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$$

Conformal Symplectic

Conformal symplectic transformation:
$f : (M, \omega) \to (N, \omega')$ such that

$$f^*\omega' = e^c\omega,$$

with c locally constant.

Remark

dim ≥ 4: The equation itself automatically implies c is locally constant.

Globally conformal symplectic

Globally conformal symplectic (GCS): Equivalence classes under $(M, \omega) \sim (M, e^c\omega)$ with c locally constant.

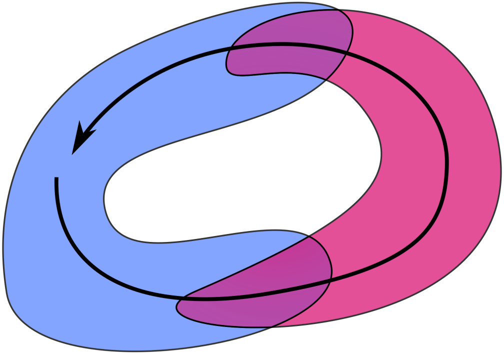
Remark

We lose a sense of scale/volume, e.g.

$$(\mathbb{CP}^n, \omega_{\mathrm{FS}}) \sim (\mathbb{CP}^n, 1000\omega_{\mathrm{FS}}).$$

(Naive embedding problems don't make sense.)

Cannot glue GCS on open sets; might come back with scale factor.



Solution: *Sheafify!*

Locally conformal symplectic, definition 1

LCS manifold: $(M, \mathcal{U} = \{(U_\alpha, \omega_\alpha)\})$

- **Cover:** $\{U_\alpha\}$ open cover of M
- **Locally symplectic:** $\omega_\alpha \in \Omega^2(U_\alpha)$ symplectic
- **Conformality:** identity on overlaps are conformal symplectic, i.e.

$$\omega_\alpha|_{U_\alpha \cap U_\beta} = e^{c_\alpha^\beta} \omega_\beta|_{U_\alpha \cap U_\beta}$$

  for locally constant functions c_α^β.
- **Maximality:** $\mathcal{U}$ maximal with these properties (or alternatively, equivalence classes)

Comparisons with symplectic

Locally, LCS = GCS, i.e. symplectic but forgetting scale.

Takeaway

Any symplectic definition or theorem which is $\mathbb{R}^+$-equivariant and can be stated/proved in a local-to-global manner is likely true for LCS manifolds.

The c_α^β are Čech cocyle with respect to open cover: obtain element of $H^1(M;\mathbb{R})$, called the **Lee class**.

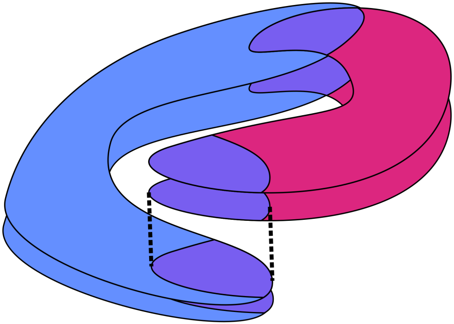
Geometrically: walk around M along loops, and symplectic form scales. Lee class is negative of holonomy in

$$\mathrm{Hom}(\pi_1(M),\mathbb{R}) \cong H^1(M;\mathbb{R}).$$

GCS iff Lee class = 0.

Locally conformal symplectic, definition 2

Glue $(U_\alpha, \omega_\alpha)$ and (U_β, ω_β) on regions where $\omega_\alpha = \omega_\beta$.



An **LCS manifold** is a flat principal $\mathbb{R}^+$-bundle with an $\mathbb{R}^+$-equivariant leafwise symplectic form.

LCS via differential forms

Pick a section s of the principal bundle: connection form is **Lee form** $\eta \in \mathcal{Z}^1(M)$ representing the Lee class. Recover tautological $\omega \in \Omega^2(M)$ with

$$\omega_p := (\omega_\alpha)_p, \qquad s(p) \in U_\alpha.$$

Locally

$$\omega|_{U_\alpha} = e^{g_\alpha}\omega_\alpha$$

with $\eta|_{U_\alpha} = dg_\alpha$.

$$d_\eta\omega := d\omega - \eta \wedge \omega = 0.$$

Locally conformal symplectic, definition 3

***Strict* LCS Manifold:** (M, η, ω)

- **Flatness of twisting:** $\eta \in \mathcal{Z}^1(M)$ (closed)
- **Non-degenerate:** $\omega \in \Omega^2(M)$ is non-degenerate
- **Twisted-closedness:**

$$d_\eta \omega := d\omega - \eta \wedge \omega = 0.$$

LCS manifold: Strict LCS manifold up to gauge equivalence

$$(\eta, \omega) \sim (\eta + df, e^f \omega).$$

Lichnerowicz-de Rham differential

Operator $d_\eta = d - \eta\wedge$ on differential forms builds a cochain complex. (See [Haller-Rybicki '99].)

- **Cohomology:** $H^*_\eta(M)$ (finite-dimensional)
- **Functoriality:** For $\phi\colon M \to N$, $\phi^*\colon H^*_\eta(N) \to H^*_{\phi^*\eta}(M)$
- **Cup product:** $H^i_\eta(M) \otimes H^j_{\eta'}(M) \to H^{i+j}_{\eta+\eta'}(M)$.
- **Künneth:** $H^i_\eta(M) \otimes H^j_{\eta'}(M') \cong H^{i+j}_{(\pi^*\eta+(\pi')^*\eta')}(M \times M')$.
- **Hodge theory:**

$$\Omega^k(M) = d_\eta\left(\Omega^{k-1}(M)\right) \oplus d^*_\eta\left(\Omega^{k+1}(M)\right) \oplus \mathcal{H}^k_\eta(M).$$

- **Poincaré duality:** If M closed, oriented,

$$(H^i_\eta(M))^* \cong H^{\dim M - i}_{-\eta}(M).$$

Comparing definitions

Guiding principles

- Principal bundle (or Atlas):
  - on-the-nose symplectic geometry built-in
  - Lee class is topological (negative holonomy)
  - topologically conceptual
- Differential forms:
  - twisted symplectic geometry
  - Lee class is geometric (represented by Lee form)
  - computational

Twisted cotangent bundles

Let $\beta \in \mathcal{Z}^1(X)$, $\pi \colon T^*X \to X$, $\lambda = \sum p_j dq_j$.

- $\eta = \pi^*\beta$
- $\omega = d_\eta \lambda = d\lambda - \eta \wedge \lambda$

Obtain LCS manifold $T^*_\beta X$.

Remark

When $\omega = d_\eta \lambda$, we call LCS structure **exact**.

- Atlas: $\omega_\alpha = d\lambda_\alpha$,

$$\lambda_\alpha|_{U_\alpha \cap U_\beta} = e^{c_\alpha^\beta} \lambda_\beta|_{U_\alpha \cap U_\beta}.$$

- Principal bundle: Leaves are equivariantly exact symplectic

LCS-ification

$(Y, \xi = \ker \alpha)$ contact

$(\mathbb{R} \times Y, d(e^t \alpha))$ symplectization

LCS-ification: LCS structure on $\mathbb{R}/T\mathbb{Z} \times Y$

- Atlas of open sets in $\mathbb{R} \times Y$ which inject under projection
- $(\eta, \omega) = (-dt, dt \wedge \alpha + d\alpha)$

Remark

This example is exact: $\omega = d_\eta \alpha$. In fact,

$$H^*_{-dt}(\mathbb{R}/T\mathbb{Z} \times Y) = 0,$$

so exactness was automatic.

Covering maps, generally

More generally, if

$$p\colon X \to Y$$

is a covering map, can pull back LCS structure on Y. Conversely, (X, ω) symplectic with deck transformations conformal symplectic, we obtain an LCS structure on Y.

Definition 4

LCS on M is GCS on $\widetilde{M}$ with conformal symplectic deck transformations.

Remark

Don't need to take $\widetilde{M}$, just $M_{[\eta]}$. LCS structure on Y is exact if and only if the GCS structure on $M_{[\eta]}$ is exact.

Non-exact examples

- GCS structure associated to any non-exact symplectic manifold (e.g. any closed symplectic manifold)
- [Banyaga '07]: Certain solvmanifolds
- LCS structures with S^2 'LCS submanifold'
  - [Apostolov-Dloussky '16]: Any compact complex surface has an LCS structure taming the complex structure; hence any compact complex surface with a rational curve has a non-exact LCS structure
  - h-principle: in dim ≥ 6, suffices to find homotopy class $[\phi\colon S^2 \to M]$ with $\phi^*[\omega] \neq 0 \in H^2_{\phi^*\eta}(S^2) = \mathbb{Z}$.

Isomorphism of LCS

Isomorphism: Diffeomorphism s.t.

- Atlas: On charts, conformal symplectic.
- Principal bundle: bundle isomorphism preserving leafwise symplectic form.
- Forms: $(\phi^*\eta', \phi^*\omega') \sim (\eta, \omega)$ (gauge-equivalence)

Remark

Everything is determined by the diffeomorphism itself, even if we have a principal bundle, because we have to send leaves to leaves.

Remark

Lee class is topological: $\phi^*[\eta'] = [\eta]$.

Lie algebra of automorphisms

Which vector fields generate LCS automorphisms?

Differential form perspective:

$$\begin{cases} \phi_t^*\eta = \eta + dg_t \\ \phi_t^*\omega = e^{g_t}\omega \end{cases}$$

Derivatives:

$$\mathcal{L}_X^\eta \omega := (d_\eta i_X + i_X d_\eta)\omega = c_X \omega$$

for c_X locally constant.

Extended Lee homomorphism

$$\mathcal{L}^{\eta}_X \omega = \boxed{d_\eta i_X \omega = c_X \omega}$$

Assignment $\mathfrak{symp} \to H^0(M)$ given by $X \mapsto c_X$ is the **extended Lee homomorphism** (independent under gauge equivalence, homomorphism of Lie algebras).

On $(\widetilde{M}, \widetilde{\omega})$ legitimately (globally conformal) symplectic,

$$\mathcal{L}_{\widetilde{X}} \widetilde{\omega} = c_X \widetilde{\omega}.$$

The sign of c_X tells whether you move up or down in the $\mathbb{R}^+$-fibers of the principal bundle.

Three flavors of infinitesimal automorphisms

- $\mathfrak{symp}$: general c_X
- $\mathfrak{symp}^{\text{flat}}$: $c_X = 0$, i.e. $i_X\omega$ is d_η-closed
  - Kernel of extended Lee homomorphism
  - Preserves leaves in the principal bundle
- $\mathfrak{ham}$: $i_X\omega$ is d_η-exact, i.e. $i_{X_H}\omega = d_\eta H$.
  - X_1 = **Lee vector field** (not gauge-invariant!)
  - If $[\eta] \neq 0$, then $H^0_\eta(M) = 0$, so X_H determines H unambiguously.

These are Lie algebras of corresponding (infinite-dimensional 'regular' Frechet) Lie groups. (See [Haller-Rybicki '99] or [Haller '02].)

Erlangen-type results

Theorem (Haller-Rybicki '99)

For a closed LCS manifold:

- $[\mathfrak{symp}, \mathfrak{symp}] = [\mathfrak{ham}, \mathfrak{ham}] = \mathfrak{ham}$
- *Ham is simple*

Furthermore, if we have two LCS structures:

- *a Lie algebra isomorphism between* $\mathfrak{ham}$ *is the derivative of an LCS-isomorphism*
- *a Lie group isomorphism between* Symp, $\mathrm{Symp}^{\mathrm{flat}}$, *or* Ham *comes from intertwining with an LCS-isomorphism*

Exploring the extended Lee homomorphism

If $c_X \neq 0$ (on every component), then LCS is exact:

$$\omega = d_\eta \left(\frac{i_X \omega}{c_X} \right).$$

Proposition

Exact LCS if and only if extended Lee homomorphism is surjective if and only if

$$H^0(M) \cong \mathfrak{symp}/\mathfrak{symp}^{\mathrm{flat}}.$$

Example

Closed GCS manifold (not exact) has extended Lee homomorphism $= 0$.

More on exactness

If $\omega = d_\eta \lambda$, the vector field Z_λ with

$$i_{Z_\lambda}\omega = \lambda$$

is the **Liouville vector field** of λ. (It is legitimately so when we work with atlases/principal bundles/universal covers, so this has a gauge-invariant formulation.)

- Z_λ has $c \equiv 1$.
- M connected: $\mathfrak{symp} = \mathfrak{symp}^{\mathrm{flat}} \oplus \langle Z_\lambda \rangle$.

Exact Examples

LCS-ification $(M = \mathbb{R}/T\mathbb{Z} \times Y, -dt, dt \wedge \alpha + d\alpha)$:

- **Liouville:** $Z_\alpha = \partial_t$
- **Lee:** $X_1 = R_\alpha$
- Because $H^2_\eta(M) = 0$, $\mathfrak{symp}^{\text{flat}} = \mathfrak{ham}$.

$T^*_\beta M = (T^*M, \pi^*\beta, d_{\pi^*\beta}\lambda)$:

- **Liouville:** $Z_\lambda = p\partial_p$
- **Lee:** $X_1 = -\beta(\partial_q) \cdot \partial_p$

Refinement of Tischler

For exact LCS $(\eta, d_\eta\lambda)$, the form $d\lambda$ (not gauge-invariant) either has kernel of dim 0 or 2; in the latter case, $\eta \neq 0$ and

$$\ker(d\lambda) = \mathrm{span}\langle Z_\lambda, X_1\rangle.$$

If $d\lambda$ degenerate everywhere (**LCS structure of the first kind**), then M admits nowhere zero closed 1-form η.

Theorem (Tischler '70)

A closed manifold with a nowhere zero closed 1-form fibers over S^1.

Theorem (Bazzoni–Marrero '06)

If $(M, \eta, d_\eta\lambda)$ is closed of the first kind and $\ker\eta$ has a closed leaf L, then $(M, \eta, d_\eta\lambda)$ is a mapping torus of $(L, \lambda|_L)$ with monodromy a strict contactomorphism.

Towards the Moser trick

The Moser trick still works as usual, so long as you know the analogue for 'cohomologous symplectic forms':

Exact homotopy of LCS structures on M:

- Differential forms: (η_t, ω_t) with $\eta_t = \eta_0 + df_t$ and $\frac{d}{dt}(e^{f_t}\omega_t)$ is d_{η_t}-exact.
- Principal bundle: Homotopy of leafwise symplectic form ω_t with $\frac{d}{dt}\omega_t - c_t\omega_t$ exact with respect to the leafwise differential for some locally constant functions c_t on M.

Moser trick

Theorem (Bande–Kotschick '09, rephrased)

Exact homotopies of LCS structures $\mathcal{M}_t$ on a closed manifold M can be realized by a family of LCS isomorphisms

$$\phi_t \colon \mathcal{M}_0 \xrightarrow{\sim} \mathcal{M}_t$$

with $\phi_0 = \mathrm{id}$.

They work with differential forms; could also use equivariant standard Moser trick (WLOG may assume $c_t \equiv 0$ using fiberwise translation in principal bundle).

Submanifolds and neighborhoods

Isotropic, Lagrangian, coisotropic, symplectic, contact-type submanifolds: all make sense in LCS geometry (their definitions are invariant under scaling the symplectic form).

(Relative) Moser trick gives neighborhood theorems. (See [Lê–Oh '16] and [Otiman–Stanciu '17])

Example: Lagrangian neighborhood theorem

Suppose $L \subset (M, \eta, \omega)$ is a Lagrangian submanifold of an LCS manifold. Let $\beta = \eta|_L$. Then

$$\mathrm{Op}_M(L) \cong \mathrm{Op}_{T^*_\beta L}(L).$$

General homotopies?

Can we homotope the Lee class? If we have a homotpy (η_t, ω_t) of strict LCS forms, then

$$\begin{aligned} 0 &= \frac{d}{dt} d_{\eta_t}\omega_t \\ &= d_{\eta_t}\dot{\omega}_t - \dot{\eta}_t \wedge \omega_t \end{aligned}$$

Obstruction: $[\dot{\eta}_t \wedge \omega_t] = [\dot{\eta}_t] \wedge [\omega_t] \in H^3_{\eta_t}(M)$.

Extra Slide 1

Thank you! Questions?

Extra Slide 2

Extra Slide 3