Rognes' connectivity conjecture and the Koszul dual of Steinberg

j/w Jeremy Miller, Rohit Nagpal, Jennifer Wilson

R PID, F field

Result 1: Rognes' conj and alg K-theory

$K(R)$ K-theory spectrum

$F_n K(R) \subset K(R)$ rank filtration
(taking only rank $\leq n$ free (= proj) R-mods)

$$E^1_{pq} = H_{p+q}(F_p K(R), F_{p-1} K(R)) \Rightarrow H_{p+q}(K(R))$$

$\rightsquigarrow$ Adams spectral sequence computes
alg. K-groups $K_i(R) = \pi_i(K(R))$

Common basis complex

$D(R^n)$ simplicial cpx

vertices = proper non zero (free) summands of R^n

$\{V_0, \dots, V_p\}$ p-simplex if it has a "common basis" B, i.e. $B = b_1, \dots, b_n$ is a basis of R^n s.t. each V_i = span subset of B

Rognes (1992)

$$F_n K(R) / F_{n-1} K(R) \simeq \Sigma^\infty \left(\mathrm{cone} \left(\begin{array}{c} (\Sigma D(R^n)) /\!\!/ GL_n(R) \\ \| \\ EGL_n(R) \times_{GL_n(R)} \Sigma D(R^n) \\ \downarrow \\ EGL_n(R) \times_{GL_n(R)} * \\ \| \\ BGL_n(R) \end{array} \right) \right)$$

i.e. $E^1_{pq} = H_{p+q}(BGL_n(R), (\Sigma D(R^n) /\!\!/ GL_n(R))$

Conj (Rognes 1992) If R Euclidean or local

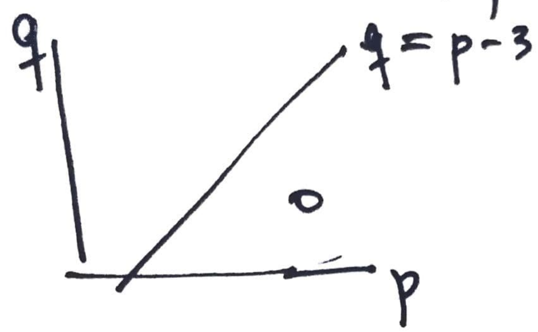
$$D(R^n) \simeq \bigvee S^{2n-3}$$

Rem: • $\dim D(\mathbb{R}^n) = 2^n - 3$

• Rognes proved $D(\mathbb{R}^n) \simeq E(\mathbb{R}^n)$
$(2n-3)$-dim

$\Rightarrow$ enough to show $D(\mathbb{R}^n)$ $(2n-4)$-ctd

• Conj $\Rightarrow$ $E^1_{pq} = 0$ for $q \leq p-3$



Progress on the conjecture:

Galatius–Kupers–Randal-Williams

$$\Sigma D(F^n) /\!/ GL_n F \longrightarrow BGL_n F$$

is $(2n-3)$-ctd for infinite fields.

Thm (Miller–P.–Wilson)

$D(F^n)$ is $(2n-4)$-ctd for all fields F.

Result 2: Koszulness of Steinberg and its dual

Tits building $T(R^n)$ simpl. cpx $\subset D(R^n)$

flags of (free) summands of R^n

$$0 \subsetneq V_0 \subsetneq \cdots \subsetneq V_p \subsetneq R^n$$

Solomon–Tits (1969)

$$T((\mathrm{Frac}\, R)^n) \cong T(R^n) \simeq \bigvee S^{n-2}$$

Steinberg module $GL_n(R) \circlearrowright St(R^n) = \tilde{H}_{n-2}(T(R^n))$

$\rightsquigarrow$ important in rep thy, alg K-thy, cohom of arithm gps

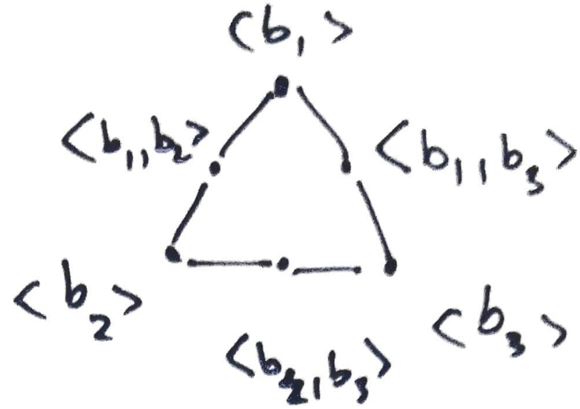
Apartment $B = b_1, \dots, b_n$ basis of R^n

$[\![b_1, \dots, b_n]\!] :=$ full subcpx of $T(R^n)$ on spans of subsets of B

$$\cong \partial(\mathrm{sd}\, \Delta^{n-1}) \cong S^{n-2}$$

"apartment" class in $\tilde{H}_{n-2}(T_*(R^n)) = St(R^n)$

Ex $n=3$



Solomon–Tits (part 2) F fields

$St(F^n)$ generated by apartments

Steinberg as a ring

$\bigoplus_{n\geq 0} St(F^n)$ forms a ring by $F^m \oplus F^n \overset{\shortparallel}{} F^{m+n}$

$$St(F^m)\otimes St(F^n) \longrightarrow St(F^{m+n})$$

$$[\![b_1,\dots,b_m]\!] \otimes [\![b_1',\dots,b_n']\!] \mapsto [\![b_1,\dots,b_m,b_1',\dots,b_n']\!]$$

not commutative $\rightsquigarrow$ account for $F^m \hookrightarrow F^{m+n}$

$\rightsquigarrow$ GL-rings

<u>$GL(R)$-modules</u> $\{M_n\}_{n \geq 0}$

w/ M_n is a $\mathbb{Z}GL_n(R)$-mod

$\longleftrightarrow$ functor $GL(R) \longrightarrow$ ab gps

$$\coprod_{n \geq 0} GL_n(R) \cong \begin{array}{c}\text{f.g. free } R\text{-mods} \\ \text{and isos}\end{array}$$

$\rightsquigarrow$ $GL(R)$-mod is symmetric monoidal cat:

$$(M \otimes N)_n = \bigoplus_{V \oplus W = R^n} M(V) \otimes N(W)$$

<u>$GL(R)$-ring</u> monoid object in $GL(R)$-mod

i.e. a $GL(R)$-mod A w/ product

$A \otimes A \longrightarrow A$ (assoc. + unit)

Define modules, tensor products, Tor's over $GL(R)$-rings.

Steinberg as $GL(R)$-ring.

$$St(R)_n = St(R^n) \qquad (\cong St((\mathrm{Frac}\,R)^n))$$

$$\circlearrowleft \; GL_n(R)$$

is skew-commutative.

Koszulness Let A be a $GL(R)$-ring w/ $A_0 = \mathbb{Z}$.

Define $\mathbb{Z}$ to be the $GL(R)$-ring

$$\mathbb{Z}_n = \begin{cases} \mathbb{Z} & n=0 \\ 0 & n>0 \end{cases}$$

$\rightsquigarrow$ $\mathbb{Z}$ is an A-module

A is Koszul if $\mathrm{Tor}_i^A(\mathbb{Z},\mathbb{Z})_n = 0$ for $i \neq n$.

Minimal free resolution: If A is Koszul

$$0 \leftarrow \mathbb{Z} \leftarrow \mathrm{Tor}_0^A(\mathbb{Z},\mathbb{Z})_0 \otimes A \leftarrow \mathrm{Tor}_1^A(\mathbb{Z},\mathbb{Z})_1 \otimes A \leftarrow \cdots$$

is exact.

Thm (Miller – P.– Wilson)

$$\Sigma T^{k+1}(R) \simeq BT^{k}(R)$$

Proof of Rognes conj for fields

$T^2(F) \overset{\text{Lem}}{=} \cancel{F}(R^n) * T(R^n)$ is $(2n-4)$-ctd

$T^k(F^n)$ $(2n-4)$-ctd $\overset{\text{Thm}}{\Longrightarrow}$ $T^{k+1}(F^n)$ $(2n-4)$-ctd

$\overset{\text{Rognes}}{\Longrightarrow}$ $D(R^n)$ $(2n-4)$-ctd

Proof of Koszul dual (and Koszulness)

$\Sigma T^2(F^n) \simeq BT(F^n)$ $\rightsquigarrow$ spectral seq

"double cpx"

$$\mathrm{Tor}_i^{St(F)}(\mathbb{Z},\mathbb{Z})_n$$

$$\underset{\text{sp sq}}{\cong} \tilde{H}_{i+n-2}(BT(F^n))$$

$$\cong \tilde{H}_{i+n-3}(T^2(F^n))$$

$$\underset{\text{Lem}}{\cong} \begin{cases} St(F^n) \otimes St(F^n) & n = i \\ 0 & n \neq i \end{cases}$$

Thm (Miller–Nagpal–P.) F field

$St(F)$ is a Koszul $GL(F)$-ring.

Thm (Miller–P.–Wilson) F field

$$\mathrm{Tor}_n^{St(F)}(\mathbb{Z},\mathbb{Z})_n \cong St(F^n) \otimes St(F^n)$$

Methods: Higher buildings

k-Tits building $T^k(R^n)$ subcpx of

$\underbrace{T(R^n) * \cdots * T(R^n)}_{k}$ of simplices w/ common basis

Ex • $\langle e_1 \rangle * \langle e_2 \rangle * \langle e_1 + e_2 \rangle \notin T^3(R^n)$

• $\langle e_1 \rangle \subset \langle e_1, e_3 \rangle * \langle e_2 \rangle \subset \langle e_2, e_1 + e_2 + e_3 \rangle \in T^2(R^3)$

b/c $e_1, e_2, e_1 + e_3$ is a common basis

Lem F field $T^2(F^n) = T(F^n) * T(F^n)$

Rognes (1992) $\bigcup T^k(R^n) \simeq D(R^n)$

<u>Cor</u> $E^1_{pq} = H_{p+q}(F_p K(R), F_{p-1} K(R))$

$$\cong H^{E_\infty}_{p+q}\big(C_*(EGL_p(R) \otimes_{GL_p(R)} St(R^p))\big)$$

(= infinitely iterated $\mathrm{Tor}^{-}(\mathbb{Z},\mathbb{Z})$)

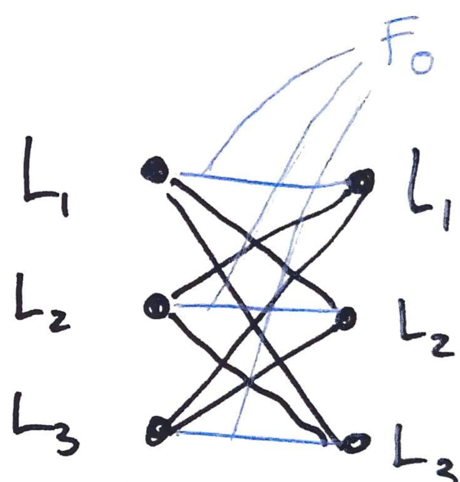
<u>Proof of Thm</u> for $n=2$

$T(F^2)$ = discrete set of lines in F^2

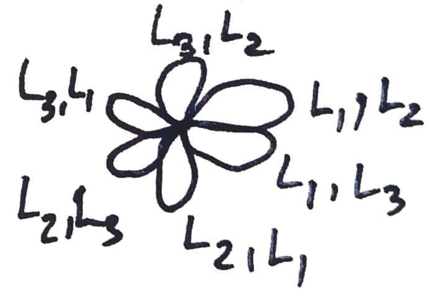
$$T^2(F^2) = T(F^2) * T(F^2)$$

$$\cup$$

$$F_0 \simeq T(F^2)$$



$$T^2(F^2)/F_0 \cong \bigvee_{L \oplus L' = F^2} S^1$$



LES:

$$0 \leftarrow \tilde{H}_0(T(F^2)) \leftarrow H_1\Big(\bigvee_{L\oplus L'=F^2} S^1\Big) \leftarrow H_1(T^2(F^n)) \leftarrow 0$$

$$\cong \qquad\qquad \cong$$

$$0 \leftarrow St(F^2) \longleftarrow \bigoplus_{L\oplus L'=F^2} St(L) \otimes St(L') \leftarrow \mathrm{Tor}_2^{St(F)}(\mathbb{Z},\mathbb{Z})_2 \leftarrow 0$$