Locally conformal symplectic structures of the first kind

Giovanni Bazzoni, Università degli Studi dell'Insubria

j. w. with D. Angella, J.C. Marrero, M. Parton

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Plan of the presentation

1 Locally conformal symplectic structures

2 Structure results for lcs manifolds of the first kind

3 Existence of lcs structures of the first kind

4 Locally conformal symplectic Lie groups and Lie algebras

5 An example

Goal of the talk

Goal

In this talk we consider locally conformal symplectic structures of the first kind. We concentrate on the structure of compact lcs manifolds. Special attention is devoted to left-invariant lcs structures of the first kind on Lie groups. This shows a nice interplay of geometric structures:

- locally conformal symplectic
- contact
- symplectic

Locally conformal symplectic structures

Definition

A **locally conformal symplectic (lcs) form** on a manifold M^{2n} ($n \geq 2$) consists of a non-degenerate 2-form $\omega \in \Omega^2(M)$ for which there exists a (closed) 1-form $\theta \in \Omega^1(M)$, the Lee form, such that

$$d\omega = \theta \wedge \omega$$

- θ is completely determined by ω
- $\omega^n \neq 0 \Rightarrow d\theta = 0$ for $n \geq 3$; we assume $\theta \neq 0$
- If $\{U_\alpha\}$ is an open cover of M with U_α contractible, $\forall \alpha \ \exists \ f_\alpha \in C^\infty(U_\alpha)$ such that $\theta_\alpha := \theta|_{U_\alpha} = df_\alpha$. Then $d(e^{-f_\alpha}\omega|_{U_\alpha}) = 0$
- Every lcK manifold has an underlying lcs structure

The conformal class of an lcs structure

Definition

Let ω be an lcs form on a manifold M. The **conformal class** of ω is

$$\mathcal{C}(\omega) = \{\omega' \in \Omega^2(M) \mid \omega' = e^f \omega \text{ for } f \in C^\infty(M)\}\,.$$

- The Lee form of $\omega' = e^f \omega \in \mathcal{C}(\omega)$ is $\theta' = \theta + df$
- ω is **globally conformal symplectic** if there exists $\omega' \in \mathcal{C}(\omega)$ with exact Lee form

Lichnerowicz cohomology

Lichnerowicz cohomology

If M is a manifold and $\theta \in \Omega^1(M)$ is closed, the *Lichnerowicz differential* is

$$d_\theta \colon \Omega^*(M) \to \Omega^{*+1}(M), \quad d_\theta\alpha = d\alpha - \theta \wedge \alpha$$

Then $d_\theta^2 = 0$ and the *Lichnerowicz cohomology*

$$H_\theta^*(M) = \frac{\ker(d_\theta \colon \Omega^*(M) \to \Omega^{*+1}(M))}{d_\theta(\Omega^{*-1}(M))}$$

is defined.

Lichnerowicz cohomology and lcs structures

Key observation

$\omega \in \Omega^2(M^{2n})$ with $\omega^n \neq 0$ defines a lcs structure $\Leftrightarrow \exists\ \theta \in \Omega^1(M)$ (closed) with $d_\theta \omega = 0$, i.e. if $[\omega]_\theta \in H^2_\theta(M)$.

Let ω be an lcs form with Lee form θ on a manifold M. One can use Lichnerowicz cohomology to distinguish lcs structures, as follows:

Definition

An lcs form ω on a manifold M is **exact** if $\exists\ \eta \in \Omega^1(M)$ such that $d_\theta \eta = \omega$, that is, if $[\omega]_\theta = 0 \in H^2_\theta(M)$; it is **non exact** otherwise.

Being exact is an invariant of the conformal class of the lcs structure: the "primitive" of $\omega' = e^f \omega \in \mathcal{C}(\omega)$ is $\eta' = e^f \eta$.

Exact lcs structures

Exact lcs structures are not “too” hard to find. In fact, they exist:

- on open manifolds (Fernandes-Frejlich(-Martínez Torres))
- on closed manifolds M with $H^1(M;\mathbb{R}) \neq 0$ and having a non-degenerate 2-form
  (Eliashberg-Murphy, Bertelson-Meigniez)

Automorphisms of lcs structures

Let (M, ω) be an lcs manifold with Lee form θ.

Definition

- A vector field $X \in \mathfrak{X}(M)$ is an **(infinitesimal) automorphism** of (M, ω) if $\mathcal{L}_X \omega = 0$. We write $X \in \mathfrak{X}(M, \omega)$ in this case.
- The **Lee morphism** is $\ell \colon \mathfrak{X}(M, \omega) \to \mathbb{R}$, $\ell(X) = \theta(X)$.

Using automorphisms, one can distinguish lcs structures as follows:

Definition

ω is **of the first kind** if the Lee morphism is non-zero; **of the second kind** otherwise.

Lcs structures of first and second kind

If ω is an lcs structure of the first kind, there exists a unique $U \in \mathfrak{X}(M,\omega)$ such that $\ell(U) = 1$. U is the **anti-Lee field**.

Some remarks

- An lcs structure of the first kind is exact; the converse is not true
- let M be compact with an lcs structure of the first kind and let θ be the Lee form; then $[\theta] \neq 0 \in H^1_{dR}(M)$, and $b_1(M) \geq 1$
- the lcs structure underlying a Vaisman structure is of the first kind
- being of the first kind *is not* an invariant of the conformal class of an lcs structure
- there are no general existence results for lcs structures of the first kind

Locally conformal symplectic structures of the first kind

Theorem [Vaisman, 1984]

- Let ω be a lcs structure of the first kind on M^{2n}, with Lee form θ, and let $U \in \mathfrak{X}(M,\omega)$ be the anti-Lee field. Set $\eta := -\imath_U\omega$. Then
  - $d\eta$ has rank $2n-2$
  - $\theta \wedge \eta \wedge (d\eta)^{n-1} \neq 0$
  - $\omega = d_\theta \eta$
- Suppose M^{2n} is endowed with two nowhere vanishing 1-forms θ and η with $d\theta = 0$, $\mathrm{rank}(d\eta) < 2n$ and $\theta \wedge \eta \wedge (d\eta)^{n-1} \neq 0$. Set $\omega := d_\theta \eta$. Then ω is an lcs structure of the first kind on M, with Lee form θ.

Mapping tori and lcs structures of the first kind

Let P be a smooth manifold and let $\varphi\colon P \to P$ be a diffeomorphism. $\mathbb{Z}$ acts on $P \times \mathbb{R}$ by $(m,(p,t)) \mapsto (\varphi^m(p), t+m)$; the action is free and properly discontinuous, hence the quotient space P_φ is a smooth manifold. $P \to P_\varphi \to S^1$ is a fiber bundle.

Definition

P_φ is the **mapping torus** of P and φ.

Proposition

Let (P,η) be a contact manifold and let $\varphi\colon P \to P$ be a strict contactomorphism, i.e. a diffeomorphism with $\varphi^*\eta = \eta$. Then P_φ has a natural lcs structure of the first kind.

Banyaga's theorem

Theorem [Banyaga, 2002]

Let ω be an lcs structure of the first kind on a compact manifold M. Then there exist a contact manifold (P, η) and a strict contactomorphism $\varphi\colon P \to P$ such that M is diffeomorphic to P_φ

- A compact manifold fibering over S^1 is diffeomorphic to a mapping torus
- The diffeomorphism $M \cong P_\varphi$ destroys the given lcs structure of M.

Structure results for lcs manifolds of the first kind

Theorem [−, Marrero, 2018]

Let M be a compact connected manifold endowed with a lcs structure of the first kind (ω, θ), let U be the anti-Lee vector field and $\eta = -\imath_U \omega$. Suppose

- $\mathcal{F} := \{\theta = 0\}$ has a compact leaf L with inclusion $i \colon L \hookrightarrow M$.

Then

- $i^* \eta$ is a contact form on L
- there exists a strict contactomorphism $\phi \colon L \to L$
- the flow of U induces an isomorphism between L_ϕ and M.

Structure results for lcs manifolds of the first kind

Definition

Let ω be a lcs structure on M. The **Lee vector field** is $V \in \mathfrak{X}(M)$, defined by $\imath_V \omega = \theta$.

In fact, $V \in \ker \ell \subset \mathfrak{X}(M, \omega)$.

Theorem [Vaisman, 1984]

Let ω be an lcs structure of the first kind on M compact; let U and V be the anti-Lee and Lee field. The distribution spanned by U and V integrates to a foliation $\mathcal{G}$ on M. Under certain regularity assumptions, the space of leaves $N := M/\mathcal{G}$ has a symplectic structure and $p\colon M \to N$ is a principal T^2-bundle.

Martinet theorem and some equivariant extensions

Recall the following results on the existence of contact forms:

Theorem [Martinet, 1971]

On a closed, oriented manifold of dimension 3 there exists a contact form.

Theorem [Kamishima-Tsuboi, 1991; Niederkrüger, 2005]

Let L be an oriented closed manifold of dimension 3 and let an orientation-preserving action of S^1 on L be given. Then there exists an S^1-invariant contact form on L.

Theorem [Carlson, 2012]

Let L be an oriented closed manifold of dimension 3 and suppose that a finite group Γ of prime order acts on L preserving the orientation. Then there exists a Γ-invariant contact form on L.

Existence of lcs structures of the first kind

For the existence of lcs structures of the first kind, we have:

Theorem [–, Marrero, 2018]

Let M be oriented, and 4-dimensional, θ a closed 1-form on M without singularities and L a compact leaf of $\mathcal{F} := \{\theta = 0\}$.

- If M is not compact, it admits a gcs structure of the first kind, globally conformal to the symplectization of L.
- If M is compact then M may be identified with a mapping torus L_ϕ. Moreover:
  - If there exists an action $\psi \colon S^1 \times L \to L$ which preserves the orientation induced on L and $\phi = \psi_\lambda$, for some $\lambda \in S^1$, then M admits a lcs structure of the first kind.
  - If $\phi \colon L \to L$ preserves the orientation induced on L, the discrete subgroup of transformations of M, $\Gamma = \{\phi^k \mid k \in \mathbb{Z}\}$, is finite and its order is prime, then M also admits an lcs structure of the first kind.

Locally conformal symplectic Lie groups

Definition

A Lie group G of dimension $2n$ ($n \geq 2$) is **locally conformal symplectic (lcs)** if there exists $\omega \in \Omega^2(G)^G$, with $\omega^n \neq 0$, such that there exists $\theta \in \Omega^1(G)^G$ such that

$$d_\theta \omega := d\omega - \theta \wedge \omega = 0\,.$$

We call G an **lcs Lie group**.

Definition

An lcs structure ω on G is **of the first kind** if there exists $U \in \mathfrak{X}(G)^G$, the **anti-Lee vector field** with $\mathcal{L}_U \omega = 0$ and $\theta(U) = 1$. Then $\eta := -\imath_U \omega \in \Omega^1(G)^G$ satisfies $\omega = d_\theta \eta$.

Locally conformal symplectic Lie algebras

Definition

Let $\mathfrak{g}$ be a Lie algebra of dimension $2n$ ($n \geq 2$). A **locally conformal symplectic (lcs)** on $\mathfrak{g}$ is $\omega \in \Lambda^2\mathfrak{g}^*$, with $\omega^n \neq 0$, for which there exists $\theta \in \mathfrak{g}^*$ such that

$$d_\theta\omega := d\omega - \theta \wedge \omega = 0\,.$$

We call $\mathfrak{g}$ an **lcs Lie algebra**.

Definition

An lcs structure ω on $\mathfrak{g}$ is **of the first kind** if there exists $U \in \mathfrak{g}$, the **anti-Lee vector** with $\mathcal{L}_U\omega = 0$ and $\theta(U) = 1$. Then $\eta := -\imath_U\omega \in \mathfrak{g}^*$ satisfies $\omega = d_\theta\eta$.

Symplectic and contact structures on Lie algebras

Definition

Let $\mathfrak{s}$ be a Lie algebra, $\dim \mathfrak{s} = 2n$. A **symplectic structure** on $\mathfrak{s}$ is $\sigma \in \Lambda^2\mathfrak{s}^*$, closed and non-degenerate, i.e. $\sigma^n \neq 0$

Definition

Let $\mathfrak{h}$ be a Lie algebra, $\dim \mathfrak{h} = 2n-1$. A **contact structure** on $\mathfrak{h}$ is $\eta \in \mathfrak{h}^*$ such that $\eta \wedge (d\eta)^{n-1} \neq 0$. The Reeb vector is $R \in \mathfrak{h}$, defined by $\imath_R d\eta = 0$ and $\eta(R) = 1$.

A few facts

- A unimodular symplectic Lie algebra is solvable
- The only semisimple Lie algebras admitting contact structures are $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$

Reductive lcs Lie algebras

Proposition [Angella, –, Parton, 2020]

Let $(\mathfrak{g}, \omega, \theta)$ be a reductive lcs Lie algebra. Then either $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathbb{R}$ or $\mathfrak{g} = \mathfrak{sl}(2) \oplus \mathbb{R}$, and the lcs structure is of the first kind.

$\mathfrak{su}(2) \oplus \mathbb{R}$ is the Lie algebra of $S^3 \times S^1$.

Where to look for?

We concentrate therefore on **nilpotent** and, more generally, **solvable** lcs Lie algebras.

Semidirect products and central extensions

Given a Lie algebra $\mathfrak{g}$, there are two natural procedures to endow $\mathfrak{g} \oplus \mathbb{R}$ with a Lie algebra structure.

Definition

- Given a derivation $D\colon \mathfrak{g} \to \mathfrak{g}$, the **semidirect product** Lie algebra $\mathfrak{g} \rtimes_D \mathbb{R}$ is the vector space $\mathfrak{g} \oplus \mathbb{R}$ with Lie algebra structure

$$[(X,0),(Y,0)] = [X,Y]_{\mathfrak{g}}, \quad [(0,1),(X,0)] = D(X).$$

- Given a cocycle $\sigma \in \Lambda^2\mathfrak{g}^*$, the **central extension** Lie algebra $\mathbb{R} \circledcirc_\sigma \mathfrak{g}$ is the vector space $\mathbb{R} \oplus \mathfrak{g}$ with Lie algebra structure

$$[(0,X),(0,Y)] = (\sigma(X,Y),[X,Y]_{\mathfrak{g}}), \quad [(1,0),(X,0)] = 0.$$

Lcs Lie algebras from contact Lie algebras

Proposition [–, Marrero, 2018]

Lcs Lie algebras of the first kind are in 1-1 correspondence with contact Lie algebras endowed with a contact derivation.

- If $(\mathfrak{h}, \eta)$ is a contact Lie algebra and $D\colon \mathfrak{h} \to \mathfrak{h}$ is a contact derivation, i.e. $D^*\eta = 0$, then $\mathfrak{g} = \mathfrak{h} \rtimes_D \mathbb{R}$ is endowed with a lcs structure of the first kind.
- Suppose $(\mathfrak{g}, \omega, \theta)$ is an lcs Lie algebra of the first kind; let U be the anti-Lee field and $\eta = -\imath_U \omega$. Set $\mathfrak{h} = \ker(\theta)$ and let η be the restriction of η to $\mathfrak{h}$. Then $(\mathfrak{h}, \eta)$ is a contact Lie algebra, endowed with a contact derivation D. Moreover, $\mathfrak{g} \cong \mathfrak{h} \rtimes_D \mathbb{R}$.

Contact Lie algebras from symplectic Lie algebras

If $(\mathfrak{h}, \eta)$ is a contact Lie algebra, then:

- $\mathcal{Z}(\mathfrak{h}) = 0$, or
- $\mathcal{Z}(\mathfrak{h}) = \langle R \rangle$

We consider contact Lie algebras with non-trivial center.

Proposition

Contact Lie algebras with non-trivial center are in 1-1 correspondence with central extensions of symplectic Lie algebras.

- If $\sigma \in \Lambda^2 \mathfrak{s}^*$ is a symplectic structure on $\mathfrak{s}$, the central extension $\mathfrak{h} = \mathbb{R} \odot_\sigma \mathfrak{s}$ has a contact structure
- if $(\mathfrak{h}, \eta)$ is a contact Lie algebra with Reeb vector R such that $\mathcal{Z}(\mathfrak{h}) = \langle R \rangle$, then the Lie algebra $\mathfrak{s} = \mathfrak{h}/\langle R \rangle$ has a symplectic structure σ and $\mathfrak{h} \cong \mathbb{R} \odot_\sigma \mathfrak{s}$.

Lcs extensions

symplectic $\leftrightsquigarrow$ contact $\leftrightsquigarrow$ lcs of the first kind

Definition

A derivation D of $(\mathfrak{s},\sigma)$ is **symplectic** if

$$\sigma(DX,Y)+\sigma(X,DY)=0 \quad \forall\, X,Y\in\mathfrak{s}.$$

Theorem [–, Marrero, 2018]

There exists a 1-1 correspondence between lcs Lie algebras of the first kind $(\mathfrak{g},\omega,\theta)$ of dimension $2n+2$ with central Lee vector and symplectic Lie algebras $(\mathfrak{s},\sigma)$ of dimension $2n$ endowed with a symplectic derivation.

The correspondence

Take a symplectic Lie algebra $(\mathfrak{s}, \sigma)$ and a symplectic derivation $D\colon \mathfrak{s} \to \mathfrak{s}$. On $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{s} \oplus \mathbb{R}$ define the Lie bracket

$$[(a, X, \alpha), (b, Y, \beta)]_{\mathfrak{g}} = (\sigma(X, Y), \alpha D(Y) - \beta D(X) + [X, Y]_{\mathfrak{s}}, 0). \tag{1}$$

Then $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}})$ is a Lie algebra. Define $\theta, \eta \in \mathfrak{g}^*$ by

$$\theta(a, X, \alpha) = \alpha \quad \text{and} \quad \eta(a, X, \alpha) = a. \tag{2}$$

Setting $\omega = d_\theta \eta$, (ω, θ) is a lcs structure of the first kind on $\mathfrak{g}$ with central Lee vector $V = (1, 0, 0) \in \mathfrak{g}$.

Definition

$\mathfrak{g} = \mathbb{R} \oplus \mathfrak{s} \oplus \mathbb{R}$ endowed with the Lie algebra structure (1) and the lcs structure of the first kind (2) is the **lcs extension** of $(\mathfrak{s}, \sigma)$ by the derivation D.

Symplectic double extensions (Dardié, Medina, Revoy)

You need:

- a symplectic Lie algebra $(\mathfrak{s}_0, \sigma_0)$
- a derivation $D_0 \colon \mathfrak{s}_0 \to \mathfrak{s}_0$
- a vector $Z_0 \in \mathfrak{s}_0$

Recipe:

- $D_0^* \sigma_0 \in Z^2(\mathfrak{s}_0, \mathbb{R})$; put $\mathfrak{h}_0 = \mathbb{R} \circledcirc_{D_0^* \sigma_0} \mathfrak{s}_0$
- the linear map $A \colon \mathfrak{h}_0 \to \mathfrak{h}_0$, $(a, X) \mapsto (-\sigma_0(Z_0, X), -D_0(X))$ is a derivation $\Leftrightarrow d(i_{Z_0}\sigma_0) = -(D_0^*)^2 \sigma_0$
- assuming it is so, $\mathfrak{s} = \mathfrak{h}_0 \rtimes_A \mathbb{R}$ is a symplectic Lie algebra with symplectic form

$$\sigma((a, X, \alpha), (b, Y, \beta)) = a\beta - \alpha b + \sigma_0(X, Y)$$

Symplectic double extensions

Definition

$(\mathfrak{s}, \sigma)$ is the **symplectic double extension** of $(\mathfrak{s}_0, \sigma_0)$ by D_0 and Z_0.

Theorem [Medina-Revoy, 1989; Dardié-Medina, 1991]

Every nilpotent symplectic Lie algebra can be obtained by a sequence of symplectic double extensions starting from $\mathbb{R}^2$.

Lcs nilpotent Lie algebras

Facts

Suppose $\mathfrak{g}$ is a nilpotent Lie algebra. One has:

- $\mathcal{Z}(\mathfrak{g}) \neq 0$
- if (ω, θ) is a lcs structure on $\mathfrak{g}$, then
  - $\exists\ \eta \in \mathfrak{g}^*$ with $\omega = d_\theta \eta \Rightarrow$ every lcs structure on $\mathfrak{g}$ is of the first kind
  - $V \in \mathcal{Z}(\mathfrak{g})$

Theorem [–, Marrero, 2018]

- Every lcs nilpotent Lie algebra of dimension $2n+2$ may be obtained as the lcs extension of a $2n$-dimensional symplectic nilpotent Lie algebra $\mathfrak{s}$ by a symplectic nilpotent derivation.
- In turn, $\mathfrak{s}$ may obtained from $\mathbb{R}^2$ by a sequence of $n-1$ symplectic double extensions by nilpotent derivations.

6D lcs nilpotent Lie algebras

Lie algebra	θ	η	Symplectic	Complex
$\mathfrak{h}_3$	e^5	e^6	$\times$	$\checkmark$
$\mathfrak{h}_9$	e^4	e^6	$\checkmark$	$\checkmark$
$\mathfrak{h}_{18}$	e^3	e^6	$\times$	$\times$
$\mathfrak{h}_{19}^+$	e^1	e^6	$\times$	$\times$
$\mathfrak{h}_{19}^-$	e^1	e^6	$\times$	$\checkmark$
$\mathfrak{h}_{20}$	e^2	e^6	$\times$	$\times$
$\mathfrak{h}_{22}$	e^3	e^6	$\checkmark$	$\times$
$\mathfrak{h}_{24}$	e^3	e^6	$\checkmark$	$\times$
$\mathfrak{h}_{27}$	e^2	e^6	$\checkmark$	$\times$
$\mathfrak{h}_{31}$	e^1	e^6	$\times$	$\times$
$\mathfrak{h}_{32}$	e^1	e^6	$\times$	$\times$

An example

- Start with the abelian Lie algebra $\mathbb{R}^2$ with symplectic form σ and symplectic derivation

$$D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

- let $\mathfrak{g}$ be the lcs extension of $(\mathbb{R}^2, \sigma)$ by D: $\mathfrak{g} = (\mathbb{R} \circledcirc_\sigma \mathbb{R}^2) \rtimes_D \mathbb{R}$
- G, the unique connected, simply connected Lie group with Lie algebra $\mathfrak{g}$, has an lcs (in fact, gcs) structure of the first kind
- the structure constants of $\mathfrak{g}$ are rational numbers. By a results of Malcev, there exists a lattice $\Gamma \subset G$. Then $N = \Gamma \backslash G$ is a compact nilmanifold with an lcs structure of the first kind, not gcs.

Grazie mille!

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