On eta quotients whose derivatives are also eta quotients

Zafer Selcuk Aygin,
Instructor,
Northern Lakes College,
selcukaygin@gmail.com

November 7, 2021

Table of Contents

Introduction

Definition

Throughout the talk we let $z \in \mathbb{H}$ and $q = e^{2\pi iz}$.

The Dedekind eta function is defined by the infinite product

$$\eta(z) = e^{\pi iz/12} \prod_{n=1}^{\infty}(1 - e^{2\pi inz}) = q^{1/24} \prod_{n=1}^{\infty}(1 - q^n)$$

and an eta quotient of level N is defined to be of the form

$$f(z) = \prod_{0<t|N} \eta(tz)^{r_t}$$

where t runs over all positive divisors of the integer N and the exponents r_t are integers. The weight attached to this eta quotient is $k = \frac{1}{2} \sum_{0<t|N} r_t$.

Example

We use

$$\eta_N[r_1, \ldots, r_t, \ldots, r_N](z) = \prod_{0<t|N} \eta(tz)^{r_t}$$

as a shorthand notation for an eta quotient of level N.

If an eta quotient $f(z)$ is of level N, then $f(z)$ and $f(dz)$ will be eta quotients of level dN for each $d \in \mathbb{N}$.

When counting the number of eta quotients in specific levels we exclude eta quotients arising like this.

Example

Let us consider the eta quotient

$$f(q) = \frac{\eta(q^4)^8}{\eta(q)^8}$$
$$= q + 8q^2 + 44q^3 + 192q^4 + 718q^5 + 2400q^6$$
$$+ 7352q^7 + 20992q^8 + 56549q^9 + 145008q^{10} + O(q^{11}).$$

Then we compute

$$f'(q) = 1 + 16q + 132q^2 + 768q^3 + 3590q^4 + 14400q^5 + 51464q^6$$
$$+ 167936q^7 + 508941q^8 + 1450080q^9 + O(q^{10})$$
$$\approx \frac{\eta(q^2)^{20}}{q\eta(q)^{16}}.$$

Example

On the other hand

Jacobi's function

$$\frac{\frac{\eta(q^2)^{20}}{\eta(q)^{16}}}{\frac{\eta(q^4)^8}{\eta(q)^8}} = \frac{\eta(q^2)^{20}}{\eta(q)^8\eta(q^4)^8} = 1 + 8\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} - 8\sum_{n=1}^{\infty} \frac{4nq^{4n}}{1-q^{4n}}$$

$$= q\frac{d}{dq}\log\left(q\prod_{n\geq 1}\frac{(1-q^{4n})^8}{(1-q^n)^8}\right)$$

$$= q\frac{d}{dq}\log\left(\frac{\eta(q^4)^8}{\eta(q)^8}\right)$$

$$= q\frac{d}{dq}\log(f(q)).$$

Example

Thus, as predicted, we have

$$q\frac{d}{dq}\frac{\eta(q^4)^8}{\eta(q)^8} = \frac{\eta(q^2)^{20}}{\eta(q)^{16}}, \text{ or in alternative notation:}$$

$$q\frac{d}{dq}\eta_4[-8,0,8](q) = \eta_4[-16,20,0](q).$$

Example

Some examples:

$$q\frac{d}{dq}\log(\eta_4[-8,0,8](q)) = \eta_4[-8,20,-8](q) \text{ (Jacobi)},$$
$$q\frac{d}{dq}\log\left(\eta_8[-4,2,-2,4](q)\right) = \eta_8[-4,6,6,-4](q) \text{ (Fine)},$$
$$q\frac{d}{dq}\log\left(\eta_{12}[-4,4,4,-4,-4,4](q)\right) = \eta_{12}[-4,10,-4,-4,10,-4](q) \text{ (Fine)},$$
$$q\frac{d}{dq}\log\left(\eta_9[-3,0,3](q)\right) = \eta_9[-3,10,-3](q) \text{ (Borwein and Garvan)}.$$

The relation between eta quotients, their derivatives and Eisenstein series

Definition

Let us define

$$E_2(z) = -\frac{1}{24} + \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} = -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma(n) q^n$$

where $\sigma(n) = \sum_{d|n} d$. It is known that

$$E_2(z) - tE_2(tz) \in M_2(\Gamma_0(N)),$$

whenever t is a positive divisor of N.

Suppose now that $f(z)$ is an eta quotient of level N and weight k. We have

$$f(z) = \prod_{t|N} \eta^{r_t}(tz) \quad \text{where } \sum_{t|N} r_t = 2k.$$

Applying logarithmic differentiation we obtain

$$\begin{aligned} q\frac{d}{dq} \log f(z) &= \sum_{t|N} r_t t E_2(tz) \\ &= -2kE_2(z) + \sum_{t|N} r_t \left(E_2(z) - tE_2(tz)\right) \end{aligned}$$

Theorem (A., Toh (J. Math. Anal. Appl., 2019))

Let $f(z) = \prod_{t|N} \eta^{r_t}(tz)$ *be an eta quotient of level N and weight k. Then*

$$q\frac{d}{dq}f(z) \text{ is an eta quotient iff}$$

$$k = 0 \text{ and } \sum_{1<t|N} r_t\left(E_2(z) - tE_2(tz)\right) \text{ is an eta quotient.}$$

Therefore finding the eta quotients whose derivatives are eta quotients is equivalent to finding eta quotients in the space

$$E_N = \left\{ \sum_{t|N} r_t E_2(tz) \,\middle|\, r_t \in \mathbb{Q}, \sum_{t|N} r_t/t = 0 \right\}.$$

Corollary (A., Toh (J. Math. Anal. Appl., 2019))

Let $f(z)$ be an eta quotient in E_N, i.e.

$$f(z) = \prod_{0<t|N} \eta^{s_t}(tz) = \sum_{1<t|N} r_t \left(E_2(z) - tE_2(tz)\right)$$

where $r_t \in \mathbb{Q}$. Denote $r_1 = -\sum_{1<t|N} r_t$ and let $c \in \mathbb{Z}$ be such that $cr_t \in \mathbb{Z}$ for all $t \mid N$. Then we have

$$q\frac{d}{dq}\prod_{t|N}\eta^{cr_t}(tz) = c\prod_{t|N}\eta^{s_t+cr_t}(tz).$$

Algorithms

Theorem (Ligozat (1975), Cohen & Stromberg (2016))

Let $f(z) = \prod_{0<t|N} \eta^{r_t}(tz)$ be an eta quotient.

a) *We have*

$$v_{a/c}(f) = \frac{N}{24\gcd(c^2, N)} \sum_{t|N} \frac{\gcd(t,c)^2 \cdot r_t}{t}.$$

b) *We have*

$$f(Mz) = \Big(\frac{(-1)^k P}{d}\Big)(cz+d)^k f(z)$$

if and only if

$$v_{1/1}(f), v_{1/N}(f) \in \mathbb{Z},$$

where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, $k = \sum_{t|N} r_t/2$, $P = \prod_{t|N} t^{r_t}$.

A set of inequivalent cusps for $\Gamma_0(N)$ is given by

$$R(N) = \bigcup_{c|N} \left\{ \frac{a}{c} : a (\mathrm{mod} \ \gcd(c, N/c)), \ \gcd(a, c) = 1 \right\}.$$

Theorem (Valence formula)

Let $N, k \in \mathbb{N}$. Suppose $f(z) = \prod_{0<t|N} \eta(tz)^{r_t}$ is an eta quotient of level N and weight k. Let $v_r(f(z))$ be the order of vanishing of $f(z)$ at the cusp $r \in R(\Gamma_0(N))$. Then we have

$$\sum_{r \in R(N)} v_r(f(z)) = \frac{2kN}{24} \sum_{c|N} \frac{\phi(\gcd(c, N/c))}{\gcd(c^2, N)}.$$

Furthermore, if $f(z)$ is holomorphic, we have

$$0 \le v_r(f(z)) \le \frac{2kN}{24} \sum_{c|N} \frac{\phi(\gcd(c, N/c))}{\gcd(c^2, N)}.$$

Theorem (A., Toh (J. Math. Anal. Appl., 2019))

There are precisely 203 distinct eta quotients whose derivatives are also eta quotients in levels less than or equal to 100.

Number of distinct identities at each level

Level	4	6	8	9	12	16	18	20	24	36
No. of Identities	3	10	4	1	100	4	12	12	32	25

	weight 0 eta quotient	logarithmic derivative
$f_{6,1a}(z)$	$\eta_6[12, -48, 36, 0](z)$	$\eta_6[12, -6, -4, 2](z)$
$f_{6,1b}(z)$	$\eta_6[-12, 3, 0, 9](z)$	$2\eta_6[-6, 12, 2, -4](z)$
$f_{6,1c}(z)$	$\eta_6[-12, 0, -4, 16](z)$	$3\eta_6[-4, 2, 12, -6](z)$
$f_{6,1d}(z)$	$\eta_6[0, -3, 4, -1](z)$	$6\eta_6[2, -4, -6, 12](z)$
$f_{6,2a}(z)$	$\eta_6[-1, 5, -5, 1](z)$	$\eta_6[7, -5, -5, 7](z)$
$f_{6,2b}(z)$	$\eta_6[-5, 1, -1, 5](z)$	$\eta_6[-5, 7, 7, -5](z)$
$f_{6,3a}(z)$	$\eta_6[4, -8, -4, 8](z)$	$\eta_6[4, -2, 4, -2](z)$
$f_{6,3b}(z)$	$\eta_6[-2, 1, 2, -1](z)$	$2\eta_6[-2, 4, -2, 4](z)$
$f_{6,4a}(z)$	$\eta_6[3, -3, -9, 9](z)$	$\eta_6[3, 3, -1, -1](z)$
$f_{6,4b}(z)$	$\eta_6[-3, 3, 1, -1](z)$	$3\eta_6[-1, -1, 3, 3](z)$

	weight 0 eta quotient	logarithmic derivative
$f_{16,1a}(z)$	$\eta_{16}[2,-5,2,-1,2](z)$	$\eta_{16}[2,-5,8,1,-2](z)$
$f_{16,1b}(z)$	$\eta_{16}[-2,1,-2,5,-2](z)$	$2\eta_{16}[-2,1,8,-5,2](z)$
$f_{16,2}(z)$	$\eta_{16}[-2,1,0,-1,2](z)$	$\eta_{16}[-2,1,6,1,-2](z)$
$f_{16,3}(z)$	$\eta_{16}[-2,5,0,-5,2](z)$	$2\eta_{16}[2,-5,10,-5,2](z)$

	weight 0 eta quotient	logarithmic derivative
$f_{18,1a}(z)$	$\eta_{18}[3,-6,-2,2,3,0](z)$	$\eta_{18}[3,-3,2,4,-1,-1](z)$
$f_{18,1b}(z)$	$\eta_{18}[-6,3,2,-2,0,3](z)$	$2\eta_{18}[-3,3,4,2,-1,-1](z)$
$f_{18,1c}(z)$	$\eta_{18}[-3,0,2,-2,-3,6](z)$	$3\eta_{18}[-1,-1,2,4,3,-3](z)$
$f_{18,1d}(z)$	$\eta_{18}[0,-3,-2,2,6,-3](z)$	$6\eta_{18}[-1,-1,4,2,-3,3](z)$
$f_{18,2a}(z)$	$\eta_{18}[-1,2,2,-6,3,0](z)$	$\eta_{18}[1,1,6,-4,-3,3](z)$
$f_{18,2b}(z)$	$\eta_{18}[2,-1,-6,2,0,3](z)$	$2\eta_{18}[1,1,-4,6,3,-3](z)$
$f_{18,2c}(z)$	$\eta_{18}[-3,0,-2,6,1,-2](z)$	$3\eta_{18}[-3,3,6,-4,1,1](z)$
$f_{18,2d}(z)$	$\eta_{18}[0,-3,6,-2,-2,1](z)$	$6\eta_{18}[3,-3,-4,6,1,1](z)$
$f_{18,3a}(z)$	$\eta_{18}[-2,1,1,-1,-1,2](z)$	$\eta_{18}[-2,1,3,3,1,-2](z)$
$f_{18,3b}(z)$	$\eta_{18}[-1,2,1,-1,-2,1](z)$	$\eta_{18}[1,-2,3,3,-2,1](z)$
$f_{18,4a}(z)$	$\eta_{18}[1,-2,0,0,-1,2](z)$	$\eta_{18}[1,-2,2,4,1,-2](z)$
$f_{18,4b}(z)$	$\eta_{18}[-2,1,0,0,2,-1](z)$	$2\eta_{18}[-2,1,4,2,-2,1](z)$

	weight 0 eta quotient	logarithmic derivative
$f_{20,1a}(z)$	$\eta_{20}[-7,1,1,-5,15,-5](z)$	$\eta_{20}[-7,16,-5,3,-4,1](z)$
$f_{20,1b}(z)$	$\eta_{20}[-1,-1,7,5,-15,5](z)$	$\eta_{20}[-5,16,-7,1,-4,3](z)$
$f_{20,1c}(z)$	$\eta_{20}[5,-15,5,-1,-1,7](z)$	$5\eta_{20}[1,-4,3,-5,16,-7](z)$
$f_{20,1d}(z)$	$\eta_{20}[-5,15,-5,-7,1,1](z)$	$5\eta_{20}[3,-4,1,-7,16,-5](z)$
$f_{20,2a}(z)$	$\eta_{20}[-1,4,-8,5,0,0](z)$	$\eta_{20}[5,1,-2,-1,-1,2](z)$
$f_{20,2b}(z)$	$\eta_{20}[-8,4,-1,0,0,5](z)$	$4\eta_{20}[-2,1,5,2,-1,-1](z)$
$f_{20,2c}(z)$	$\eta_{20}[-5,0,0,1,-4,8](z)$	$5\eta_{20}[-1,-1,2,5,1,-2](z)$
$f_{20,2d}(z)$	$\eta_{20}[0,0,-5,8,-4,1](z)$	$20\eta_{20}[2,-1,-1,-2,1,5](z)$
$f_{20,3a}(z)$	$\eta_{20}[7,-20,8,5,0,0](z)$	$\eta_{20}[7,-5,2,-3,5,-2](z)$
$f_{20,3b}(z)$	$\eta_{20}[8,-20,7,0,0,5](z)$	$4\eta_{20}[2,-5,7,-2,5,-3](z)$
$f_{20,3c}(z)$	$\eta_{20}[-5,0,0,-7,20,-8](z)$	$5\eta_{20}[-3,5,-2,7,-5,2](z)$
$f_{20,3d}(z)$	$\eta_{20}[0,0,-5,-8,20,-7](z)$	$20\eta_{20}[-2,5,-3,2,-5,7](z)$

Recently, D. Choi, B. Kim, S. Lim gave **the complete list for squarefree and prime square levels**. Their list agrees with our results, namely, their results consists of the 10 pairs at level 6, 3 pairs at level 4 and 1 pair at level 9. See, "Pairs of eta-quotients with dual weights and their applications", Adv. Math. 355 (2019) 106779. Additionally, they show that:

Theorem (D. Choi, B. Kim, S. Lim (Adv. Math., 2019))

Let N be a level having a primitive eta quotient whose derivative is also an eta quotient then $N = 2^a 3^b 5^c 7^d$ for some $a, b, c, d \in \mathbb{N}_0$.

Recall:

Number of distinct identities at each level

Level	4	6	8	9	12	16	18	20	24	36
No. of Identities	3	10	4	1	100	4	12	12	32	25

Our approach

Recall that we want to find all eta quotients in

$$E_N = \left\{ \sum_{t|N} r_t E_2(tz) \middle| r_t \in \mathbb{Q}, \sum_{t|N} r_t/t = 0 \right\}.$$

To exclude repeated results we define

$$F_{N,t} = \left\{ f(tz) | f(z) \in E_{N/t} \right\}$$

and

$$P_N = E_N \setminus \left(\bigcup_{1<t|N} \left(F_{N,t} \cup E_{N/t} \right) \right).$$

And if $f(z) \in P_N$ then we call f an E-primitive modular form.

Goal

Let $f(z) \in P_N$. Our goal is to find an upper bound for

$$\sum_{a/c \in R(N)} v_{a/c}(f(z)),$$

say $U(N)$, and compare $U(N)$ with $L(N)$.

Recall that, by the Valence Formula, the sum of vanishings of an eta quotient at cusps of $\Gamma_0(N)$ is a fixed number for each N given by

$$L(N) = \frac{4kN}{24} \sum_{c|N} \frac{\phi(\gcd(c, N/c))}{\gcd(c^2, N)}.$$

It is known that

$$E_2(Mz) = E_2\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 E_2(z) - \frac{6ic}{\pi}(cz+d)$$

for all $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Let us denote

$$E_{2,t}(z) = E_2(tz).$$

Noting that the width of the cusp a/c in $\Gamma_0(N)$ is given by $\frac{N}{\gcd(c^2,N)}$, define

$$q_{c,N} = e^{2\pi iz \gcd(c^2,N)/N}.$$

Then for all $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ we have

$$(cz+d)^{-2} E_{2,t}(Mz) = \sum_{n \geq 0} a_n(c,t) \omega_{M,t}^n q_{c,N}^{n \gcd(t,c)^2 N/t \gcd(c^2,N)} - \frac{6ic}{(cz+d)t\pi},$$

where

$$\omega_{M,t} := \begin{cases} 1 & \text{if } c \equiv 0 \pmod{t}, \\ e^{-2\pi i gcd(t,c) df/t} & \text{if } c \not\equiv 0 \pmod{t}. \end{cases}$$

Assuming

$$\sum_{t|N} r_t/t = 0,$$

for $r_t \in \mathbb{Q}$ we have

$$(cz+d)^{-2}\sum_{t|N} r_t E_{2,t}(Mz) = \sum_{t|N}\sum_{n\geq 0} a_n(c,t) r_t \omega_{M,t}^{n} q_{c,N}^{n\,\gcd(at,c)^2 N/t\,\gcd(c^2,N)}.$$

Note that $a_n(c,t) \in \mathbb{Q}$.

Prime power case

The case $N = p^m$

Let

$$f(z) = \sum_{t|N} r_t E_{2,t}(z) \in E_2(p^m),$$

then for all $c \mid p^m$ we have

$$(cz+d)^{-2} f(Mz) = \sum_{t|p^m} \sum_{n \geq 0} a_n(c,t) r_t \omega_{M,t}^n q_{c,p^m}^{n \gcd(t,c)^2 p^m / t \gcd(c^2, p^m)},$$

all divisors of p^m are of the form p^i $(0 \leq i \leq m)$, that is we have

$$(p^i z + d)^{-2} f(Mz) = \sum_{j=0}^{m} \sum_{n \geq 0} a_n(p^i, p^j) r_{p^j} \omega_{M,p^j}^n q_{p^i,p^m}^{n \gcd(p^j,p^i)^2 p^m / p^j \gcd(p^{2i}, p^m)}.$$

On the other hand we have

$$\omega_{M,p^j} = \begin{cases} 1 & \text{if } i \geq j, \\ e^{-2\pi i d f / p^{j-i}} & \text{if } i < j, \end{cases}$$

and

$$\gcd(p^j, p^i)^2 p^m / p^j \gcd(p^{2i}, p^m) = \begin{cases} p^j & \text{if } i \geq j \text{ and } i \geq m/2, \\ p^{2i-j} & \text{if } i < j \text{ and } i \geq m/2, \\ p^{m+j-2i} & \text{if } i \geq j \text{ and } i < m/2, \\ p^{m-j} & \text{if } i < j \text{ and } i < m/2. \end{cases}$$

We put these together and obtain

$$(p^i z+d)^{-2} f(Mz) = \sum_{j=0}^{m} \sum_{n\geq 0} a_n(p^i,p^j) r_{p^j} \omega_{M,p^j}^{n} q_{p^i,p^m}^{n\,\mathrm{gcd}(p^j,p^i)^2 p^m/p^j\,\mathrm{gcd}(p^{2i},p^m)}$$

$$= \begin{cases} \sum_{i\geq j} \sum_{n\geq 0} a_n(p^i,p^j) r_{p^j} q_{p^i,p^m}^{np^j} \\ +\sum_{i<j} \sum_{n\geq 0} a_n(p^i,p^j) r_{p^j} e^{-2\pi i n d f/p^{j-i}} q_{p^i,p^m}^{np^{2i-j}} & \text{if } i\geq m/2, \\ \sum_{i\geq j} \sum_{n\geq 0} a_n(p^i,p^j) r_{p^j} q_{p^i,p^m}^{np^{m+j-2i}} \\ +\sum_{i<j} \sum_{n\geq 0} a_n(p^i,p^j) r_{p^j} e^{-2\pi i n d f/p^{j-i}} q_{p^i,p^m}^{np^{m-j}} & \text{if } i< m/2. \end{cases}$$

Now let $v_{1/p^0}(f)>1$, that is, the coefficient of q_{1,p^m} is 0. Then we have $a_1(p^0,p^m) r_{p^m} e^{-2\pi i d f/p^m}=0$, that is, we have $r_{p^m}=0$. By similar arguments we obtain:

Lemma

If $f(z) \in E_{p^m}$ *and* $v_{a/p^i}(f) > 1$ *for an* $0 \leq i \leq m$*, then*

$$\begin{aligned} r_{p^m} &= 0 && \text{if } i < m/2, \\ r_1 &= 0 && \text{if } i > m/2, \\ r_1, r_{p^m} &= 0 && \text{if } i = m/2. \end{aligned}$$

Notice that,

- if $r_{p^m} = 0$, then $f(z) \in E_{p^{m-1}}$; and
- if $r_1 = 0$, then there exists a $g(z) \in E_{p^{m-1}}$ such that $f(z) = g(pz)$.

Therefore, in either case f is not E-primitive.

Theorem

Let p be a prime and $m \in \mathbb{N}$. If $f(z) \in P_{p^m}$, then

$$\sum_{a/c \in R(p^m)} v_{a/c}(f(z)) < |R(p^m)|.$$

We have

$$L(p^m) = \frac{p^m + p^{m-1}}{6},$$
$$|R(p^m)| = p^{[(m-1)/2]}(p^{(m-1)-2[(m-1)/2]} + 1).$$

Comparison of $L(p^m)$ with $|R(p^m)|$

p^m	$L(p^m)$	$\|R(p^m)\|$	p^m	$L(p^m)$	$\|R(p^m)\|$	p^m	$L(p^m)$	$\|R(p^m)\|$
2^1	$1/2$	2	3^1	$2/3$	2	5^1	1	2
2^2	1	3	3^2	2	4	5^2	5	6
2^3	2	4	3^3	6	6	5^3	25	10
2^4	4	6						
2^5	8	8						

p^m	$L(p^m)$	$\|R(p^m)\|$	p^m	$L(p^m)$	$\|R(p^m)\|$
7^1	$4/3$	2	11^1	2	2
7^2	$28/3$	8			

Corollary

There are no eta quotients in P_{p^m} except possibly in the following cases:

$$p = 2, m = 1, 2, 3, 4,$$
$$p = 3, m = 1, 2,$$
$$p = 5, m = 1, 2,$$
$$p = 7, m = 1.$$

Recall the number of distinct identities at each level

Level	2^2	$2 \cdot 3$	2^3	3^2	$2^2 \cdot 3$	2^4	$2 \cdot 3^2$	$2^2 \cdot 5$	$2^3 \cdot 3$	$2^2 \cdot 3^2$
No. of Id's	3	10	4	1	100	4	12	12	32	25

Thanks!

Lemma

Let $m > 2$ be positive even integer. Let $f(z) \in E_2(p^m)$ and assume $v_{a/p^{m/2}} > 1$ for all $a \in \{a : 1 \leq a \leq p^{m/2}, \gcd(a, p^{m/2}) = 1\}$, then

$$r_1, r_{p^m} = 0.$$

Proof

By Lemma **??**, for all $a \in \{a : 1 \leq a \leq p^{m/2}, \gcd(a, p^{m/2}) = 1\}$, we have

$$a_n(p^{m/2}, 1) r_1 + a_n(p^{m/2}, p^{m/2}) e^{2\pi i d/p^{m/2}} r_{p^m} = 0. \tag{1}$$

Proof - Continued

On the other hand since $ad \equiv 1 \pmod{p^{m/2}}$, as a runs through the set $\{a : 1 \le a \le p^{m/2}, \gcd(a, p^{m/2}) = 1\}$, so will d. Thus we have

$$\sum_{\substack{a \pmod{p^{m/2}},\\ \gcd(a,p^{m/2})=1}} e^{2\pi i d/p^{m/2}} = \sum_{\substack{d \pmod{p^{m/2}},\\ \gcd(d,p^{m/2})=1}} e^{2\pi i d/p^{m/2}} = \mu(p^{m/2}). \quad (2)$$

Thus when $m > 2$, combining (1) and (2) we obtain

$$\begin{aligned} 0 &= \sum_{\substack{a \pmod{p^{m/2}},\\ \gcd(a,p^{m/2})=1}} (a_n(p^{m/2}, 1) r_1 + a_n(p^{m/2}, p^{m/2}) e^{2\pi i d/p^{m/2}} r_{p^m}) \\ &= \varphi(p^{m/2}) a_n(p^{m/2}, 1) r_1. \end{aligned}$$

Thus we obtain $r_1 = 0$. Then using (1) we obtain $r_{p^m} = 0$.

Thanks!