Scattering Theory of Locally Symmetric Spaces

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Outline

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1. Review of geometric scattering theory
2. Scattering geodesics in locally symmetric spaces of $\mathbb{Q}$-rank one
3. Scattering flats in locally symmetric spaces of higher $\mathbb{Q}$-rank
4. Main theorems and overview of proof method
5. Future Directions

Introduction

Geometric scattering theory was introduced by physicists Ludvig Faddeev and Victor Popov to establish a rigorous framework for the scattering processes that show up in Quantum field Theory. Scattering theory compares the asymptotic behavior of an evolving system as t tends to $-\infty$ with its asymptotic behavior as t tends to ∞. It is especially fruitful for studying systems constructed from a simpler system by the imposition of a disturbance (also called perturbation or scatterer) provided that the influence of the disturbance on motions at large $|t|$ is negligible.

There are several frameworks for studying geometric scattering, the one that is of particular interest to Number theory and Spectral geometry is the framework developed by Lax and Phillips.

Scattering on finite area hyperbolic surfaces

Victor Guillemin was the first one who realized that just as in the case of a compact manifold, there is an analogous poisson relation between sojourn times of scattering geodesics and singularities of the scattering matrices for non compact hyperbolic surfaces. Here we will review the work of Guillemin's paper.

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Let $\mathbb{H} = \{z = x + iy | x, y \in R, y > 0\}$ is the upper half plane with the assigned hyperbolic metric $ds^2 = \frac{dx^2+dy^2}{y^2}$. The associated laplacian is given by $\Delta = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$, since $\mathbb{H}$ is a complete Riemannian manifold, the laplacian has a unique self adjoint extension which is also be denoted by Δ.

Scattering on finite area hyperbolic surfaces

Now consider a cofinite discrete torsion free subgroup Γ of PSL(2,$\mathbb{R}$) and let $X = \Gamma\backslash\mathbb{H}$ be the associated finite area non-compact hyperbolic surface with $k_1, ..., k_n$ inequivalent cusps . One then knows that, for any sufficiently large "a" , X is a disjoint union of compact subset X_a and a finite number of open sets X_i , $i = 1, 2, , , , , n$, where X_i is the cusp neighbourhood for the corresponding cusp k_i and so that each X_i is isometric to the set $\{-1/2 \leq Re(z) \leq 1/2||Im(z)| \geq a\}$ in the upper half plane.

Scattering geodesics on finite area hyperbolic surfaces

Definition

A geodesic $\gamma(t)$ in X is called a scattering geodesic if it is contained in $X \backslash X_a$ for large positive as well as negative times t. A scattering geodesics that is contained in X_i for $t \ll t_0$ and in X_j for $t_1 \ll t$ is called a geodesic scattered between cusp ends X_i to X_j . The associated **sojourn time** T_γ is the total amount of time the geodesic spends in the compact core X_a, starting from the first time it entered X_a until the time when it exits.

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Asume the cusp k_i is at ∞ and the cusp k_j will then be a vertex of the fundamental domain lying on the real axis given by the point $(x_j, 0)$ with the cusp neighborhood bounded bytwo geodesics σ_1 and σ_2 which are perpendicular to the real axis at $(x_j, 0)$.

Scattering geodesics on finite area hyperbolic surfaces

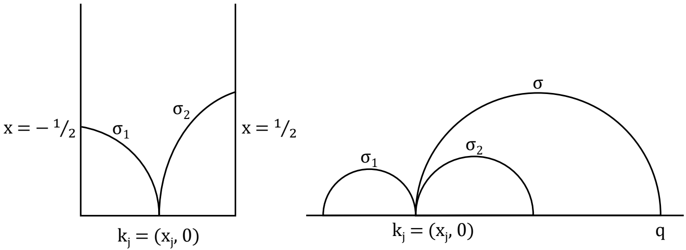

Figure: Construction of scattering geodesics.

Then for a $B \in \Gamma$ such that $Bq = \infty$, the image of $B\sigma$ in $\Gamma \backslash \mathbb{H}$ is a scattering geodesic. This shows that there are countable number of scattering geodesics running between two fixed cusp neighborhoods.

Scattering geodesics on finite area hyperbolic surfaces

We further choose an isometry Ψ mapping the vertical strip $\{-1/2 \leq Re(z) \leq 1/2\}$ onto the j-th cusp neighborhood such that $\Psi(\infty) = k_j$.
Associated to the i-th cusp, we have the **Eisenstein Series** $E_\infty(z, s)$ given by,

$$E_\infty(z, s) = \sum_{B \in \Gamma_\infty \backslash \Gamma} (Im(Bz))^s$$

Now we set $s = 1/2 + i\tau$, and let $E(z, \tau) = E_\infty(z, 1/2 + i\tau)$. Then observe that the zero-th Fourier coefficient in the expansion of $E(z, \tau)$ in the j-th cusp neighborhood is given by the integral,

$$e^{-2i\tau ln(a)} \int_{-1/2}^{1/2} E(\Psi z, \tau) dx = C_{ij}(\tau) y^{1/2 - i\tau}$$

Scattering geodesics on finite area hyperbolic surfaces

Where, $C_{ij}(\tau)$ is the ij-th entry of the scattering matrix.

Theorem (Guillemin)

Let $\mathcal{T}_{ij}$ be the set of sojourn times for geodesics that are scattered from the i-th cusp neighborhood to the j-th cusp neighborhood. Define the following integral,

$$F(\tau) = \int_{-\infty}^{\infty} (1 + w^2)^{-(1/2 + i\tau)} dw \tag{1}$$

Then for $Im(\tau) \leq -3/2$, one has

$$C_{ij}(\tau) = aF(\tau) \sum_{T_\sigma \in \mathcal{T}_{ij}} e^{-T_\sigma(1/2 + i\tau)} \tag{2}$$

For a general τ, the right-hand side is supposed to be the meromorphic continuation of this series.

Scattering geodesics in Riemannian manifolds

Scattering geodesics in Riemannian manifolds

Definition

Let S be a complete Riemannian manifold. A geodesic $\gamma(t)$ in S is a scattering geodesic if it is eventually minimizing in both directions, in the sense that there exists $t_1 < t_2 \in \mathbb{R}$ such that $\gamma|_{(-\infty,t_1]}$ and $\gamma|_{[t_2,\infty)}$ are both isometric embedding onto its image.

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Consider the case $S = \Gamma\backslash X$ a non-compact finite volume locally symmetric space with $X = G/K$, where G is a semisimple lie group, $K \subset G$ is a maximal compact subgroup and $\Gamma \subset G$ is an arithmetic subgroup. Such a space S admits a decomposition into a compact core and finite number of ends running to infinity. In this case, the **sojourn time** of a scattering geodesic can be defined as the time it spends in the compact core.

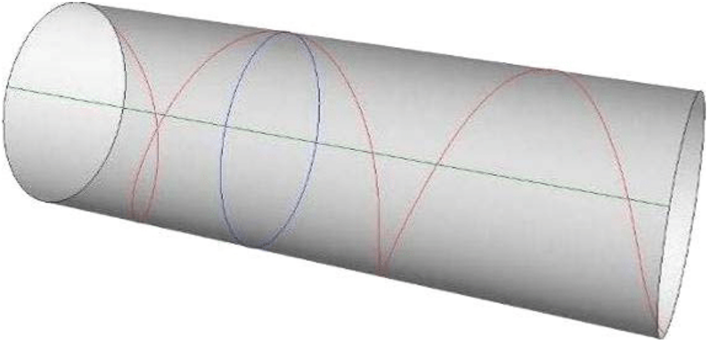

Figure: Geodesics on a cylinder

When S is a rank one locally symmetric space(this would be the case when G has rank one), Lizhen Ji and Maciej Zworski constructed a class of scattering geodesics which move to infinity in both directions and are distance minimizing near both infinities. An associated sojourn time was defined for such a scattering geodesic, which is the time it spends in a fixed compact region. One of their main results was that the frequencies of oscillation coming from the singularities of the Fourier transforms of scattering matrices on $\Gamma\backslash X$ occur at sojourn times of scattering geodesics on the locally symmetric space.

Scattering geodesics in S

Scattering geodesics in S

Theorem (JZ)

Assume that the $\mathbb{Q}$-rank of S is equal to one. Let $\sigma(t)$ be a scattering geodesic in S between the ends associated with two rational parabolic subgroups Q_1 and Q_2. Then $\sigma(t)$ lies in a smooth family of scattering geodesics of the same sojourn time parametrized by a common finite covering space X_{12} of the boundary locally symmetric spaces X^{Q_1} and X^{Q_2} and the set of sojourn times of all scattering geodesics forms a discrete sequence of points in $\mathbb{R}$ of finite multiplicities.

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Theorem (JZ)

Assume that the $\mathbb{Q}$-rank of S is equal to one. Let $C_{ij}(\lambda)$ denote the scattering matrix associated to the ith and j-th end of S. Then we have $\{Singular\ Support(\hat{C}_{ij}(\lambda))\} = \mathcal{T}$, where $\mathcal{T}$ is the set of Sojorun times associated to scattering geodesic running between the ith and jth end of S.

Scattering Flats in S

Now consider a locally symmetric space of rank greater than one given by $S = \Gamma\backslash X = \Gamma\backslash G/K$ as before. Let $\mathfrak{g}$ denote the Lie algebra of G.

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Definition

Let $\mathfrak{a}$ be an abelian subalgebra of $\mathfrak{g}$, denote by Σ the set of roots corresponding to the adjoint action of $\mathfrak{a}$ on $\mathfrak{g}$, then a scattering flat is a flat submanifold F of $\Gamma\backslash X$ of dimension equal to rank($\mathfrak{a}$), given by a smooth immersion $\Psi : \mathfrak{a} \longrightarrow \Gamma\backslash X$, such that for any choice of a full subset of positive roots Σ^+ of Σ with the associated positive chamber $\mathfrak{a}^+$, we have that the restriction of Ψ to the shifted Weyl chamber $\mathfrak{a}^+(\Sigma, H)$ is an isometric embedding into $\Gamma\backslash X$, where $H \in \mathfrak{a}^+$ with $|H| >> 0$ and
$\mathfrak{a}^+(\Sigma, H) = \{X \in \mathfrak{a} | \beta(X - H) > 0 \forall \beta \in \Sigma^+\}$

Review of Lie algebra and parabolic subgroups of $SL(3, \mathbb{R})$

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We will denote by $\mathfrak{g}$ the lie algebra of $SL(3,\mathbb{R})$ consisting of 3 by 3 traceless matrices with real entries. Let H denote the cartan subalgebra consisting of diagonal matrices in $\mathfrak{g}$, we will identify H with the subspace of $\mathbb{R}^3$ given by $\{(h_1,h_2,h_3)\in\mathbb{R}^3 | h_1+h_2+h_3=0\}$.

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Denote by $\alpha_{ij} : H \longrightarrow \mathbb{R}$ as the functionals defined by $\alpha_{ij}(h_1, h_2, h_3) = h_i - h_j$. Further let $\mathfrak{g}^{\beta} = \{Y \in \mathfrak{g} | [h, Y] = \beta(h)Y \; \forall h \in H\}$ where $\beta : H \longrightarrow \mathbb{R}$ is a nonzero linear functional.

Review of Lie algebra and parabolic subgroups of $SL(3, \mathbb{R})$

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The lie algebra $\mathfrak{g}$ admits the following root space decomposition,

$$\mathfrak{g} = H \oplus_{i \neq j} g^{\alpha_{ij}} \tag{3}$$

where E_{ij} denote the three by three matrix all of whose entires are zero, except for the ij-th entry which is 1.

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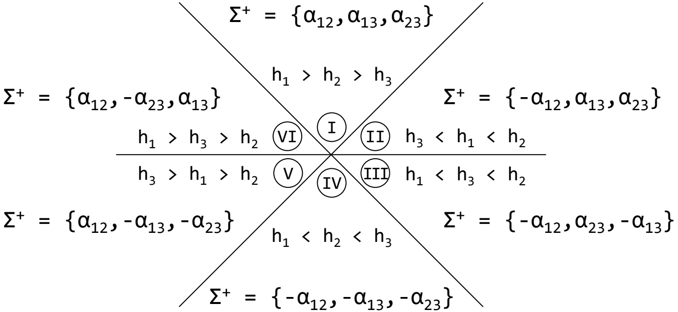
Note that for the adjoint action of H on $\mathfrak{g}$, $\Sigma^{++} = \{\alpha_{12}, \alpha_{23}\}$ serves as a set of simple roots and the corresponding set of positive roots is given by $\Sigma^{+} = \{\alpha_{12}, \alpha_{23}, \alpha_{13}\}$. We further define $\tau : H \longrightarrow \mathbb{R}$, to be half the sum of the three positive roots, more explicitly for $h = (h_1, h_2, h_3) \in H$, we have $\tau(h) = h_1 - h_3$. Associated to this root space decomposition , we have the Weyl group $\mathcal{W} = S_3$, which acts on the Cartan subalgebra H by permuting coordinates.

Review of Lie algebra and parabolic subgroups of $SL(3, \mathbb{R})$

Cartan Subalgebra $H = \left\{(h_1,h_2,h_3) \in \mathbb{R}^3 \mid \sum_{i=1}^{3} h_i = 0\right\}$



The Weyl group acts transitively on these chambers.

Review of Lie algebra and parabolic subgroups of $SL(3, \mathbb{R})$

Each chamber in H gives rise to a minimal parabolic subgroup in $SL(3, \mathbb{R})$ with a common split component H and any arbitrary minimal parabolic subgroup of $SL(3, \mathbb{R})$ is K-conjugate to one of these six parabolic subgroups where K is $SO(3)$.
Let N be the set of upper triangular unipotent matrices in $SL(3, \mathbb{R})$, A the subgroup of diagonal matrices of $SL(3, \mathbb{R})$ with positive diagonal entries. Then we denote by P_0 the minimal parabolic subgroup of $SL(3, \mathbb{R})$ consisting of upper traingular matrices along with the Langlands decomposition $P_0 = M_0AN$, where $M_0 = \{\pm Id_{3\times 3}\}$.
Apart from P_0, $SL(3, \mathbb{R})$ has two other standard maximal parabolic subgroups, which we denote by P_1 and P_2, any other parabolic subgroup is conjugate to one of these standard ones.

Scattering Flats in $SL(3,\mathbb{Z})\backslash SL(3,\mathbb{R})/SO(3)$

Let $G = SL(3,\mathbb{R})$ and $K = SO(3)$, we pick the arithmetic subgroup of G given by $\Gamma = SL(3,\mathbb{Z})$. We will look at two dimensional scattering flats in the locally symmetric space $S = \Gamma\backslash G/K$. Since, there is only one Γ-conjugacy class of minimal parabolic subgroups of $SL(3,\mathbb{R})$ with our chosen representative P_0, w.l.o.g we can say that any such scattering flats scatters between the Siegel end associated to P_0 and itself.

Choose $\gamma \in \Gamma - P_0$. Then γ admits a Bruhat decomposition with respect to P_0 given by $\gamma = u_2 z \gamma_a w u_1$ with $u_1, u_2 \in N$, $\gamma_a \in A$ and $z \in M_0, w \in S_3$

Scattering Flats in $SL(3,\mathbb{Z})\backslash SL(3,\mathbb{R})/SO(3)$

Note that such a γ gives rise to a family of scattering flats with a common associated sojourn vector given by $\log(\gamma_a) \in H$. Each scattering flat in this family scatters between the Siegel ends corresponding to the parabolic subgroups P_0 and $P_1 = \gamma P_0 \gamma^{-1}$.

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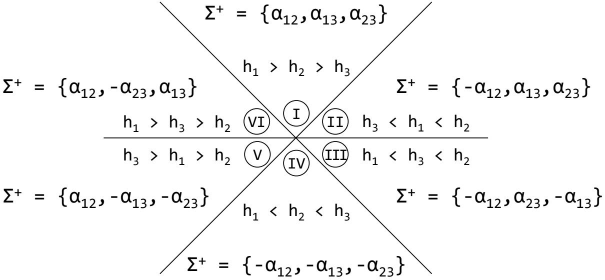
In case P_0 and P_1 correspond to adjacent chambers we can choose a maximal parabolic subgroup Q containing both P_0, P_1 along with Langlands decomposition $Q = M_Q A_Q N_Q$ and two $\mathbb{Q}$ rank one rational parabolic subgroups Q_0, Q_1 of M_Q, such that the boundary symmetric space X^Q can be naturally identified with the upper half plane $\mathbb{H}$ and the associated boundary locally space S_Q with the quotient $SL(2,\mathbb{Z})\backslash\mathbb{H}$.

chamber I corresponds to P_0,chamber II gives parabolic subgroup $P_1 = wP_0w^{-1}$, with $w = (12) \in S_3$.

Cartan Subalgebra $H = \{(h_1,h_2,h_3) \in \mathbb{R}^3 \mid \sum_{i=1}^{3} h_i = 0\}$



one can find a maximal parabolic subgroup P of $SL(3,\mathbb{R})$ which contains both P_0 and P_1. The family of scattering flats in S associated to the pair $\{P_0, P_1\}$ then projects onto a family of scattering geodesic in $SL(2,\mathbb{Z})\backslash\mathbb{H}$ with a common sojourn time whose A_P component corresponds to the common wall shared between chambers I and II.

Scattering Flats in $SL(3,\mathbb{Z})\backslash SL(3,\mathbb{R})/SO(3)$

Theorem

If two minimal parabolic subgroups P_0 and P_1 correspond to adjacent chambers(with respect to a common split component), then the family of scattering flats in S corresponding to the pair $\{P_0, P_1\}$ projects onto a family of scattering geodesics in $SL(2,Z)\backslash\mathbb{H}$ running between ends corresponding to Q_0 and Q_1 with a common sojourn time given by $|log(\gamma_a)|$, where $|\bullet|$ is the norm on the Lie algebra $\mathfrak{a}$ associated to the Killing form.

For future reference, denote the set of sojourn times associated to scattering geodesics running between the single cusp of $SL(2,Z)\backslash\mathbb{H}$ as $\mathcal{T}$.

Scattering Flats in $SL(3,\mathbb{Z})\backslash SL(3,\mathbb{R})/SO(3)$

We will now look at the rank two scattering matrices associated to S. Since, there is only one association class of minimal parabolic subgroups of $SL(3,\mathbb{R})$, these matrices only depend on $w \in S_3$ and $\lambda \in H^* \otimes_{\mathbb{R}} \mathbb{C}$ and denoted by $C(w,\lambda)$.
Also denote by $C(s)$ the scattering matrix associated with the unique cusp of $SL(2,\mathbb{Z})\backslash\mathbb{H}$.

Scattering Flats in $SL(3,\mathbb{Z})\backslash SL(3,\mathbb{R})/SO(3)$

Let $w = (12) \in S_3$, with $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in H^* \otimes_{\mathbb{R}} \mathbb{C}$ with $Re(\lambda) >> 0$ and $a, b \in \{1, 2, 3\}$ chosen such that $a < b$ and $w(a) > w(b)$, we have for $Im(\lambda_a - \lambda_b + 1) \leq -3$ and $\tilde{s} = (1/2)(\lambda_a - \lambda_b + 1)$,

$$C(w, \lambda) = F(\tilde{s}) \sum_{T \in \mathcal{T}} e^{-T(1/2 + i\tilde{s})}$$

where,

$$F(\tau) = \int_{-\infty}^{\infty} (1 + w^2)^{-(1/2 + i\tau)} dw$$

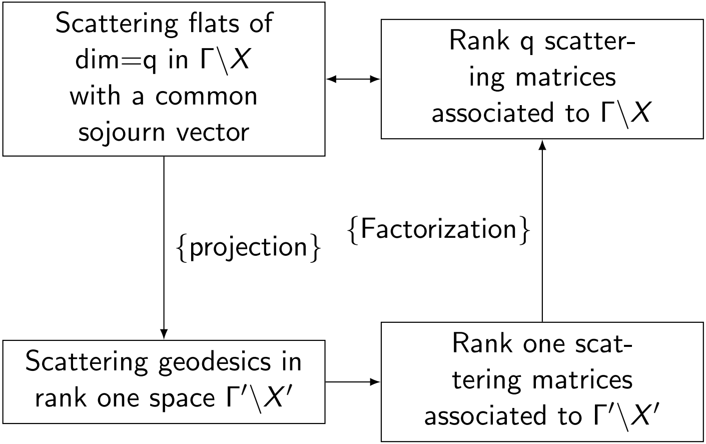


Figure: Correspondence between scattering flats and scattering matrices.

Thank you!

Main References

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