Twisted structures on complex non-Kähler surfaces

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Taming LCS

Compact complex curves are all projective, therefore have a lot of holomorphic objects, admit a Kähler metric.
In dimension 2, situation changes: some K3 surfaces have no holomorphic foliation, Inoue-Bombieri have no curves. Some Inoue-Bombieri have no LCK structure.

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In dimension 2, situation changes: some K3 surfaces have no holomorphic foliation, Inoue-Bombieri have no curves. Some Inoue-Bombieri have no LCK structure.
Is there a structure common for all compact complex surfaces, in particular for non Kähler surfaces ?
$S = (M^4, J)$ compact complex surface.

Definition (Taming LCS)

A **taming** LCS structure is $\omega \in \Omega^2(M)$ s.t.

$$(\omega)^{1,1} > 0, \qquad d_\alpha \omega := d\omega - \alpha \wedge \omega = 0, \qquad d\alpha = 0.$$

α is called the **Lee form**.

First existence result

Example (LCK structures)

A LCK structure is a LCS structure ω s.t. $\omega = \omega^{1,1}$;
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Theorem 1 (Apostolov-D 2016)

Any compact complex surface S admits taming LCS.

Remark

- (Siu, 80's; Buchdahl and Lamari 2000's) S supports a Kähler metric ($[\alpha] = 0 \in H^1_{dR}(M)$) iff $b_1(S)$ is even.
- (Belgun, 98) $\exists$ non-Kähler surfaces with no LCK structures.

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It is a different family of LCS structures

Cannot tame because of rational curves. Many non-Kähler surfaces do admit rational curves. For example Kato surfaces.

Moduli space of LCS taming

Theorem 2 (Apostolov-D. 2019)

Suppose S with $b_1(S) = 1$ (e.g. Class VII). Then

$$\tau(S) := \left\{ [\alpha] \in H^1_{dR}(M) \,|\, \alpha \text{ Lee form of a taming LCS} \right\}$$

is either an open subset of $]-\infty, 0[\subset \mathbb{R} \simeq H^1_{dR}(S)$ or is a single point $\{pt\}$ in $]-\infty, 0[$. In the latter case S is obtained by blow-ups of an Inoue-Bombieri surface.

Context for Theorems 1 and 2

Class VII surfaces: $b_1(S) = 1$ and $\dim_{\mathbb{C}} \left(H^0(S, K_S^{\otimes m}) \right) = 0$ for all $m \in \mathbb{N}^\star$.

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i.e. $\exists\ \mathbb{S}^3 \subset U \subset S$ s.t. $S \setminus \mathbb{S}^3$ is connected and
$(\mathbb{S}^3 \subset U, J) \cong \mathbb{S}^3 \subset U_0 \subset \mathbb{C}^2$. (Ma.Kato)

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When $b_2(S) > 0$ a minimal surface with a GSS is called Kato surface
If true $\Rightarrow$ end of Kodaira classification $\Rightarrow$ up to finite cover $S \cong (\mathbb{S}^1 \times \mathbb{S}^3)k\sharp\overline{\mathbb{C}P^1}$.

Context for Theorems 1 and 2

A.Teleman proved that for $b_2 = 1, 2, 3$ there is at least one rational curve, so it ends the classification for $b_2(S) = 1$.

Ideas for the proof of Theorem 1 & 2

Question

Is it possible to construct other objects with these LCS taming structures ? LCK ? currents ? psh functions ?

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Theorem (M.Brunella)

Every Kato surface admits a LCK structure.

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Sketch of proof:

1. Flat (holomorphic) line bundles and degree

- $\alpha \in \Omega^1(M)$, $d\alpha = 0$ and $L_\alpha := \mathbb{R} \times M$ with $\nabla^\alpha := d + \alpha$ a flat connection:

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- $\mathcal{L}_\alpha = \mathbb{C} \times M$ with holomorphic structure from ∇^α. For the hermitian structure $|\cdot|$ on $\mathbb{C} \times M$ the Chern curvature

$$\rho^\alpha = -\frac{1}{2} dd^c \log e^{-2f_i} = dJ\alpha \in \Omega^{1,1}(M, J).$$

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- (Gauduchon) $(\mathcal{L}, h)$ holomorphic Hermitian line bundle and g Hermitian metric on S such that $F = g(J., .)$ satisfies $dd^c F = 0$ (aways exists) $\Rightarrow$

$$\deg_g(\mathcal{L}) = \frac{1}{2\pi} \int_S \rho^{\nabla^h} \wedge F$$

independent of the Chern connection ∇^h on $\mathcal{L}$.

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- $\mathcal{L}_\alpha := \mathbb{C} \times M$ with holomorphic structure from ∇^α. For the hermitian structure $|\cdot|$ on $\mathbb{C} \times M$ the Chern curvature is

$$\rho^\alpha = -\frac{1}{2} dd^c \log e^{-2f_i} = dJ\alpha \in \Omega^{1,1}(M, J).$$

- (Gauduchon) $\exists\ g$ Hermitian metric with $dd^c F = 0$.

$$\deg_g(\mathcal{L}_\alpha) = \frac{1}{2\pi}\int_S \rho^\alpha \wedge F = \frac{1}{2\pi}\int_S (dJ\alpha) \wedge F = \frac{1}{2\pi}\int_S \langle dJ\alpha, F\rangle_g dV_g$$
$$= -\frac{1}{2\pi}\int_S \langle \alpha, Jd^*_g F\rangle_g dV_g = -\frac{1}{2\pi}\int_S \langle \alpha_h, (Jd^*_g F)_h\rangle_g dV_g.$$

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1. Flat (holomorphic) line bundles and degree

- $\alpha \in \Omega^1(M)$, $d\alpha = 0$ and $L_\alpha := \mathbb{R} \times M$ with $\nabla^\alpha := d + \alpha$ a flat connection;
- $\mathcal{L}_\alpha = \mathbb{C} \times M$ with holomorphic structure from ∇^α.
- $\forall$ g Hermitian metric with $dd^c F = 0$.

$$\deg_g(\mathcal{L}_\alpha) = -\frac{1}{2\pi}\int_S \langle \alpha_h, (Jd_g^* F)_h\rangle_g dV_g = -\frac{1}{2\pi}\int_S \langle \alpha_h, \theta_h\rangle_g dV_g.$$

- (Gauduchon) $(Jd_g^* F)_h = 0 \Leftrightarrow b_1(S)$ even.

Conclusion: If $b_1(S) = 1$ then $\mathrm{sign}(\deg_g(\mathcal{L}_\alpha))$ does not depend on the Gauduchon metric and $\deg_g(\mathcal{L}_\alpha) = 0$ iff $[\alpha] = 0$.

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Ideas for the proof of Theorems 1 & 2

If $a \in \tau(S)$, let $\alpha \in a$ such that $F = \omega^{(1,1)}$ is Gauduchon and $g = F(., J)$ then for $\mathcal{L}_\alpha$,

$$\deg_g(\mathcal{L}_\alpha) = -\frac{1}{2\pi}\int_M \|\alpha_h\|^2 v_g < 0$$

Remark: We have an isomorphism of complexes between

$$\cdots \stackrel{d_\alpha}{\to} \mathcal{E}^{k-1}(M,\mathbb{R}) \stackrel{d_\alpha}{\to} \mathcal{E}^k(M,\mathbb{R}) \stackrel{d_\alpha}{\to} \cdots$$

and

$$\cdots \stackrel{d_{L_{-\alpha}}}{\to} \mathcal{E}^{k-1}(M, L_{-\alpha}) \stackrel{d_{L_{-\alpha}}}{\to} \mathcal{E}^k(M, L_{-\alpha}) \stackrel{d_{L_{-\alpha}}}{\to} \cdots$$

$\omega \to (e^{-f_i}\omega_{|U_i})$ therefore

$$H^k_\alpha(M,\mathbb{R}) \simeq H^k_{d_{L_{-\alpha}}}(M, L_{-\alpha})$$

It explains the title “twisted geometry”

Ideas for the proof of Theorems 1 & 2

2. Taming LCS and the degree:

Lemma 1

let ω a LCS which tames S and $F = \omega^{(1,1)}$, then

$$d_\alpha d_\alpha^c F = 0$$

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Proof: develop $d_\alpha \omega$ with $\omega = F + \omega^{(2,0)} + \omega^{(0,2)}$.

Lemma 2

$$d_\alpha d_\alpha^c F = 0 \iff d_g^*(\theta - \alpha) + g(\theta - \alpha, \alpha) = 0$$

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Proof: Take Hodge $\star$ of $d_\alpha d_\alpha^c F$.

Ideas for the proof of Theorems 1 & 2

Lemma 3

$S = (M, J)$, $a \in H^1_{dR}(M, \mathbb{R})$, $c = [g]$ conformal class and g Gauduchon. We have equivalence:

(i) $\forall \tilde{g} \in c$, $\exists \tilde{\alpha} \in a$ s.t. $d_{\tilde{\alpha}} d^c_{\tilde{\alpha}} \tilde{F} = 0$,

(ii) $\exists \psi > 0$ on M which satisfies

$$\mathbb{L}_{g,a}(\psi) = \Delta_g(\psi) - g(\theta^g - 2a^g_h, d\psi) + g(\theta^g - a^g_h, a^g_h) = 0$$

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Proof: Write $\alpha = a^g_h - d \log \psi$, where $\psi > 0$ in lemma 2.

Ideas for the proof of Theorems 1 & 2

Lemma 4

For every smooth $\psi > 0$,

$$\int_M \frac{\mathbb{L}_{g,a}(\psi)}{\psi} v_g = \int_M g(\theta_h^g, a_h^g) - \int_M (\|a_h^g\|^2 + \frac{1}{\psi^2}\|d\psi\|_g^2) v_g$$

In particular if ψ is a solution

$$\int_M g(a_h^g, \theta_h^g) v_g = \int_M (\|a_h^g\|_g^2 + \frac{1}{\psi^2}\|d\psi\|^2) v_g > 0$$

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Proof: difficult part, elliptic but not self-adjoint differential equation.
(Perron's principle): $\exists$ solution $L(\psi) = 0$ with $\psi > 0$ iff the principal eigenvalue $\lambda(L) = 0$;

Ideas for the proof of Theorems 1 & 2

For $b_1(S) = 1$,

$$\tau(S) \subset]-\infty, 0[$$

Moreover, $H^1_{dR}(M, \mathbb{R}) \simeq \mathbb{R}$ may be endowed with the order relation

$$a > b \iff deg_g(\mathcal{L}_\alpha \otimes \mathcal{L}^\star_\beta) = deg_g(\mathcal{L}_{a-b}) = \deg_g(\mathcal{L}_\alpha) - deg_g(\mathcal{L}_\beta) > 0$$

Corollary

If α is a Lee form of a taming LCS and $[\alpha] \neq 0 \Rightarrow H^0(S, \mathcal{L}_\alpha) = 0$.

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Corollary

If α is a Lee form of a taming LCS and $[\alpha] \neq 0 \Rightarrow H^0(S, \mathcal{L}_\alpha) = 0$.

Proposition

For $S \in VII_0^+$, $\tau(S)$ is an open not empty subset of $]-\infty, 0[$.

Proof: deformation of the structure of Goto type.

From LCS to twisted currents

How to use these LCS taming structures to obtain new objects ?

Let $\mathcal{D}'_k(M,\mathbb{R})$ be the currents of dimension k on M.

$$d_\alpha : \mathcal{D}'_k(M,\mathbb{R}) \to \mathcal{D}'_{k-1}(M,\mathbb{R})$$

is defined by

$$\forall \phi \in \mathcal{E}^{k-1}(M,\mathbb{R}), \quad d_\alpha T(\phi) := (-1)^{k-1} T(d_\alpha \phi)$$

If ϕ is a $(k-1)$-form and ψ a $(2n-k)$-form we have

$$d(\phi \wedge \psi) = d_\alpha \phi \wedge \psi + (-1)^{k-1} \phi \wedge d_{-\alpha} \psi$$

From LCS to twisted currents

Recall:

Proposition

For $S \in VII_0^+$. We have an equivalence between the following two conditions:

- *Let $0 \neq a \in H^1_{dR}(M, \mathbb{R})$. For any $\alpha \in a$, there exists a Hermitian metric g s.t.*

$$d_\alpha d^c_\alpha F = 0.$$

- *Let $0 \neq a \in H^1_{dR}(M, \mathbb{R})$. For any $\alpha \in a$, there exists a LCS taming ω s.t.*

$$d_\alpha \omega = 0.$$

From LCS to twisted currents

We have a proposition of Harvey-Lawson or Otiman type:

Proposition

Let S whose minimal model is in VII_0^+.

- *Let $\beta \in H^1_{dR}(M,\mathbb{R})$ s.t. $\exists \tau \leq 0$ of degree zero, with $T = d_\beta d^c_\beta \tau \geq 0$. Then $b \in]-\infty, 0[$ and $\tau(S) \subset]-\infty, b[$.*
- *Conversely if $\tau(S) \subset]-\infty, b[$ for some $b < 0$, then for each $\beta \in b$, there exists $\tau \leq 0$ with $0 \neq T = d_\beta d^c_\beta \tau \geq 0$.*

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Proof: $(\Downarrow)$ A computation gives

$$(d_\beta d^c_\beta \tau)(F) = \frac{t(t-1)}{2} |\alpha|^2_g F \wedge F$$

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Let $a \in \tau(S)$. As $\dim H^1_{dR}(M,\mathbb{R}) = 1$, $b = ta$, $\beta = t\alpha$.

From LCS to twisted currents

We apply T to both members

$$0 < T(F) = (d_\beta d^c_\beta \tau)(F) = \tau(d_\beta d^c_\beta F) = \frac{t(t-1)}{2} \underbrace{\tau(|\alpha|^2_g F \wedge F)}_{<0}$$

therefore $t \in]0,1[$ and $a = b/t < b$ (because $b < 0$!).
($\Uparrow$) Hahn-Banach
There is a similar result for $\tau \geq 0$.

From twisted currents to automorphic psh functions

Consider the kernel H of

$$\pi_1(S) \to \frac{\pi_1(S)}{[\pi_1(S), \pi_1(S)]} \simeq H^1(S, \mathbb{Z}) \simeq \mathbb{Z}$$

and

$$\hat{S} \to S$$

the covering associated to H, then the fiber is $\mathbb{Z}$.
Let $\hat{g} : \hat{S} \to \hat{S}$ the automorphism (up to inverse) such that

$$S \simeq \hat{S} / \langle \hat{g}^p \rangle$$

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S admits a closed 1-form β and $\tau \leq 0$ of degree 0 such that $T = d_\beta d^c_\beta \tau \geq 0 \iff \hat{S}$ admits $\hat{u} < 0$ PSH and automorphic i.e. $\hat{u} \circ \hat{g} = C\hat{u}$ for $C > 0$.

From twisted currents to automorphic psh functions

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Proof: τ is a function in L^1_{loc}, and we consider lifts $\hat{\tau}$ and $\hat{\beta}$ which are invariant by $\hat{g}$. On $\hat{S}$, there is a smooth function $\hat{f}$ such that $\hat{\beta} = d\hat{f}$. There is a constant C such that

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where $C \in \mathbb{R}^{\star}$ and may be supposed positive (change $\hat{g}$ into $\hat{g}^{-1}$).

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where $C \in \mathbb{R}^\star$ and may be supposed positive (change $\hat{g}$ into $\hat{g}^{-1}$). We have

$$dd^c(e^{\hat{f}}\tau) = e^{\hat{f}}(d_\beta d^c_\beta \tau) = e^{\hat{f}}T \geq 0.$$

Examples

Primary Hopf surfaces are defined by a contraction

$$F(z_1, z_2) = (az_1 + tz_2^m, bz_2), \quad 0 < |a| \leq |b| < 1, \quad t(a - b^m) = 0.$$

Let $\hat{u}(z_1, z_2) = \log|z_2|$ then $\hat{u} \circ \hat{g} = \hat{u} + \log|b|$

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Lelong number

$$\nu(\hat{u}, p) := \liminf_{z \to p} \frac{\hat{u}(z)}{\log|z - p]}$$

is equal to 1. We recover E by Siu theorem:
$\hat{E} = \{p \in \hat{S} \mid \nu(\hat{u}, p) > 0\}$.

Examples

Enoki surfaces are defined by the contraction

$$F(z_1, z_2) = (t^n z_1 z_2^n + P(z_2), tz_2) \quad 0 < |t| < 1$$

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Intermediate Kato surfaces defined by a contraction of type

$$F(z_1, z_2) = (\lambda z_1 z_2^s + P(z_2), z_2^k), \quad \lambda \in \mathbb{C}^\star, \quad \deg(P) \leq s, \quad |z_2| < 1$$

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We have $\hat{u} \circ \hat{g}(z_1, z_2) = \log|z_2^k| = k \log|z_2| = k\hat{u}$.

Examples

The current $\hat{T} = dd^c \hat{u}$ **does not** induce a current on S. If we want an additive PSH function we may consider the function

$$\hat{v}(z_1, z_2) = -\log(-\hat{u})$$

which is PSH automorphic additive but all Lelong numbers are zero. Therefore it is not possible to recover curves by Siu theorem.

Characterization by application of Siu theorem

Recall: Let $T \geq 0$ a closed $(1,1)$-current. For $c > 0$

$$E_c(T) = \{x \in S \mid \nu(T, x) \geq c\}$$

is an analytic set.

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For C_i an irreducible component

$$\nu(T, C_i) = \inf_{x \in C_i} \nu(T,x)$$

Characterization by application of Siu theorem

Define

$$R := T - \sum_i \nu(T, C_i)[C_i]$$

$E_c(R)$ is composed of at most a finite number of points.

Theorem

Let $S \in VII_0$. S is Hopf or Enoki $\iff$ there exists $T \geq 0$, exact $(1,1)$-current such that C is not empty.

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Proof: $(\Longrightarrow)$ see Examples.
$(\Longleftarrow)$ If there is a rational curve A such that $A^2 = 0$ it is a rational curve with double point and S is a Enoki surface, therefore we may suppose that $C_i^2 \leq -2$. T is exact, $[C_i]$ are closed, therefore $R \geq 0$ is closed. Since $S \in VII_0$, R is exact by Lamari, hence $[C] = \sum_i \nu(T, C_i)[C_i]$ is exact and $C^2 = 0$.

Need to twist to have other surfaces

Theorem (Brunella)

Let $S \in VII$ and $\hat{S} \to S$ its cyclic covering. Then, the following conditions are equivalent:
(i) The minimal model of S is a hyperbolic Kato surface (resp. a Inoue-Bombieri surface).
(ii) $\hat{S}$ admits a negative (resp. positive) PSH $\tau \leq 0$ such that $T = dd^c\tau \geq 0$ and the support of T is an analytic set of dimension 1 (resp. empty).

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Proposition

Let S whose minimal model is in VII_0^+.

- *Let $\beta \in H^1_{dR}(M, \mathbb{R})$ s.t. $\exists \tau \leq 0$ of degree zero, with $T = d_\beta d^c_\beta \tau \geq 0$. Then $b \in]-\infty, 0[$ and $\tau(S) \subset]-\infty, b[$.*
- *Conversely if $\tau(S) \subset]-\infty, b[$ for some $b < 0$, then for each $\beta \in b$, there exists $\tau \leq 0$ with $0 \neq T = d_\beta d^c_\beta \tau \geq 0$.*

Need to twist to have other surfaces

There is a gap with Brunella's theorem : $L^1_{loc} \to PH$!

Conjecture: VII_0 is divided

- Hopf and Enoki
- Inoue-Bombieri and hyperbolic Kato

Foliations

How to obtain PSH functions, PH outside polar set ?
We suppose that there exists on S a twisted logarithmic 1-form

$$\omega \in H^0(S, \Omega(-\log D) \otimes \mathcal{L}), \quad \mathcal{L} \in H^1(S, \mathbb{C}^\star)$$

where $\mathcal{L}$ is a topologically trivial flat line bundle. Let $\mathcal{F}$ be the associated foliation.

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- If $S \in VII_0^+$, not Enoki, D polar, then D contains a cycle of rational curves (Steenbrink & Van Stratten)
- By [D- Complex manifolds, 2021; 8:208–222] as application of the Donaldson trivialization theorem of the intersection form, the maximal divisor looks like the one of a Kato surface (two cycles or one cycle with possible branches)

Foliations

- all intersection points are singularities of the foliation $\mathcal{F}$. Locally
$$e^h\theta = \theta_0 + g_1(z)\frac{dz_1}{z_1} + g_2(z)\frac{dz_2}{z_2}$$
then $\lambda_1 = g_1(0) \neq 0$, $\lambda_2 = g_2(0) \neq 0$. These constants give the value of the residue along each curve. In fact we have the exact sequence
$$0 \to \Omega^1_S \otimes \mathcal{L} \to \Omega^1_S(\log D) \otimes \mathcal{L} \overset{Res}{\to} \mathcal{O}_{\tilde{D}} \otimes \mathcal{L} \to 0$$
with $\tilde{D}$ the normalization of D. But the restriction of $\mathcal{L}$ to D_i is trivial, therefore we have
$$H^0(S, \Omega^1_S(\log D) \otimes \mathcal{L}) \to \bigoplus_{i=0}^{p} H^0(D_i, \mathcal{O}_{D_i}) \simeq \mathbb{C}^p$$

Foliations

- For each singular point p of $\mathcal{F}$, $\lambda(p) = \frac{\lambda_1}{\lambda_2}$ is real. In fact, the holonomy group is generated by $h(x) = \mu x$ where $\mu = \exp(2\pi i \lambda(p))$.
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  For $\lambda(p) = a + ib$, $\mu = \exp(2\pi i a)\exp(-2\pi b)$. h is a contraction if $b \neq 0$. Therefore a leaf L , accumulates on the divisor and is closed outside D.
  Intersection matrix is negative definite, therefore D contracts onto a normal isolated point $n : S \to \bar{S}$. The singular point has $H^1(\{n(D)\}) = 0$ zero Hausdorff measure. Apply Remmert-Stein-Shiffman extension theorem: $\overline{n(L)}$ is analytic. Hence $L \subset n^{-1}(n(L))$ should be analytic. Contradiction.
- As current $\bar{\partial}_\alpha T_\theta = 2\pi i Res_\alpha(\theta) T_D$. Replacing θ by a suitable complex multiple, $Res_\alpha(\theta)_{|D_i}$ are real $Re(T_\theta) = d_\alpha \tau$.

THANK YOU !