# Some ideas about chromatic homotopy theory

Hopkins talks Harvard 2007

Jack Morava (Saw in Orsay, biology) + discussions

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Starting point: Thom towers i.d.o. books ~ 2006.

$$D(\mathbb{S}) \xrightarrow[\sim\sim\sim]{\text{"concrete realization"}} D\left(\text{QCoh sheaves} \,/\, \text{Stack of 1-dim. formal groups}\right)$$

Stable category of spectra

$\uparrow$ tensoring with MU Thom spectrum

$\rightsquigarrow$ Study $\mathcal{M}_{FG}$ moduli of 1-dim. formal groups.

* Def: $\mathcal{M}_{FG} \to \mathrm{Spec}(\mathbb{Z})$ fpqc stack

$\mathcal{M}_{FG}(S) = \{ G$ 1-dim. commutative formal Lie group $/S \}$

formal group scheme locally $/S \simeq \hat{\mathbb{A}}^1_S$

→ Looks for other one later

usual <u>Presentation</u>: $\widetilde{\mathcal{M}}_{FG}$ = moduli scheme of 1-dim. formal gp. laws

$$\begin{array}{c} \widetilde{\mathcal{M}}_{FG} \\ \Big\downarrow {\scriptstyle \text{Zariski presentation}} \\ \mathcal{M}_{FG} \end{array}$$

fix a coordinate

$$\widetilde{\mathcal{M}}_{FG}(S) = \left\{ (\mathcal{G}, \iota) \;\middle|\; \iota : \hat{\mathbb{A}}^1_S \xrightarrow{\sim} \mathcal{G} \right\}$$

$0 \longleftrightarrow$ neutral section

$\widetilde{\mathcal{M}}_{FG} = \mathrm{Spec}(\Lambda)$ $\quad$ $\Lambda$ = Lazard ring $\simeq \mathbb{Z}[t_k]_{k \geq 1}$ polynomial ring

non canonical coordinates

<u>Localization at $p$</u>: $\mathcal{M}_{FG} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ simpler presentation

by moduli of $p$-typical formal group laws

$\iota : \hat{\mathbb{A}}^1_S \xrightarrow{\sim} \mathcal{G}$ s.t. $\iota$ seen as a curve in the Cartier module is $p$-typical

i.e. $F_n \cdot \iota = 0$ if $(n, p) = 1$

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$\simeq \operatorname{Spec}\left(\mathbb{Z}_p[v_b]_{b\geq 0}\right)$

Universal formal gp. law given by the Cartier module with V-base $\gamma$ and relation

$$F.\gamma = \sum_{b\geq 0} V^b [v_b].\gamma$$

Height stratification $(\mathcal{M}_{FG}\otimes\mathbb{Z}_p)^{\leq h}$ open subst ~~gp~~ where height $\leq h$, given by $v_h \neq 0$

$\rightsquigarrow$ plenty of computations.

* Completion at $p$: $\widehat{\mathcal{M}}_{FG}$ $p$-adic completion

↳ Change the test category

$\mathrm{Nilp}_p$ = Schemes on which $p$ is loc. nilpotent

$S \mapsto \{1\text{-dim}\text{-formal gp.}/S\}$

Lubin-Tate space / $\mathrm{Spf}(\breve{\mathbb{Z}}_p)$

$$\widehat{\mathfrak{M}}^{\leq h} \setminus \widehat{\mathfrak{M}}^{\leq h-1} = \left[ \widehat{H} \backslash \mathcal{X} \right]$$

$\widehat{H}$ pro-étale group / $\mathrm{Spf}(\breve{\mathbb{Z}}_p)$

$H_{\mathbb{Z}_{p^n}} = \mathcal{O}_D^\times$ Morava stabilizer

$\circlearrowleft$ $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ via $x \mapsto \pi x \pi^{-1}$

$H$ = twisted form of $\mathcal{O}_D^\times$

⇝ plenty of computations (see Goerss-Hopkins)

## Another presentation of $\widehat{\mathfrak{M}}^{\leq h}$.

$$\begin{array}{ccl} H & & BT_{1,h} = \text{stack of 1-dim. height } h \text{ } p\text{-divgroups} / \mathrm{Spf}(\breve{\mathbb{Z}}_p) \\ \downarrow & & \downarrow \text{fpqc cover} \\ \widehat{H} & & \widehat{\mathfrak{M}}^{\leq h} \end{array}$$

non-evident result

$BT_{1,h}$ is a complicated object - simplify it.

The real thing: Generic fiber as a diamond stacks

$$\left(\widehat{\mathfrak{M}}_{FG}\right)_\eta = v\text{-stack on } \mathrm{Perf}_{\mathbb{F}_p}$$

↑ Scholze's v-topology

stack associated to the prestack

$$(R,R^+) \longmapsto \{1\text{-dim. f.g.} / R^+\}$$

~~Perf~~ aff. perf. $/\mathbb{Q}_p$

$$\begin{array}{c} (BT_{1,h})_\eta \\ \big\downarrow {\scriptstyle v\text{-cover}} \\ \left(\widehat{\mathfrak{M}}_{FG}\right)_\eta^{\leq h} \end{array}$$

given by Hodge–Tate periods

th: $\quad (BT_{1,h})_\eta \simeq \left[\, \underline{GL_h(\mathbb{Q}_p)} \backslash \mathbb{P}^{h-1,\diamond}_{\mathbb{Q}_p} \right]$

Simple description

~~Explicit~~

$\Rightarrow$ Canonical chart

$$\boxed{\mathbb{P}^{h-1,\diamond} \twoheadrightarrow \left(\widetilde{\mathcal{M}}^{Sh}_{FG}\right)_h}$$

$GL_h(\mathbb{Q}_p)$-invariant

+ explicit description of the height stratification (chromatic layers)

$$\Omega^{\diamond} \subset \mathbb{P}^{h-1,\diamond} \quad \text{open stratum}$$

Second theorem: $(BT_{1,h})_s$ on $Perf_{\mathbb{F}_p}$ special fiber

[ Th: $(BT_{1,h})_s \xrightarrow{\sim} Bun^1_h$ ]

[ stack of degree 1 rk h v.b. on curve ]

height stratification

~~Newton~~ HN. stratification

modification of deg. 1 of $\mathcal{O}^h$/curve

$$\rightsquigarrow \quad \begin{tikzcd} \mathrm{Bun}^{1}_{h} \arrow[d, "v\text{-cover}"] & \mathbb{P}^{h-1,\diamond}_{\mathbb{Q}_p} \arrow[l] \arrow[d] \\ \left(\widehat{\mathfrak{X}}^{\le h}_{FG}\right)_s & \left(\widehat{\mathfrak{X}}^{\le h}_{FG}\right)_\eta \arrow[l, "sp"'] \end{tikzcd}$$

perfection stacks on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$

+ specialization $\cancel{(BT_{1,h})_\eta \xrightarrow{sp} (BT_{1,h})_s}$

Third theorem: $(BT_{1,h})_\eta$ = stack of $\varphi$-modules over $A_{\inf}$ $(M,\varphi)$ of rk. $h$ s.t. coker $\varphi$ killed by $\xi$

$\ker\theta = (\xi)$.

→ Maybe should study p-adic sheaves on $\mathbb{P}^{h-1,\diamond}_{\mathbb{Q}_p}$ ?

⚠ : Locally free $\mathcal{O}$-modules $/\mathbb{P}^{h-1}_{\mathbb{Q}_p} \hookrightarrow (-)/\mathbb{P}^{h-1,\diamond}_{\mathbb{Q}_p}$

v.b.

Eg v.b. $/\mathrm{Spa}(\mathbb{Q}_p)^\diamond = \mathrm{Rep}_{\mathbb{Q}_p}\left(\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)\right)$

(Sen theory)