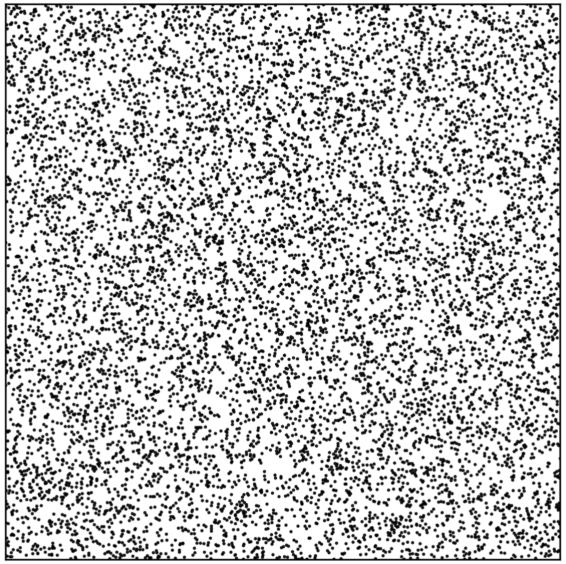
Scaling limit of Baxter permutations and Bipolar orientations

Mickaël Maazoun — Oxford University
Joint work with Jacopo Borga, University of Zürich

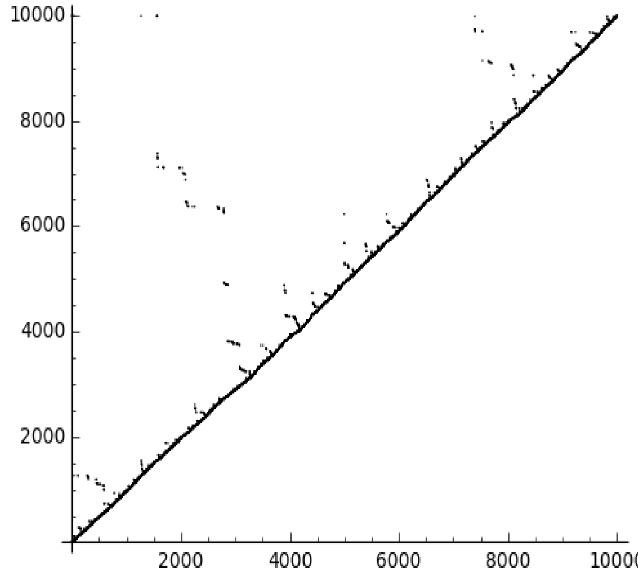
"Banff", 17 september 2020

version 2 of the slides, figures fixed

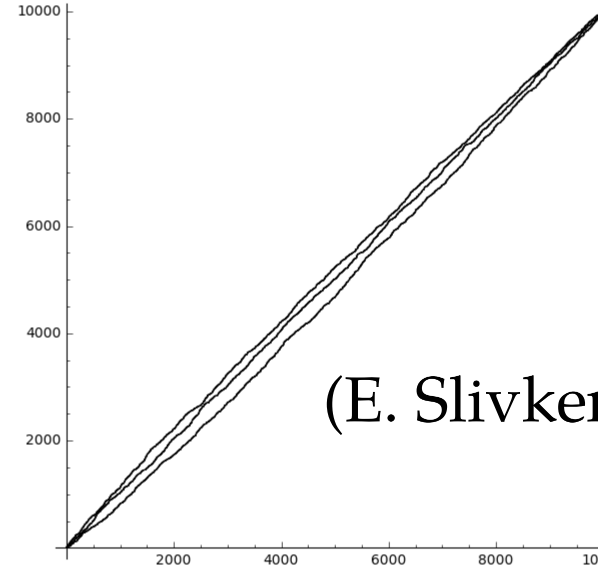
Limit shapes of uniform restricted permutations



$\mathfrak{S}_n$

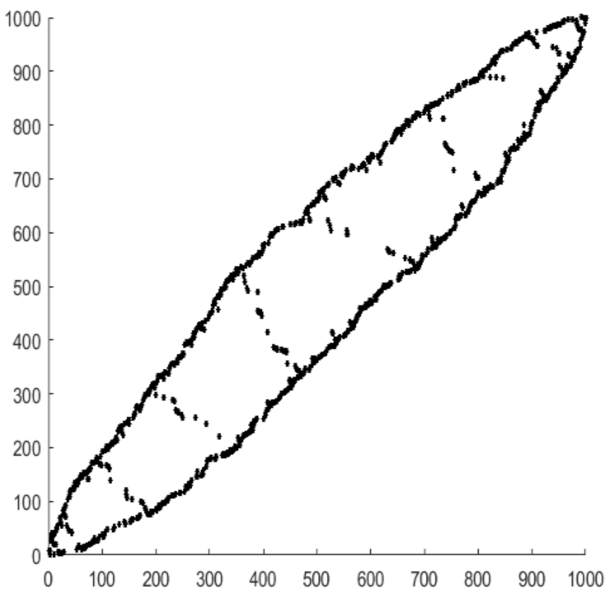


Av(231)

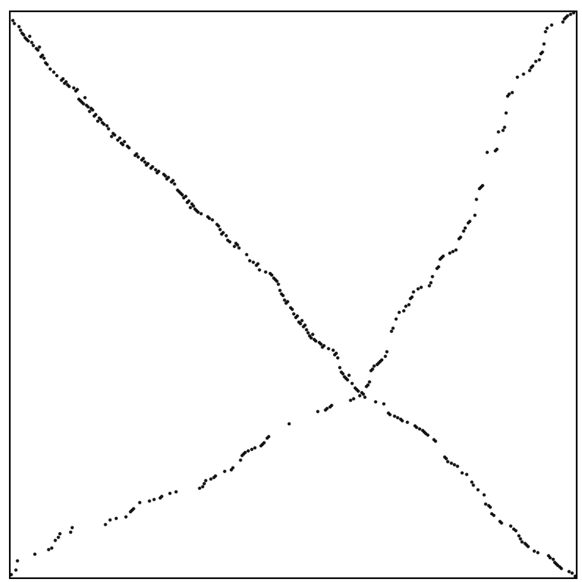


(E. Slivken)

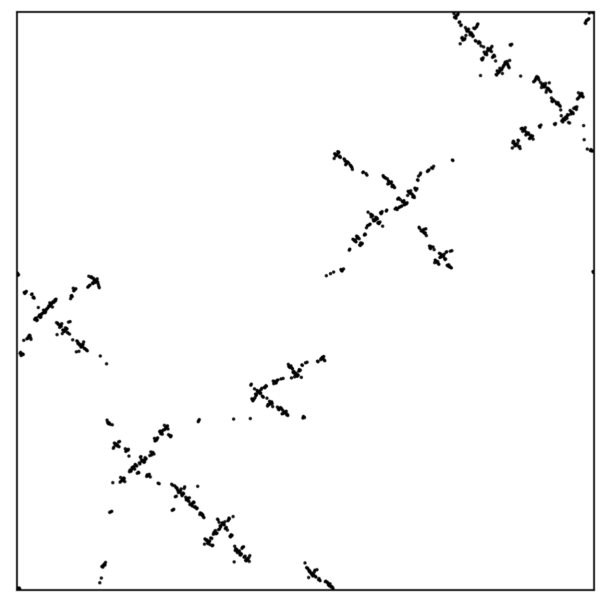
Av(4321)



Av(4231)
(Madras-Yildrim)

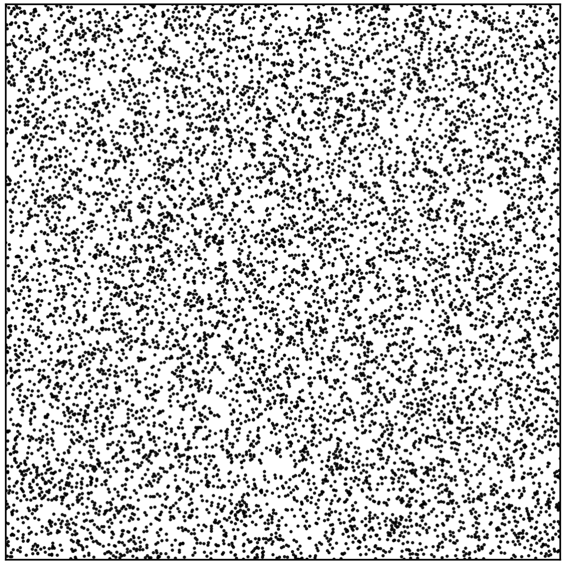


Av(2413, 3142, 2143, 34512)



Av(2413,3142)
={separables}

Limit shapes of uniform restricted permutations



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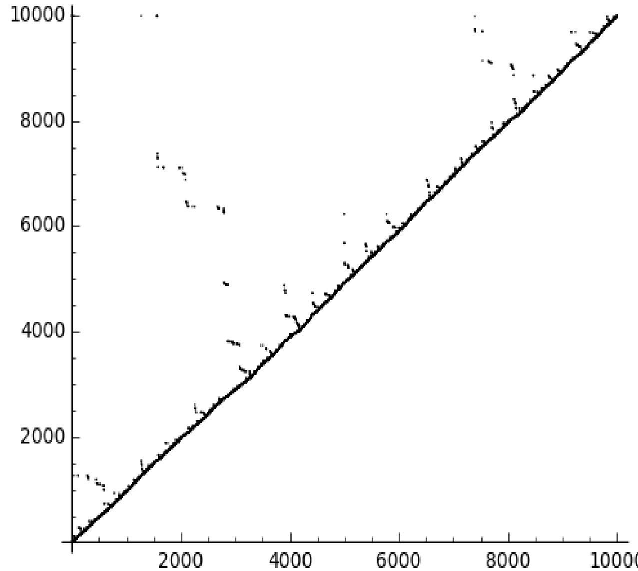

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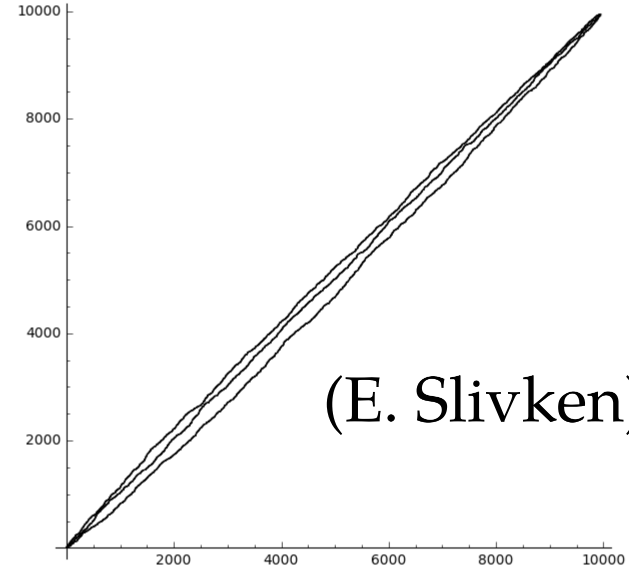

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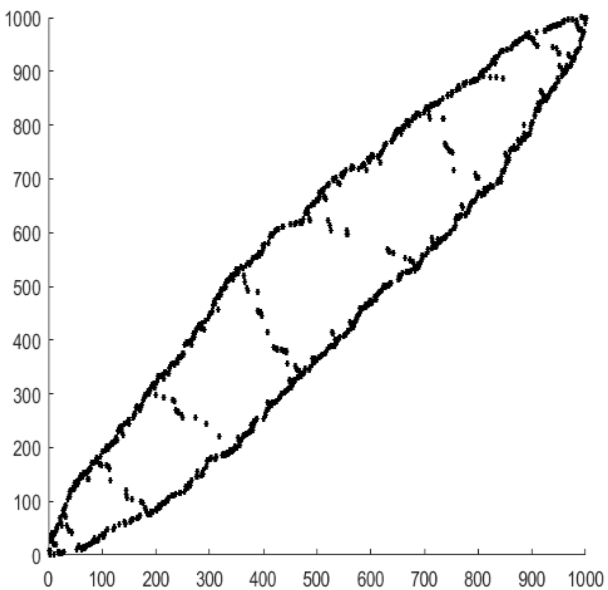

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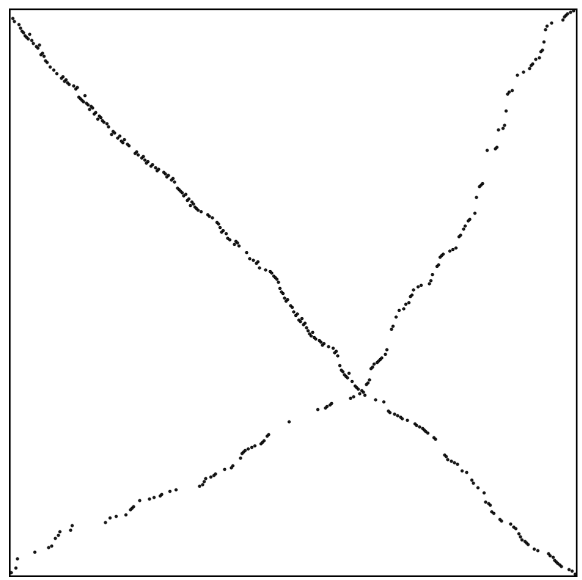

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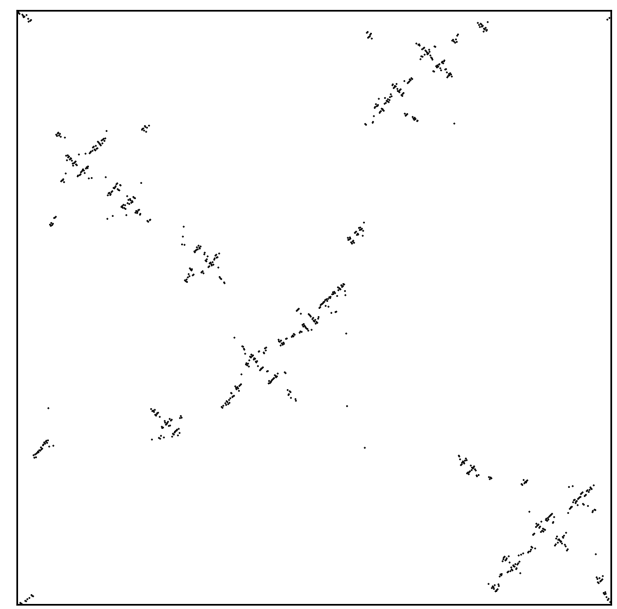

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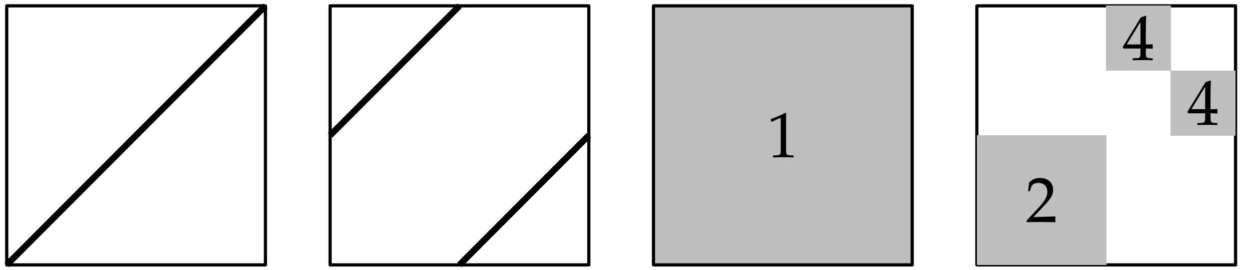
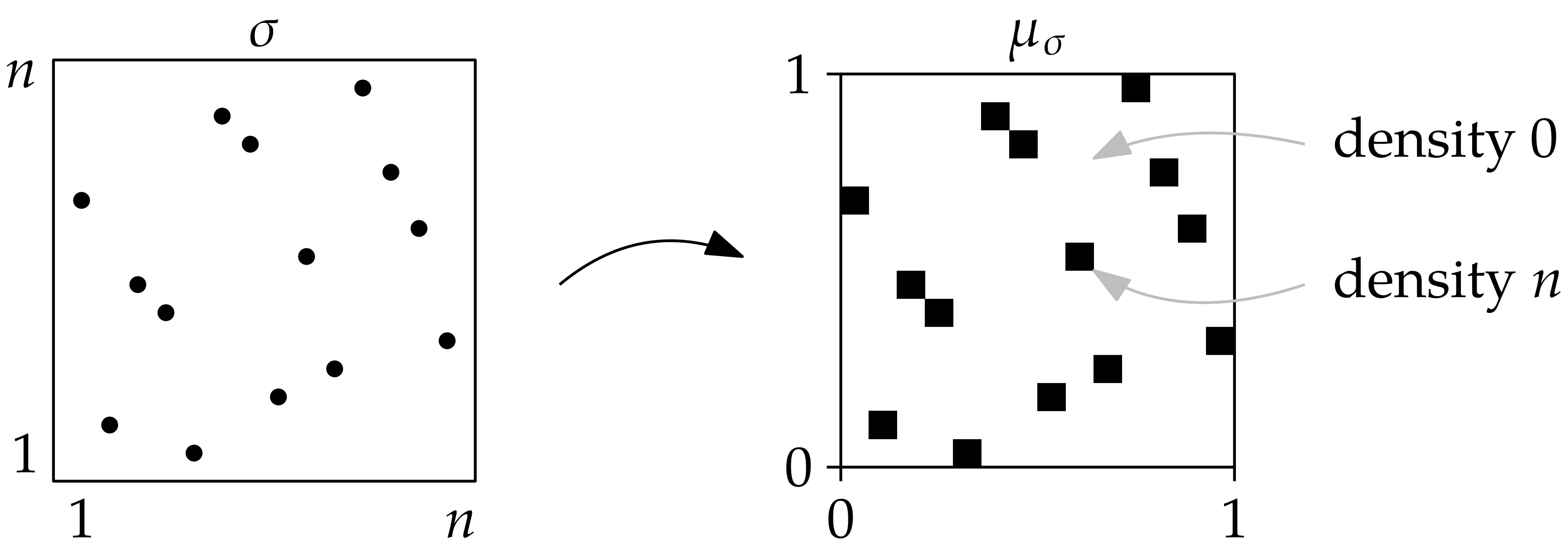
Permutons

A permuton is a probability measure on $[0,1]^2$ with both marginals uniform.



$\Longrightarrow$ compact metric space (with weak convergence).

Permutations of all sizes are densely embedded in permutons.



Baxter Permutations

A *Baxter* permutation avoids the vincular patterns $2\underline{41}3$ and $3\underline{14}2$. In other words, a permutation σ is Baxter if it is not possible to find $i < j < k$ s.t. $\sigma(j+1) < \sigma(i) < \sigma(k) < \sigma(j)$ or $\sigma(j) < \sigma(k) < \sigma(i) < \sigma(j+1)$.

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Counted by the Baxter numbers (A001181) $\sum_{k=1}^{n} \frac{\binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}}{\binom{n+1}{1}\binom{n+1}{2}} \sim \frac{2^{3n+5}}{\pi\sqrt{3}n^4}$ which count many other objects (see Felsner,Fusy,Noy,Orden 08)

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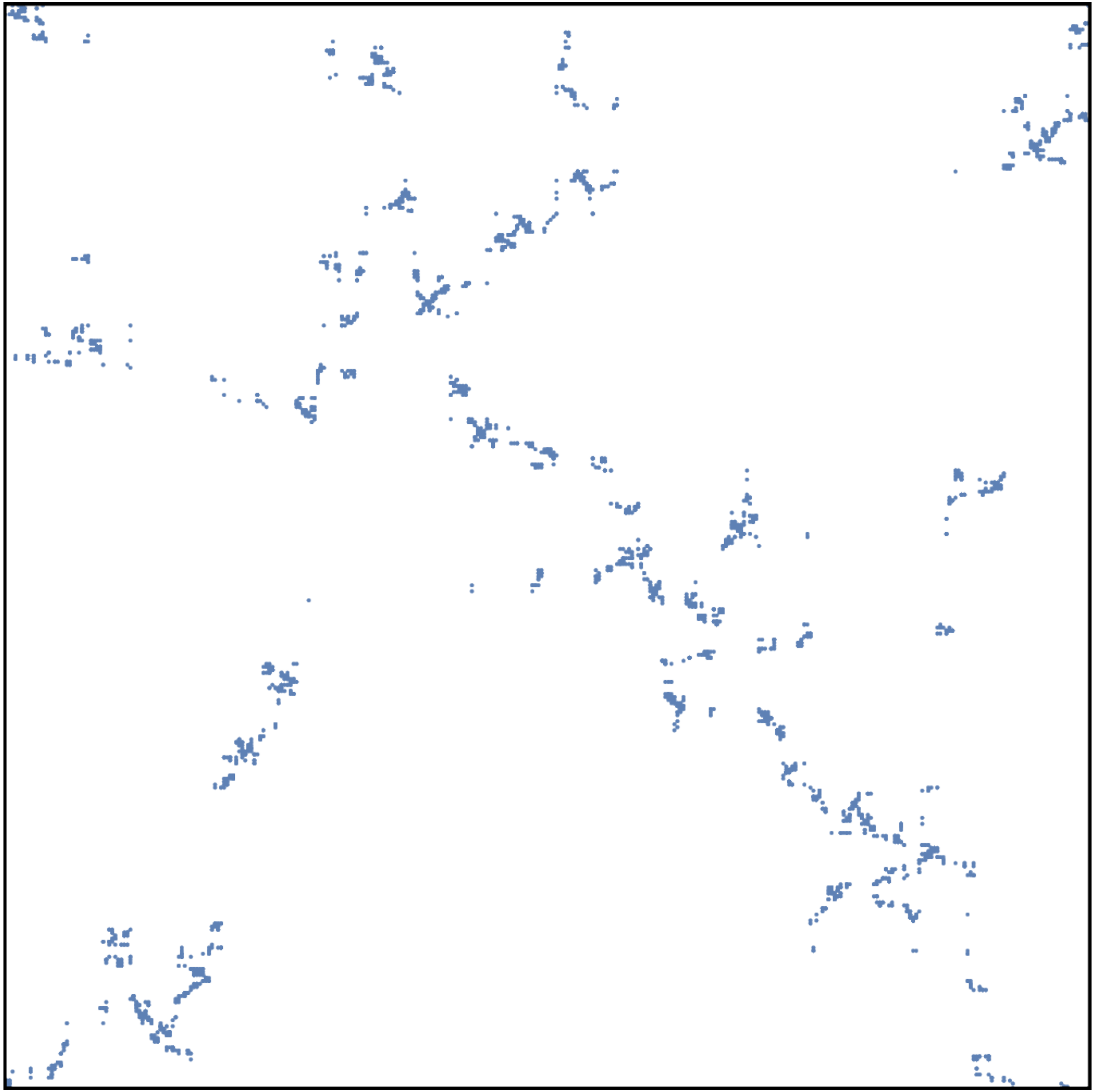
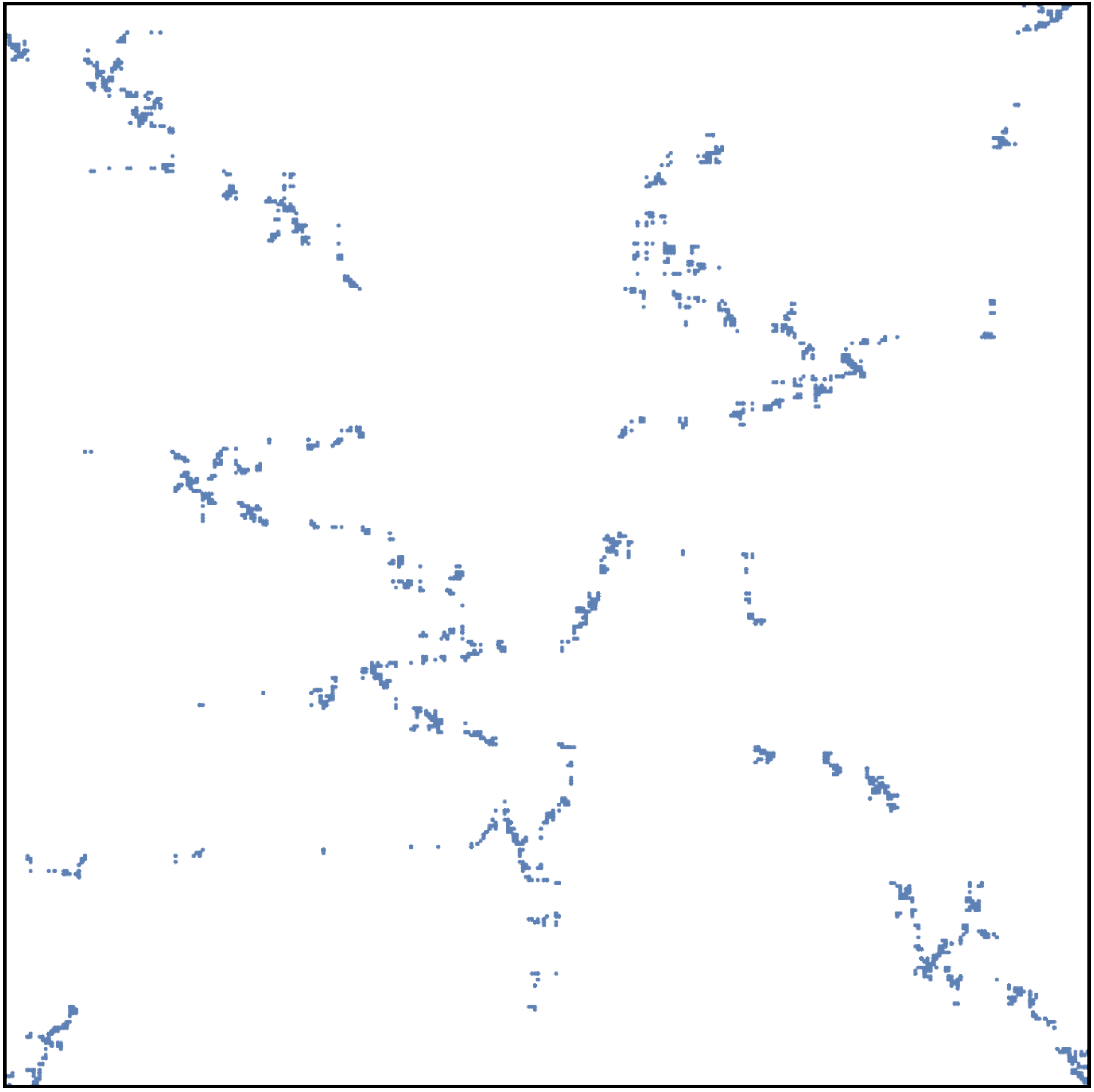
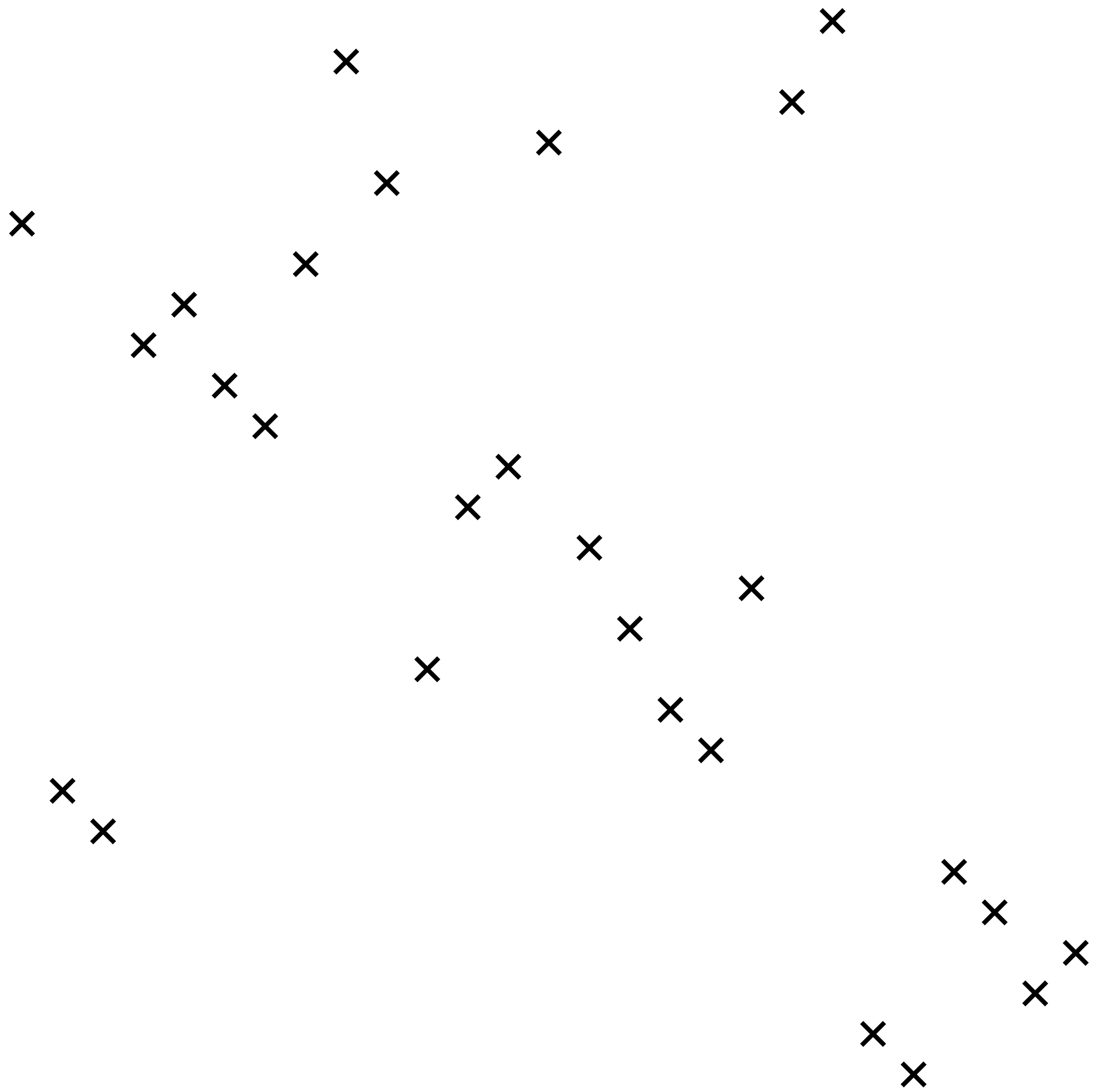
Theorem. (Borga, M) There exists a random permuton μ_B such that if σ_n is a uniform random Baxter permutation of size n, $\mu_{\sigma_n} \to \mu_B$ in distribution in the space of permutons.





Baxter permutations and bipolar oriented maps



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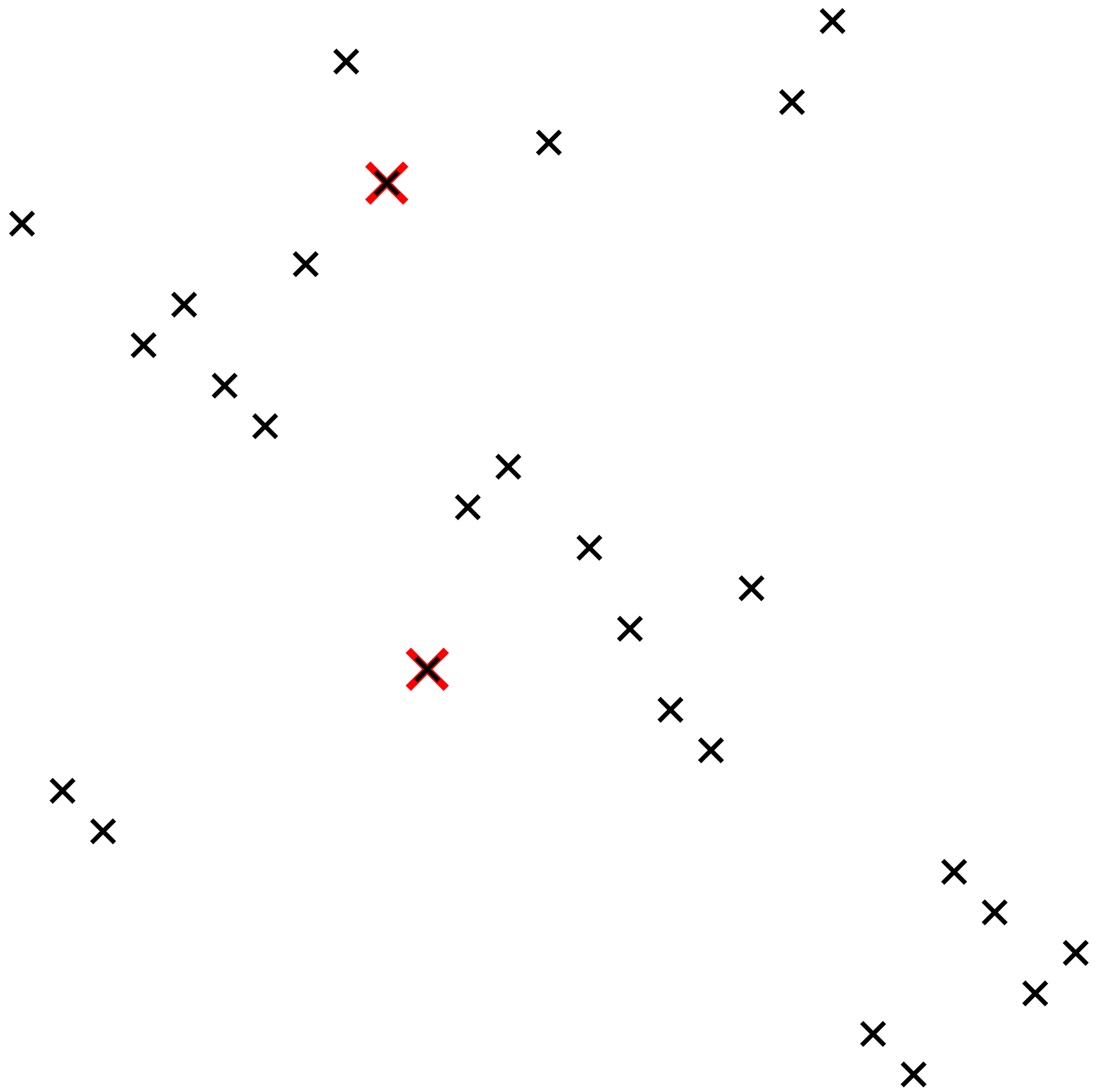
A *Baxter* permutation avoids the vincular patterns 2413 and 3142.

Baxter permutations and bipolar oriented maps



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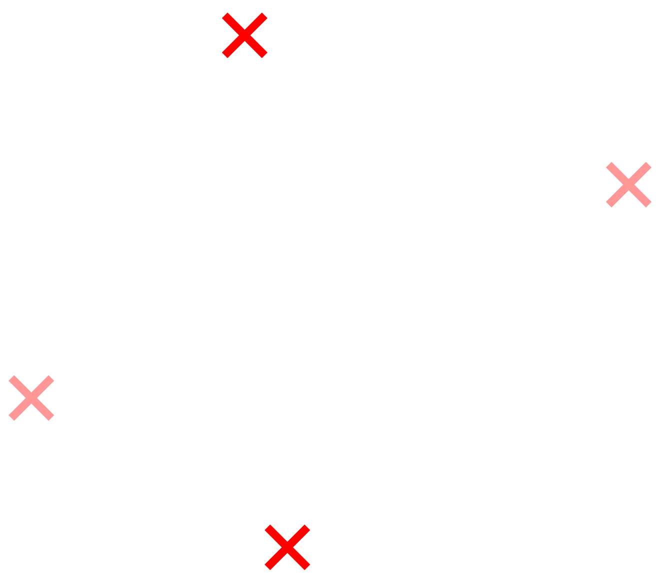

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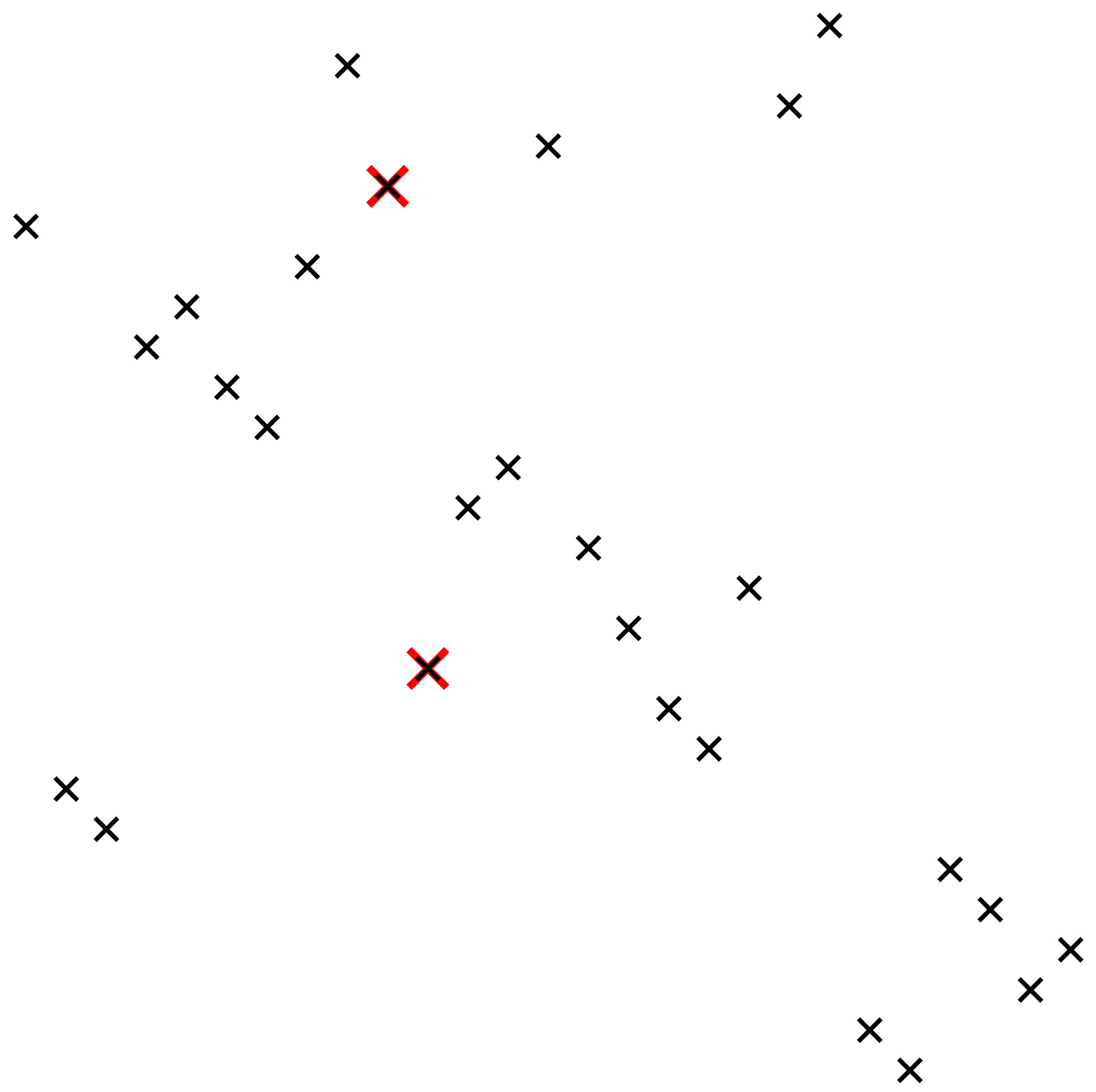

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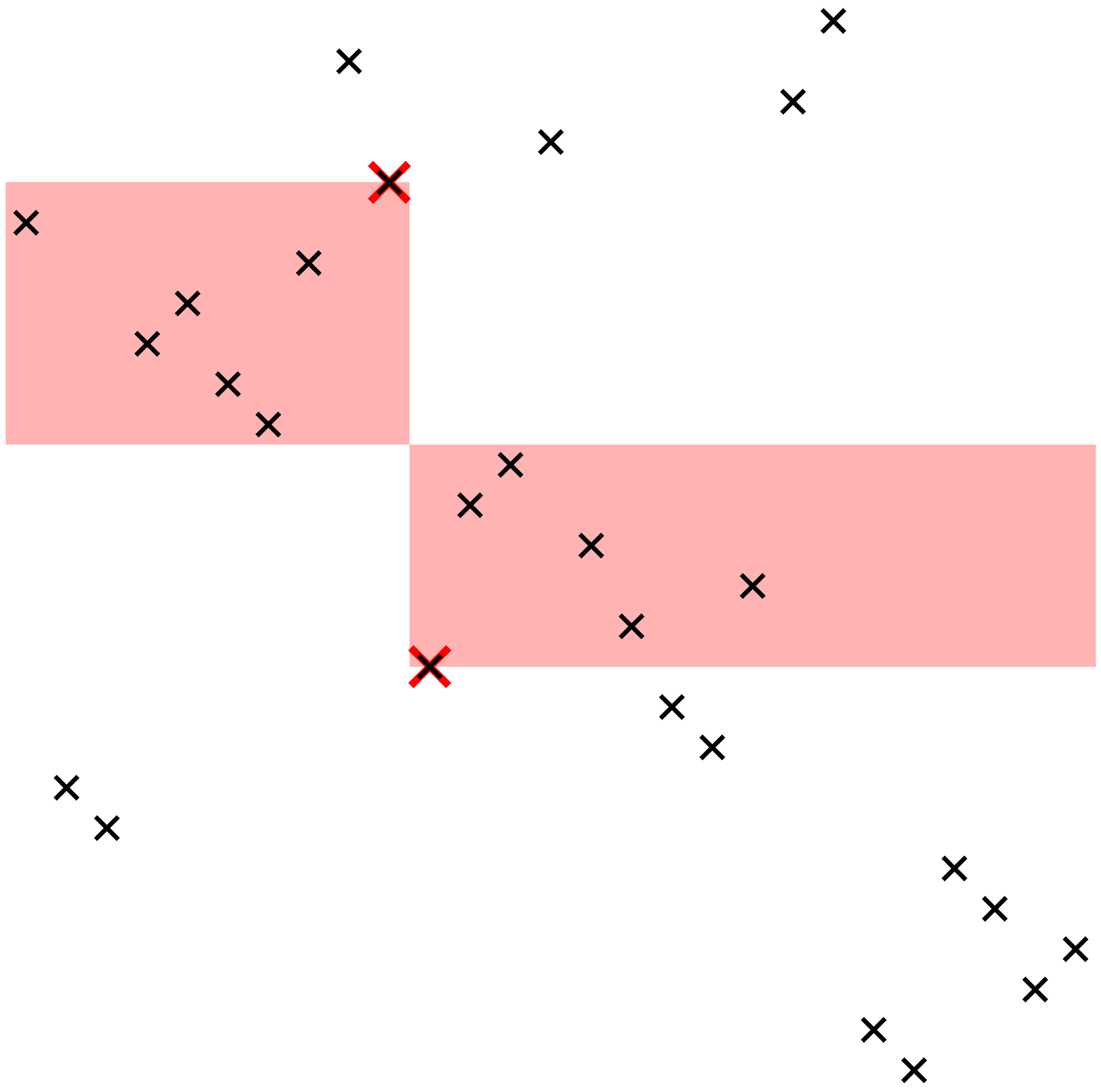

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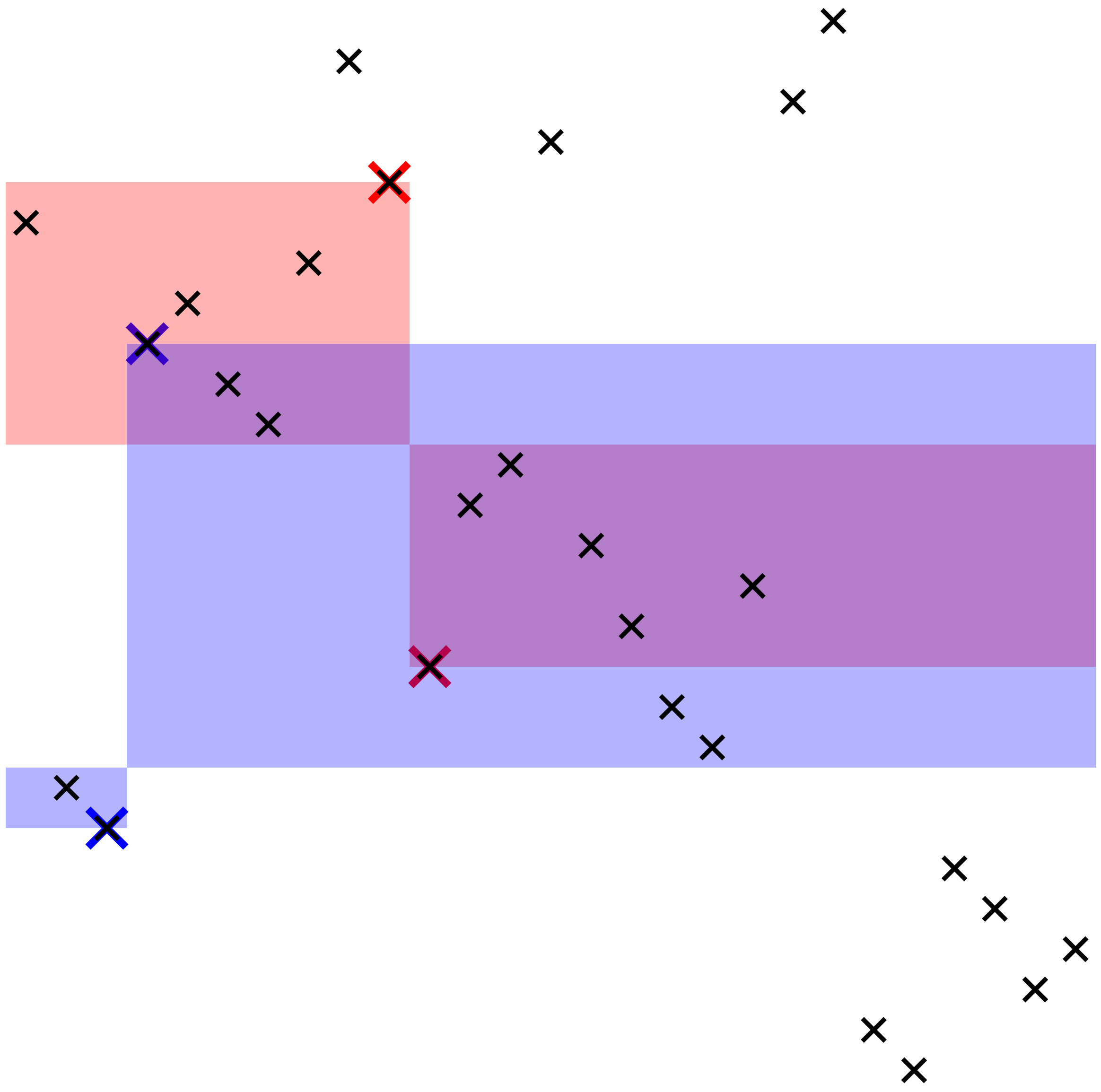

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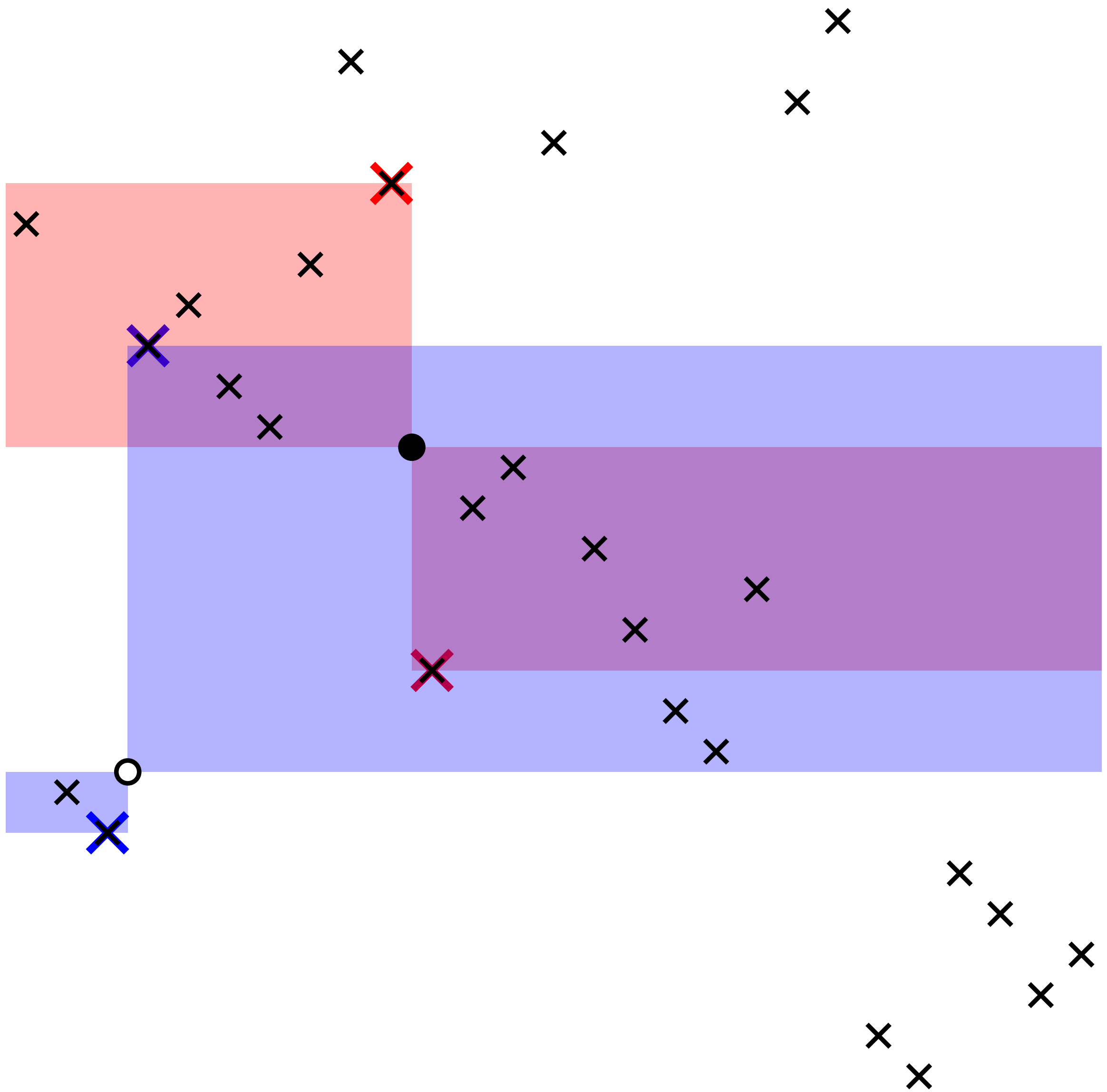

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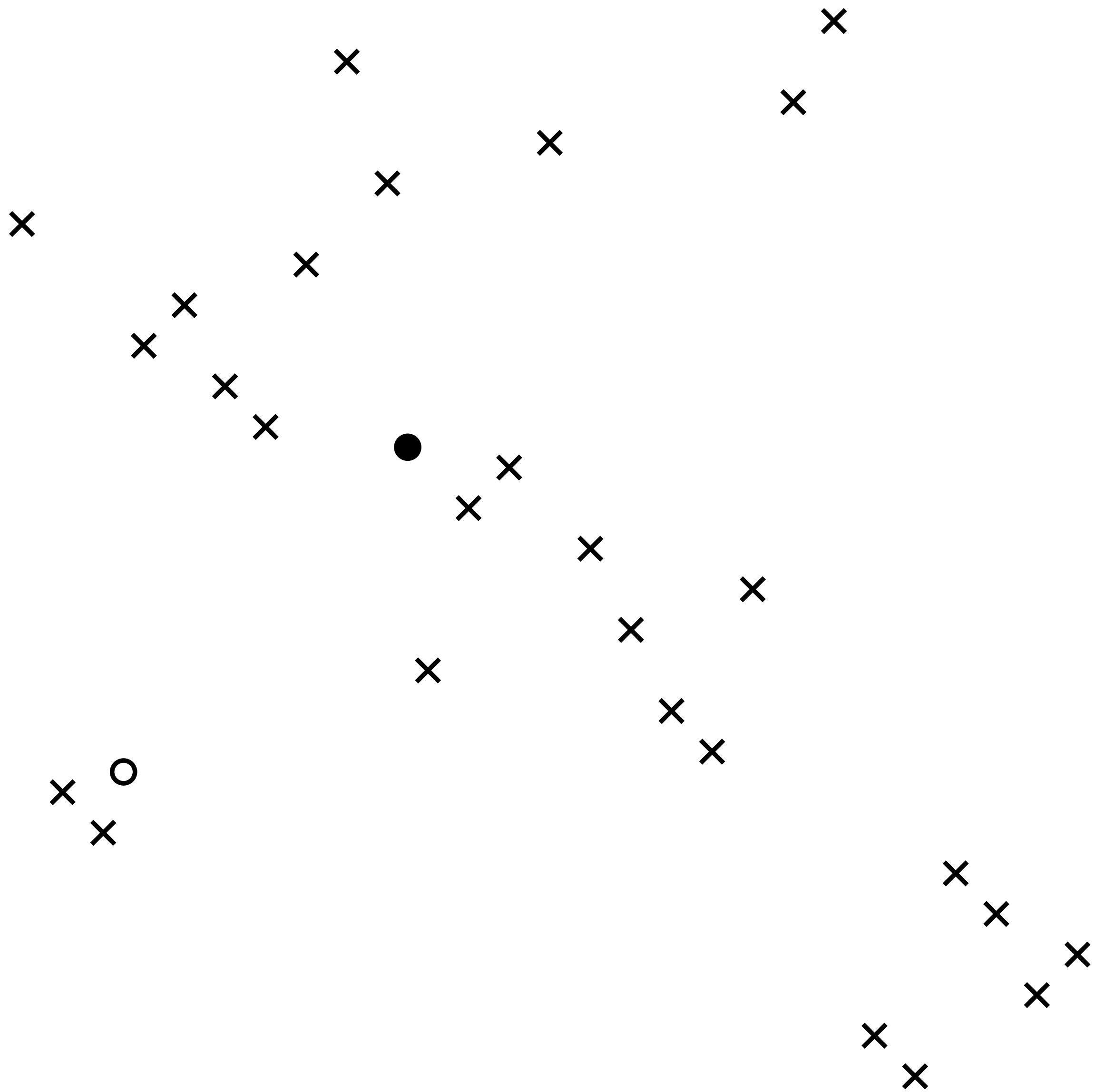

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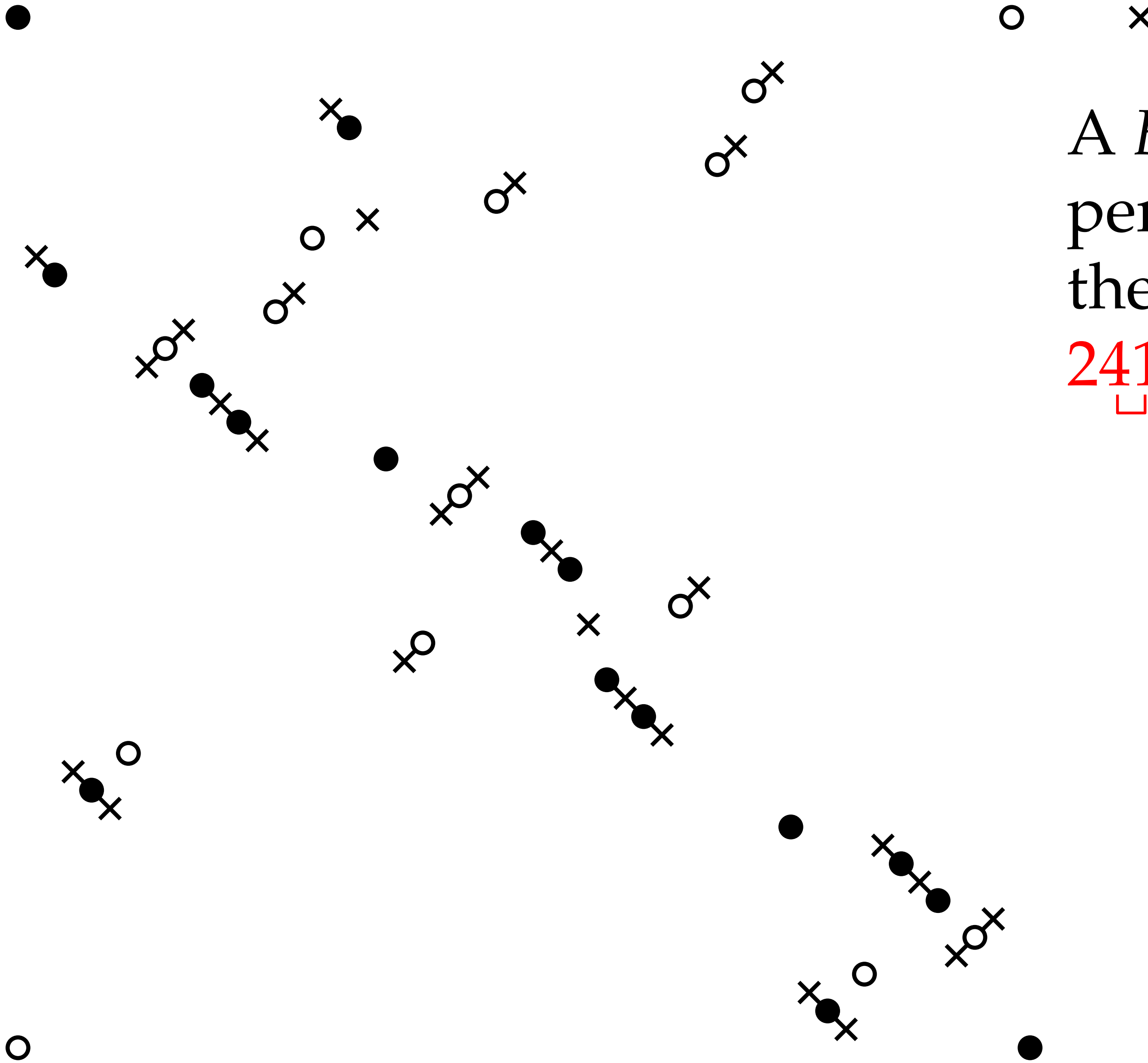

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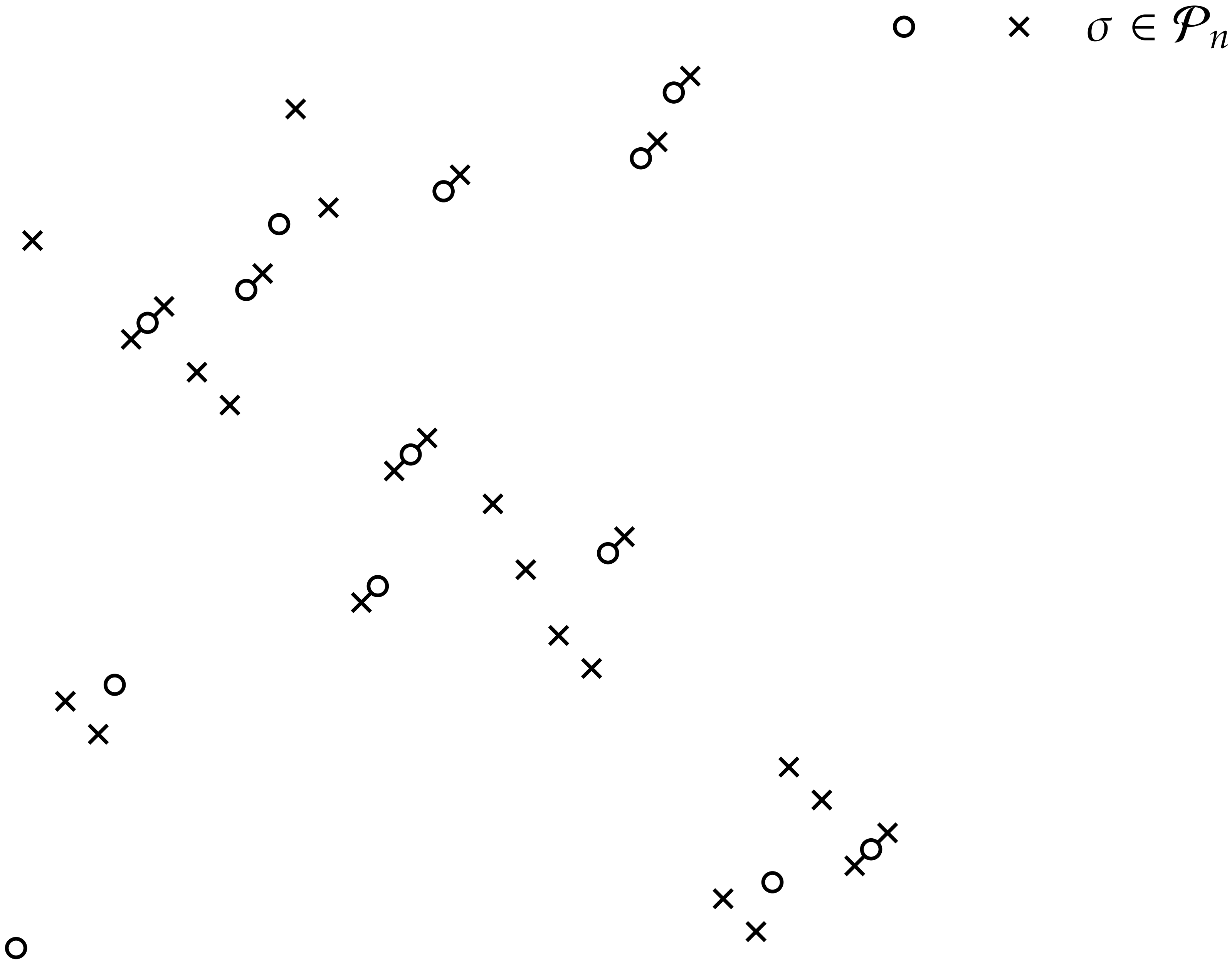

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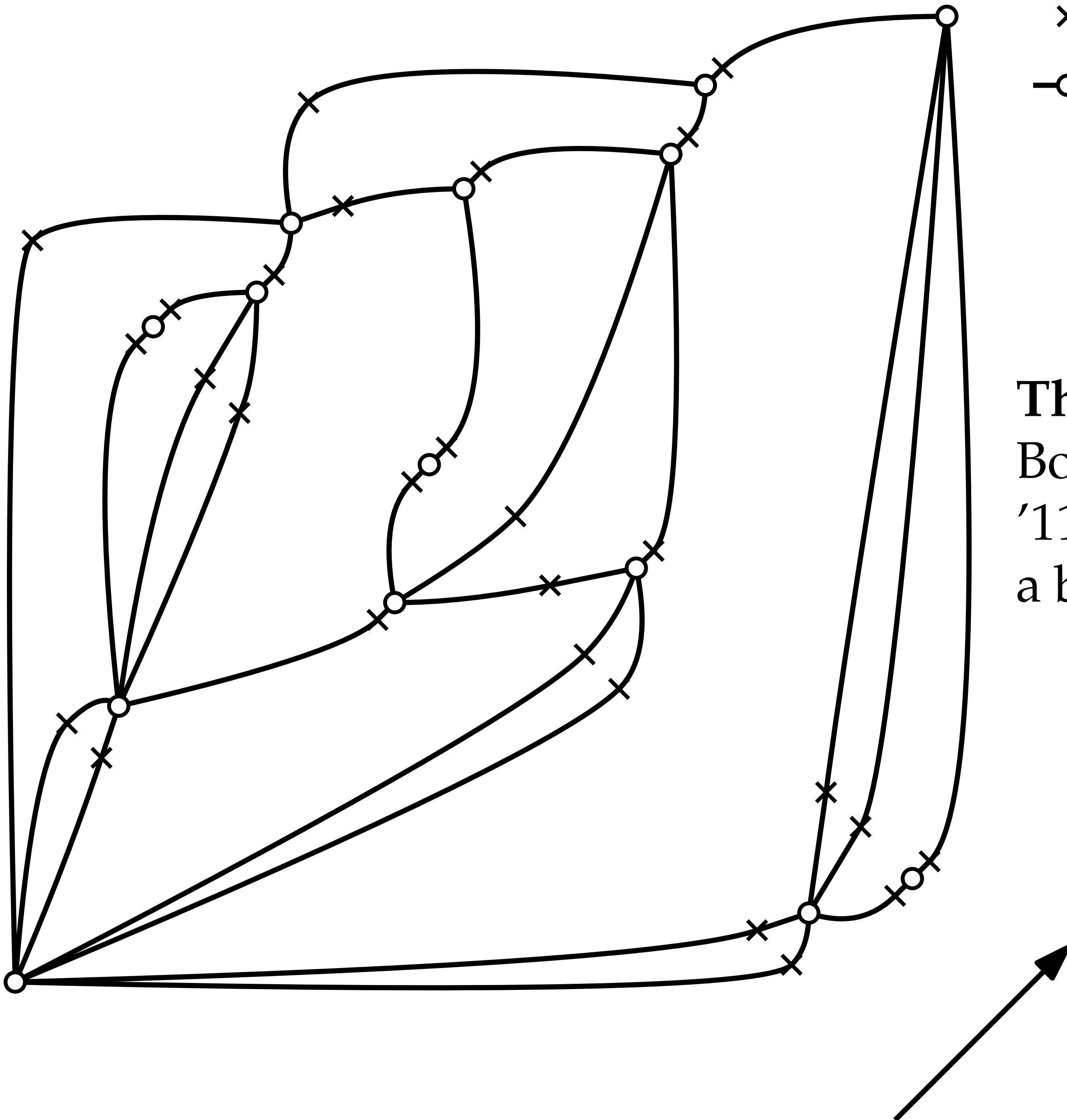

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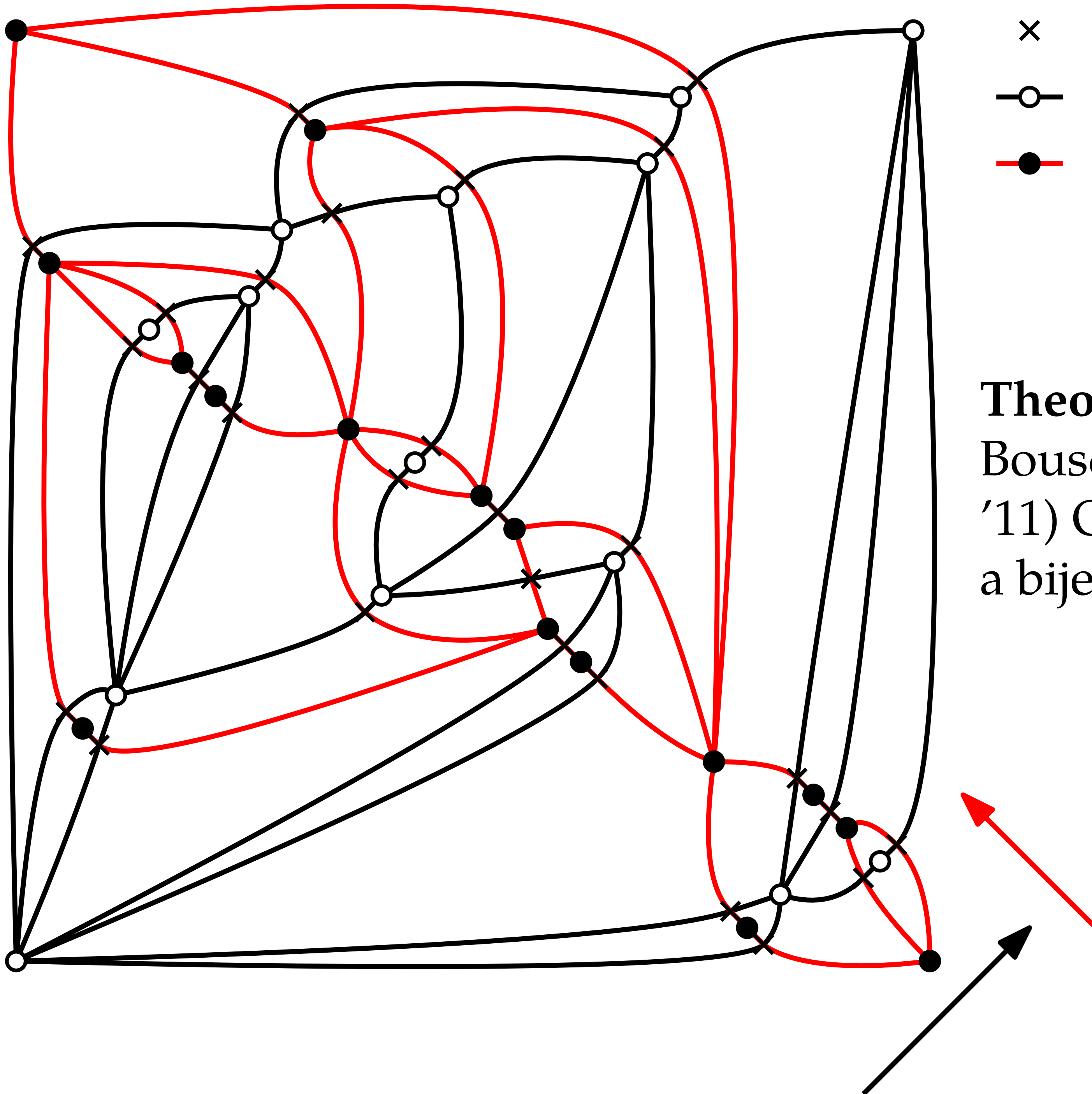
–o– $m = \mathrm{OP}^{-1}(\sigma) \in \mathcal{O}_n$

Theorem (Bonichon, Bousquet-Mélou, Fusy '11) $\mathrm{OP}^{-1} : \mathcal{P}_n \to \mathcal{O}_n$ is a bijection.

Baxter permutations and bipolar oriented maps



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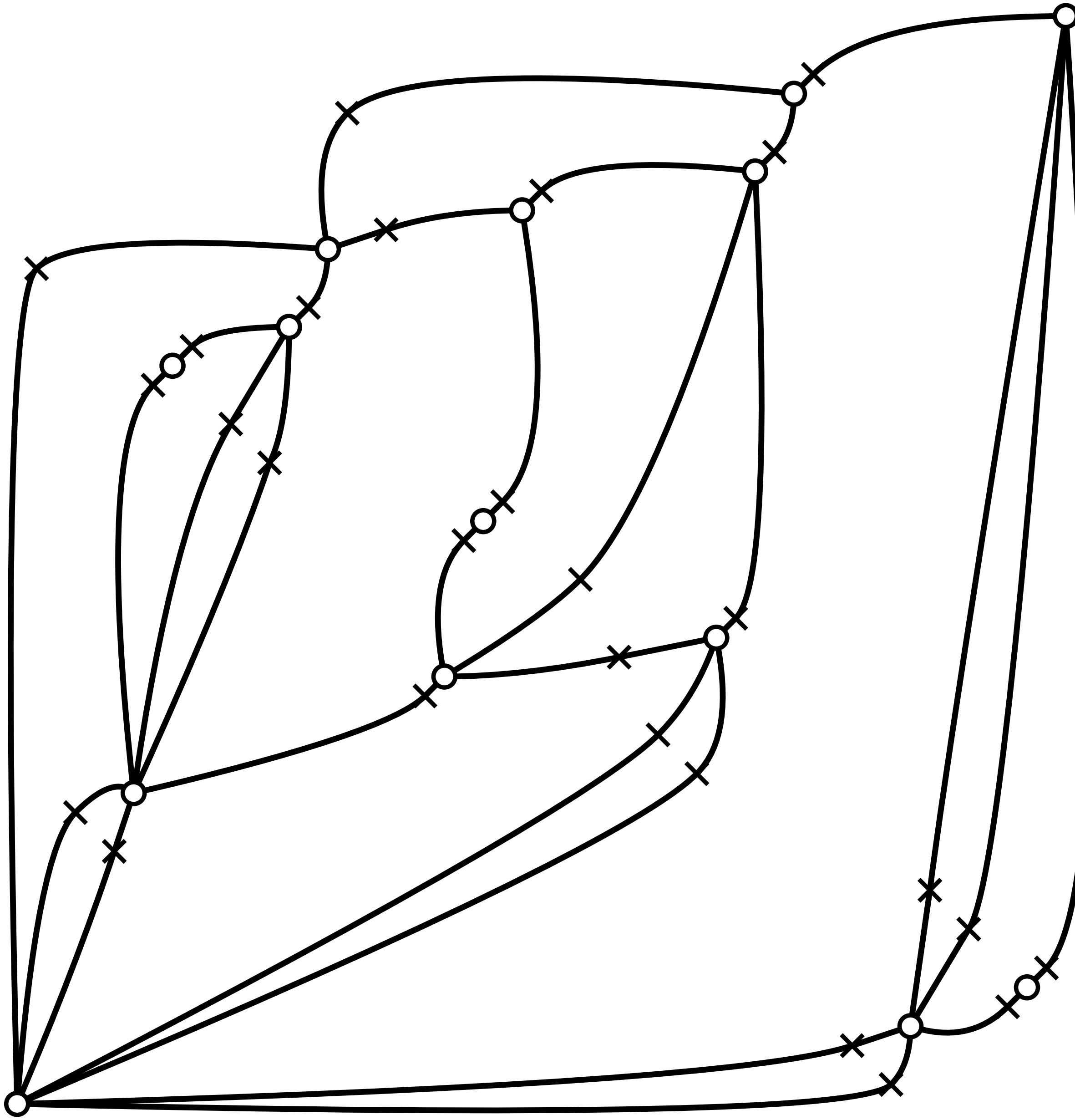

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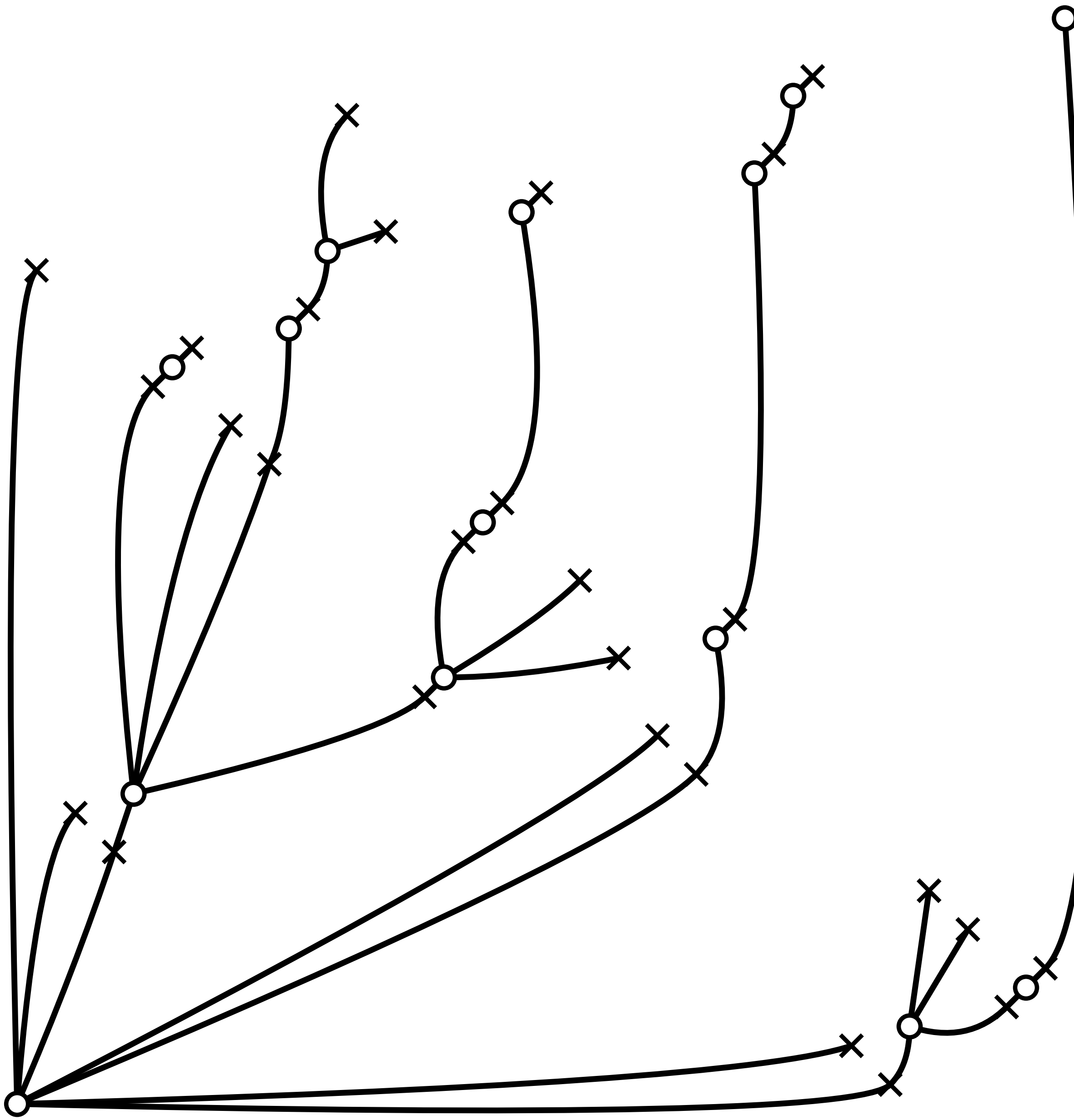

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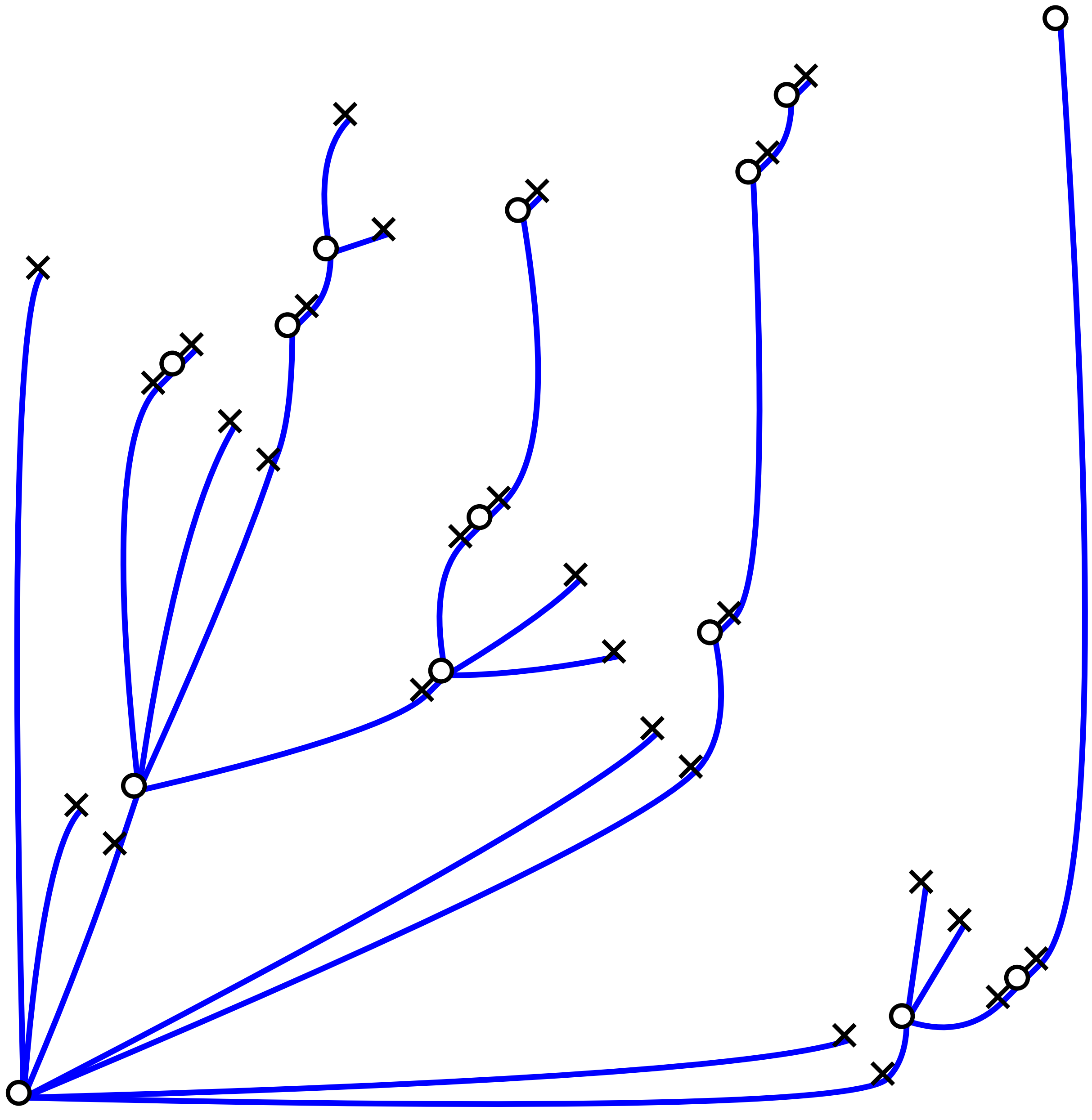

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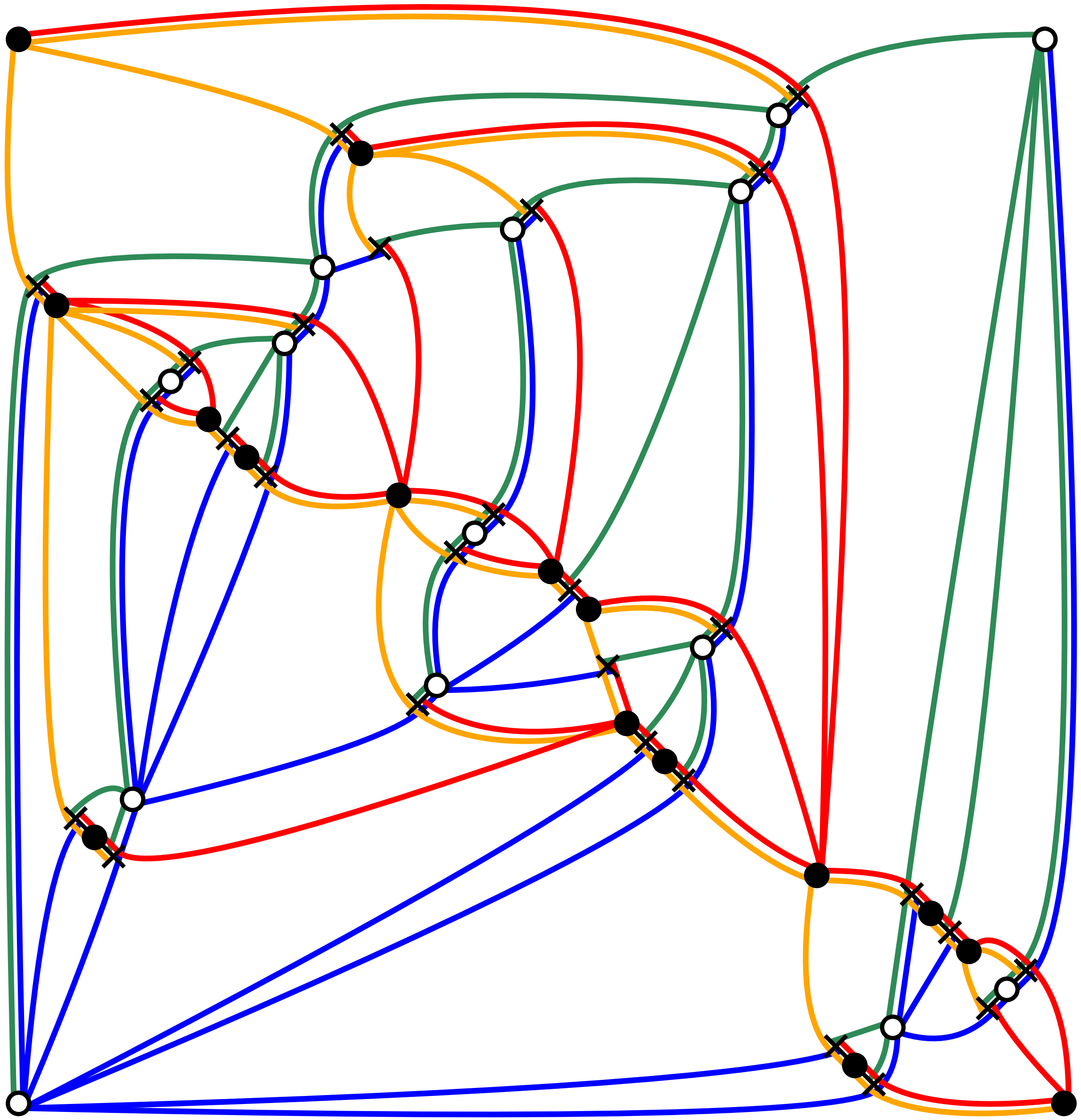

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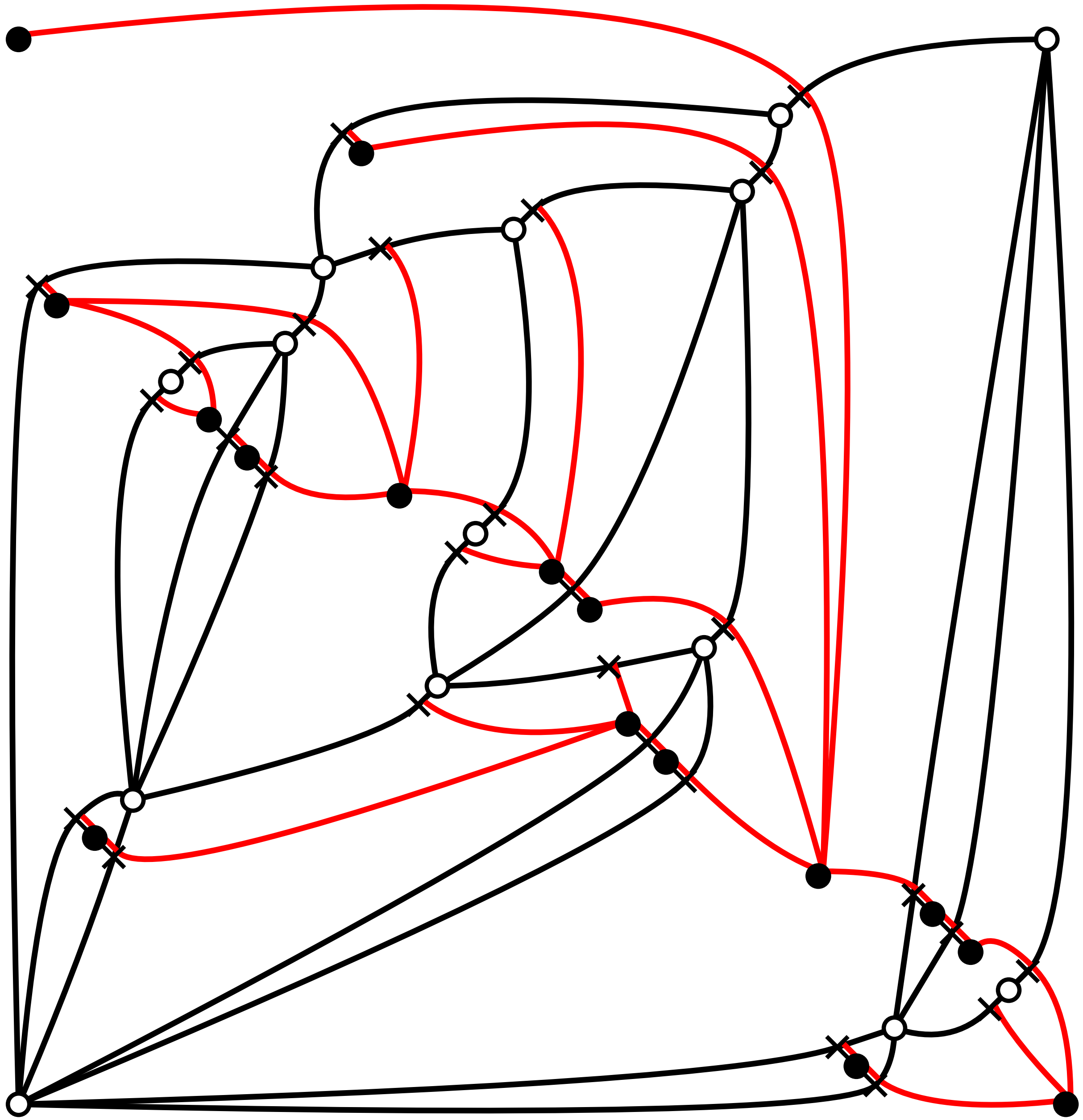

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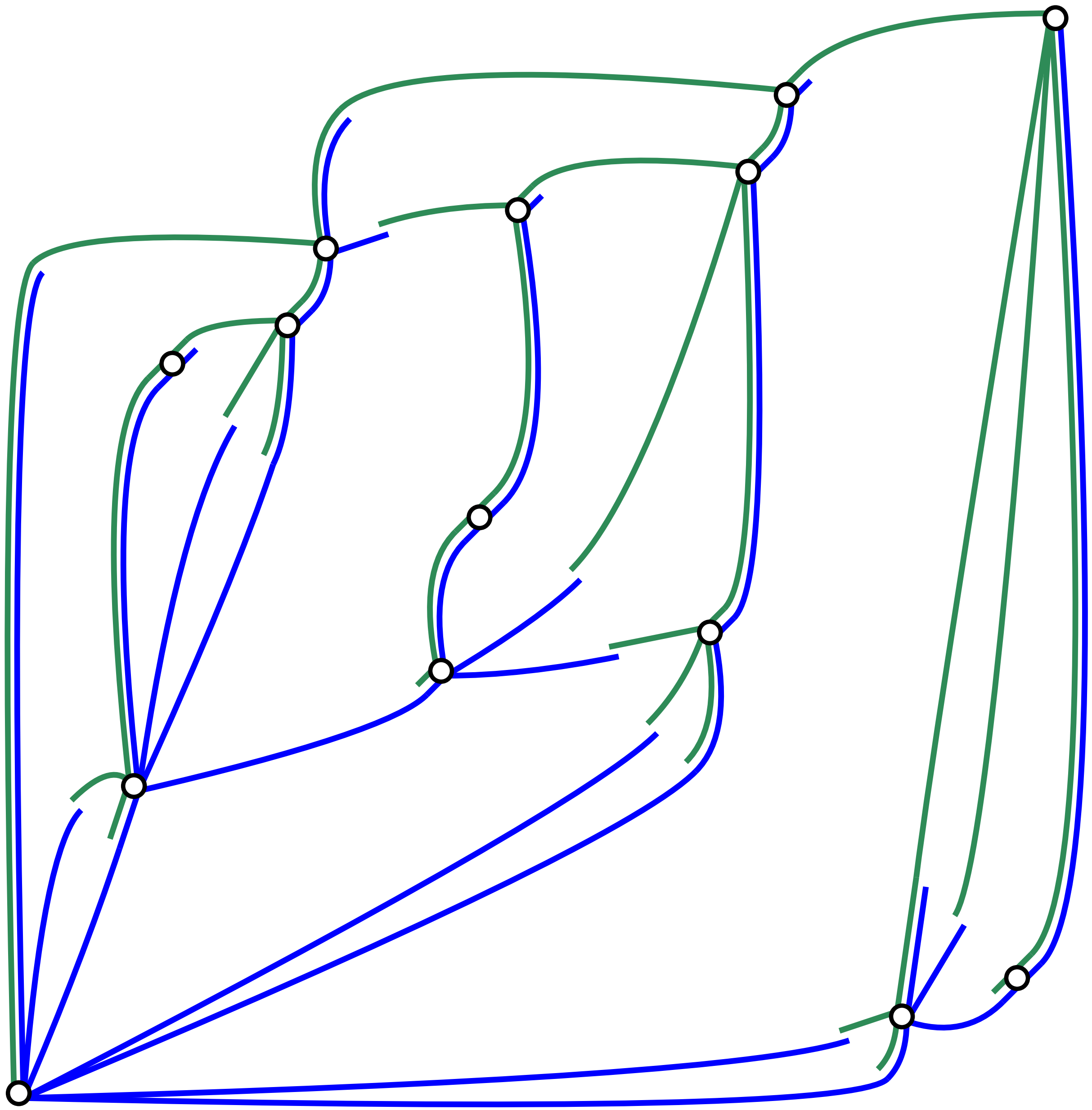

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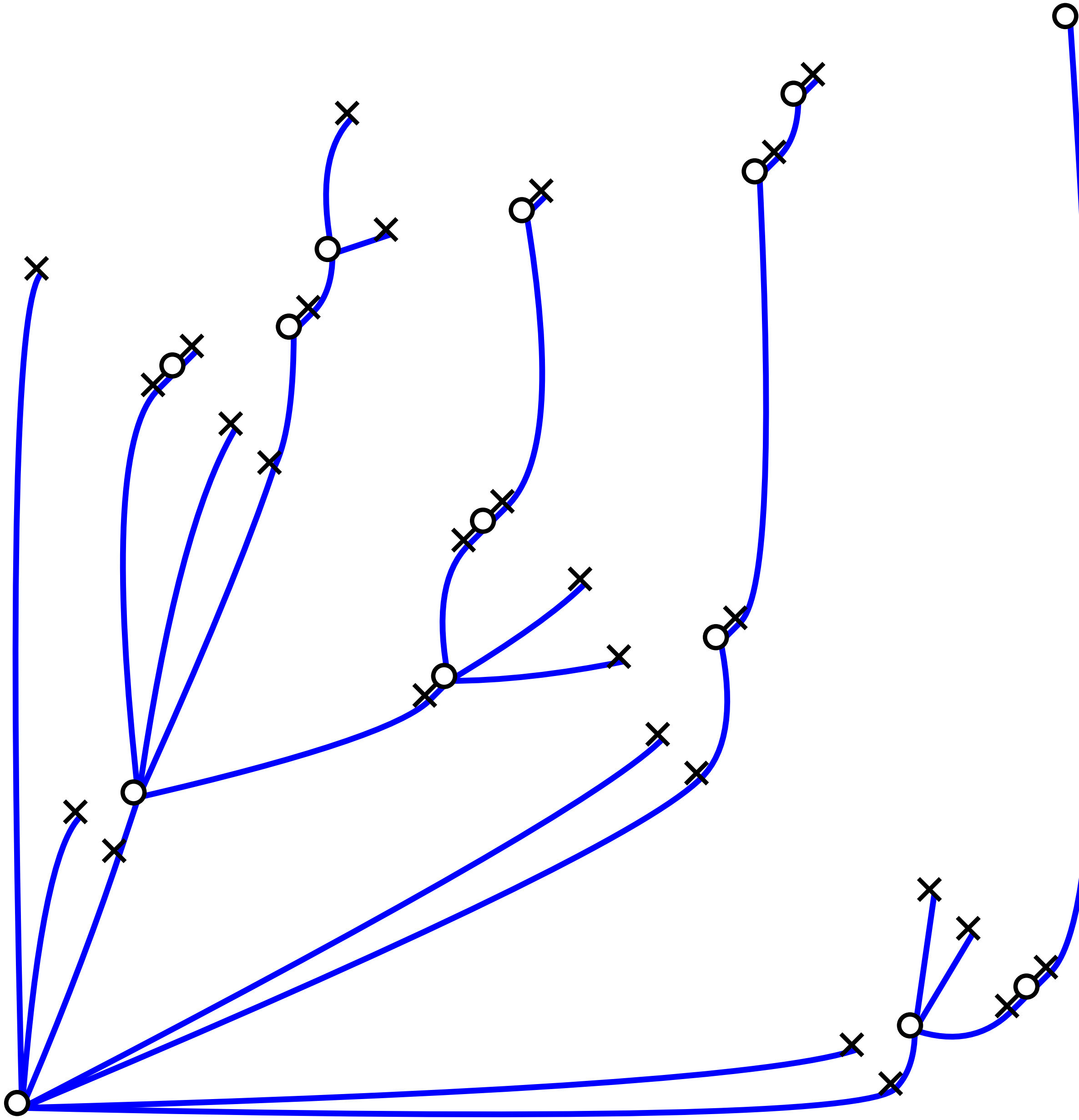

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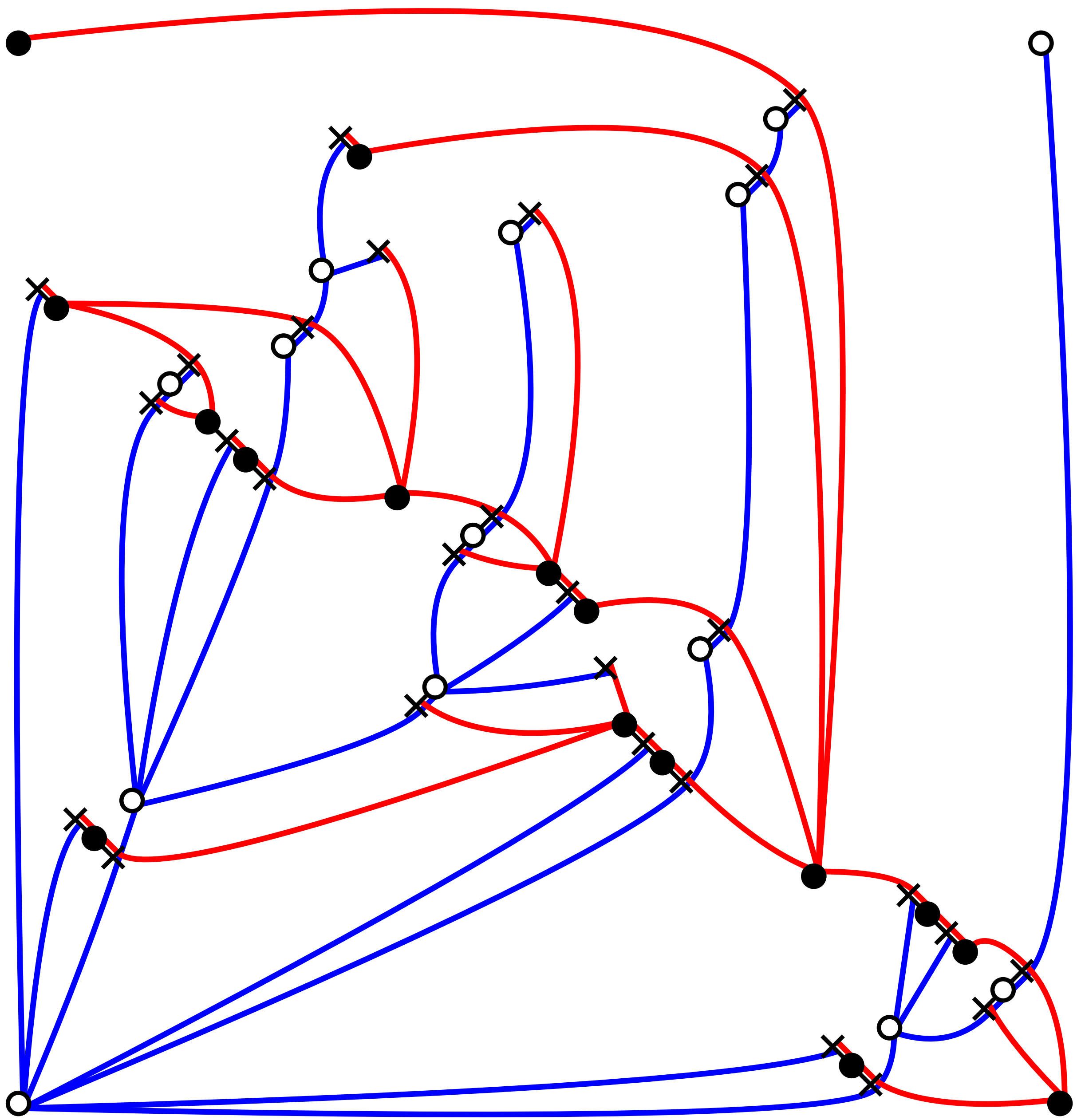

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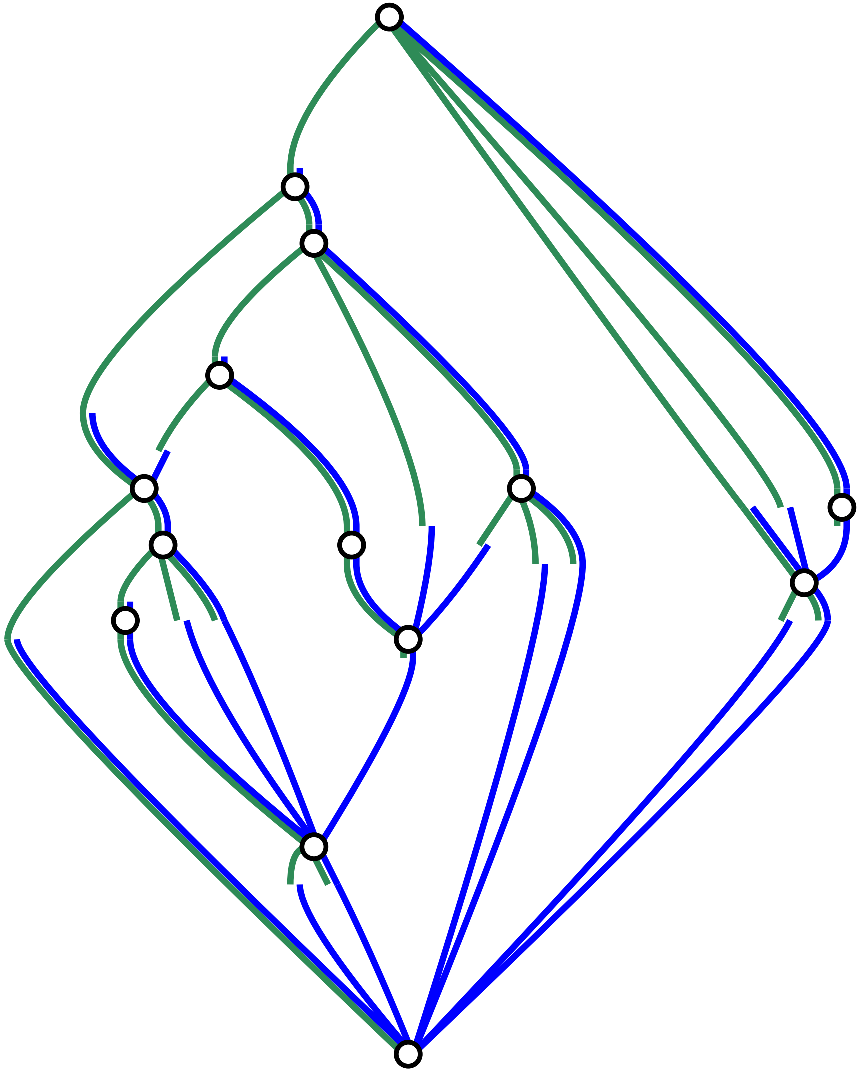
Theorem (Bonichon, Bousquet-Mélou, Fusy '11) $\mathrm{OP}^{-1} : \mathcal{P}_n \to \mathcal{O}_n$ is a bijection.

Inverse bijection: $OP(m)$ is the only permutation π such that the i-th edge in the exploration of $T(m)$ is the $\pi(i)$-th edge in the exploration of $T(m^*)$

Bipolar orientations and walks in the quadrant



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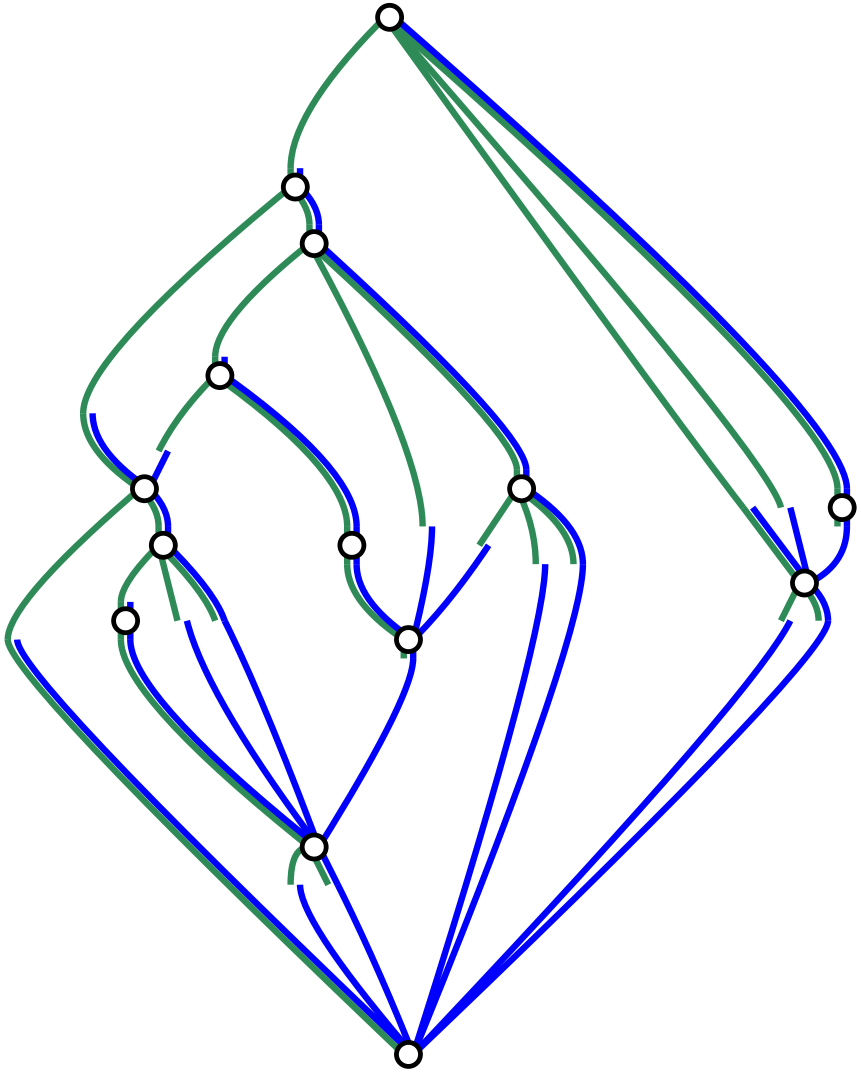

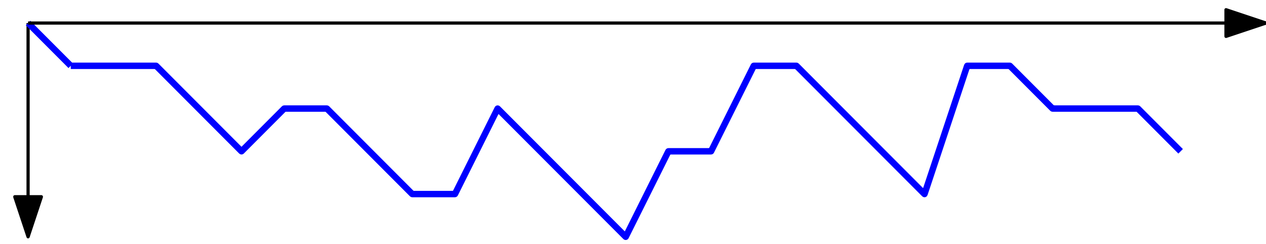

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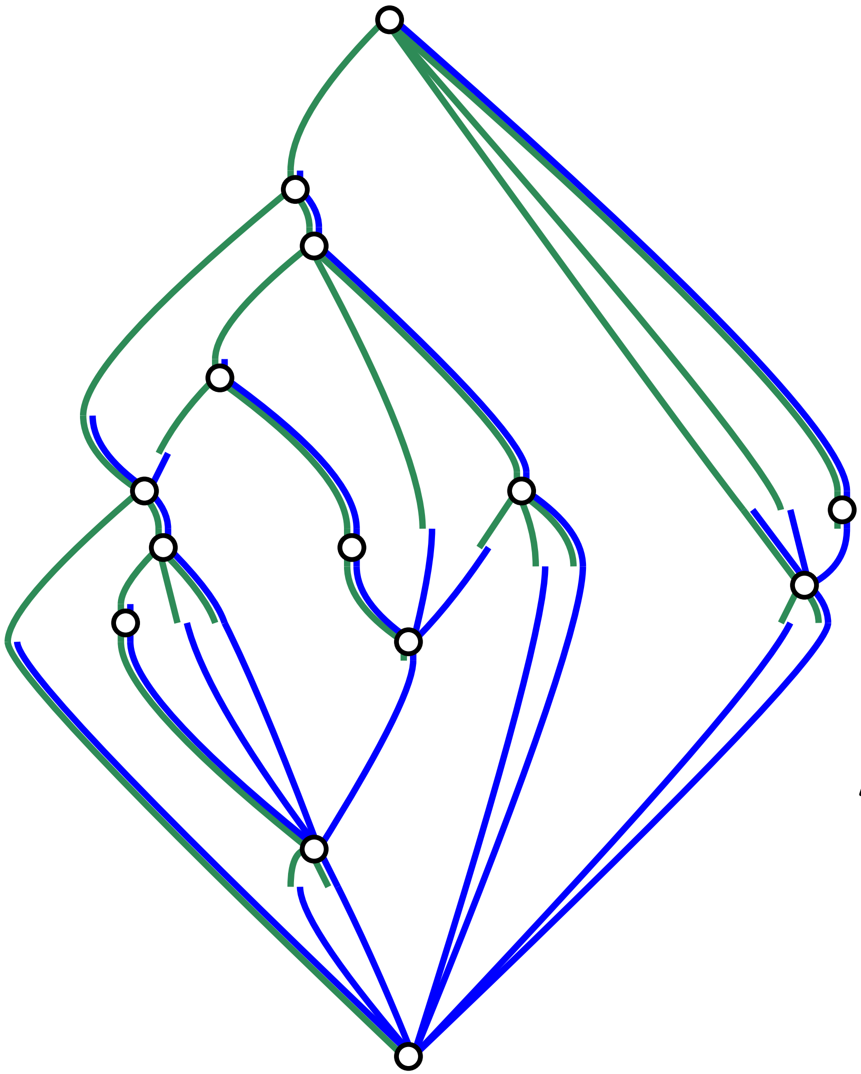

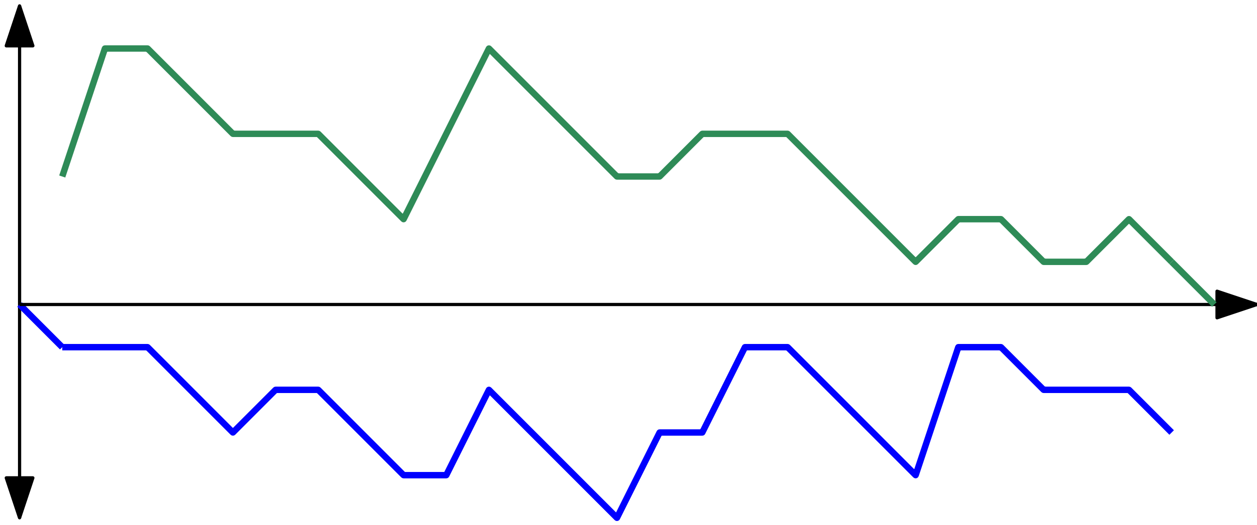

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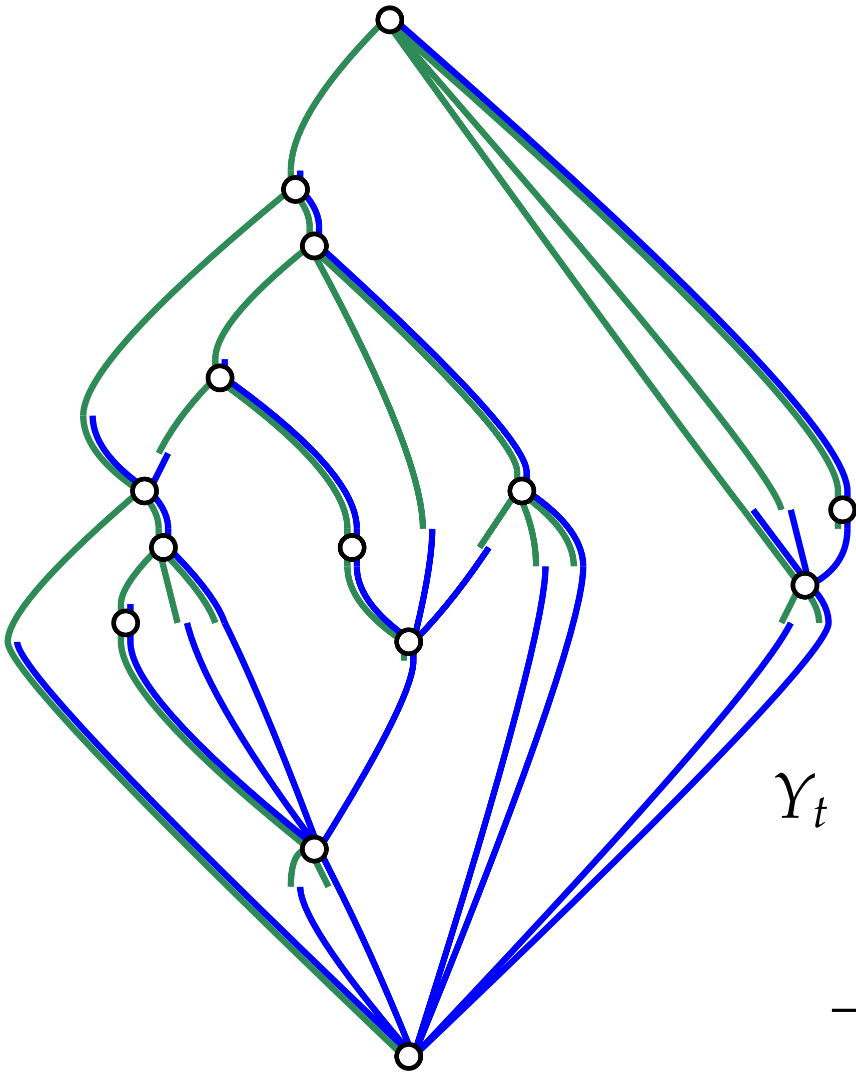
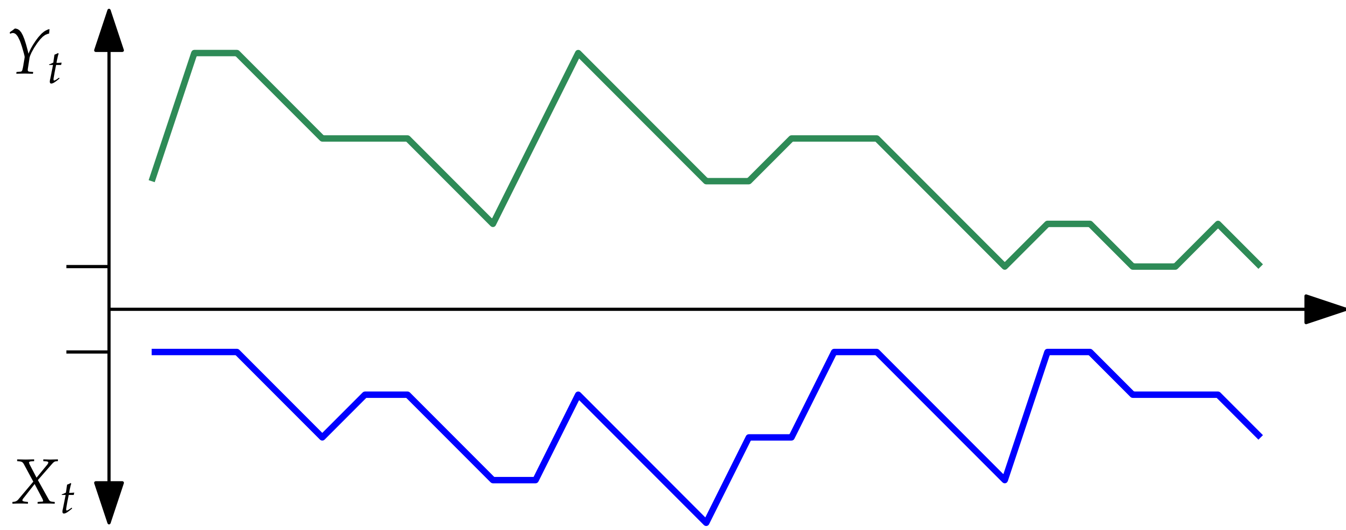
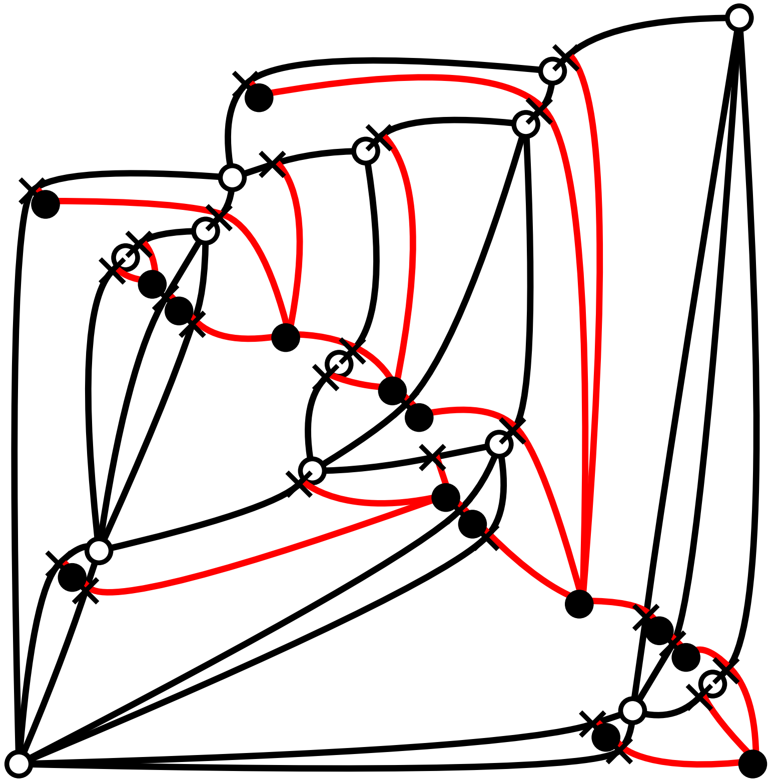
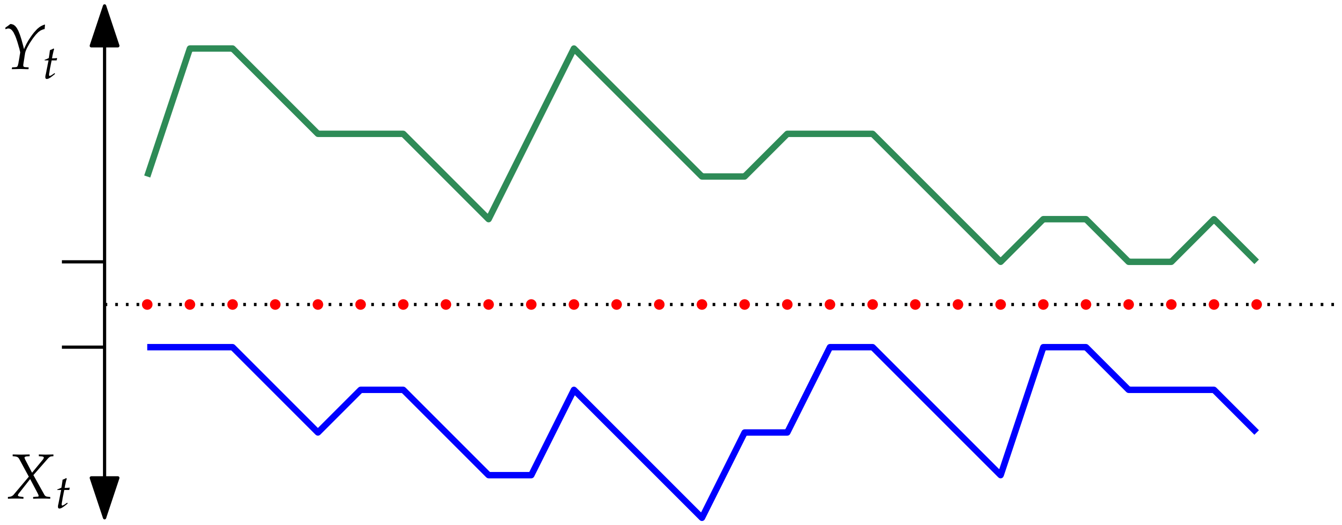


Theorem.
(Kenyon-Miller-Sheffield-Wilson, 2010)
Let $(0, X_1 + 1, X_2 + 1, \ldots X_n + 1)$ and $(0, Y_n + 1, Y_{n-1} + 1, \ldots, Y_1 + 1)$ be the height processes of $T(m)$ and $T(m^{**})$. Denote $\mathrm{OW}(m) = W = (X, Y)$. Then OW is a bijection between $\mathcal{P}_n$ and the set W_n of n-step walks in the cone from $(\mathbb{N}, 0)$ to $(0, \mathbb{N})$ and steps in $(1, -1) \cup (-\mathbb{N}) \times \mathbb{N}$.



Coalescent-walk processes





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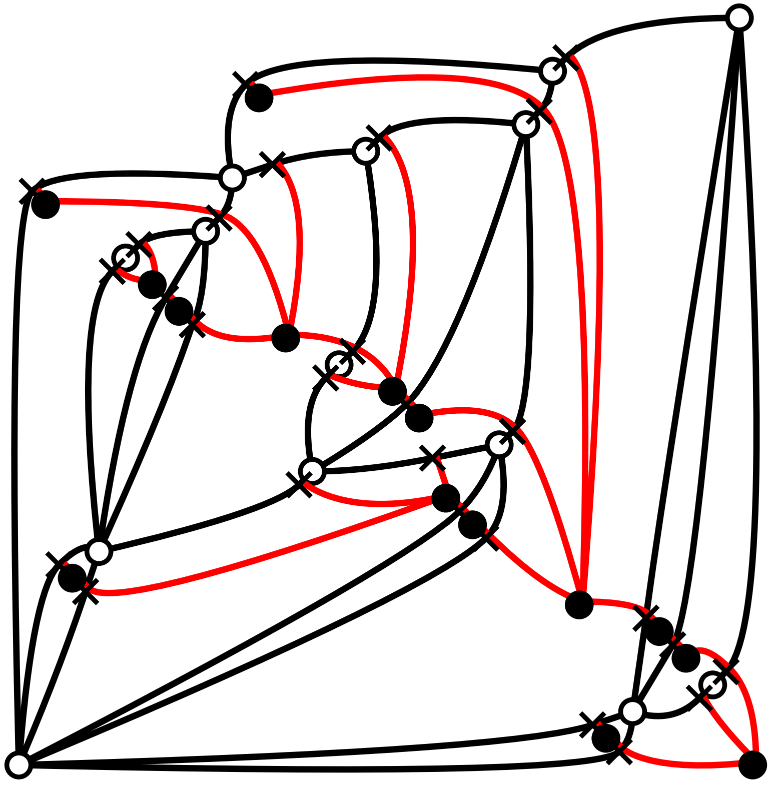

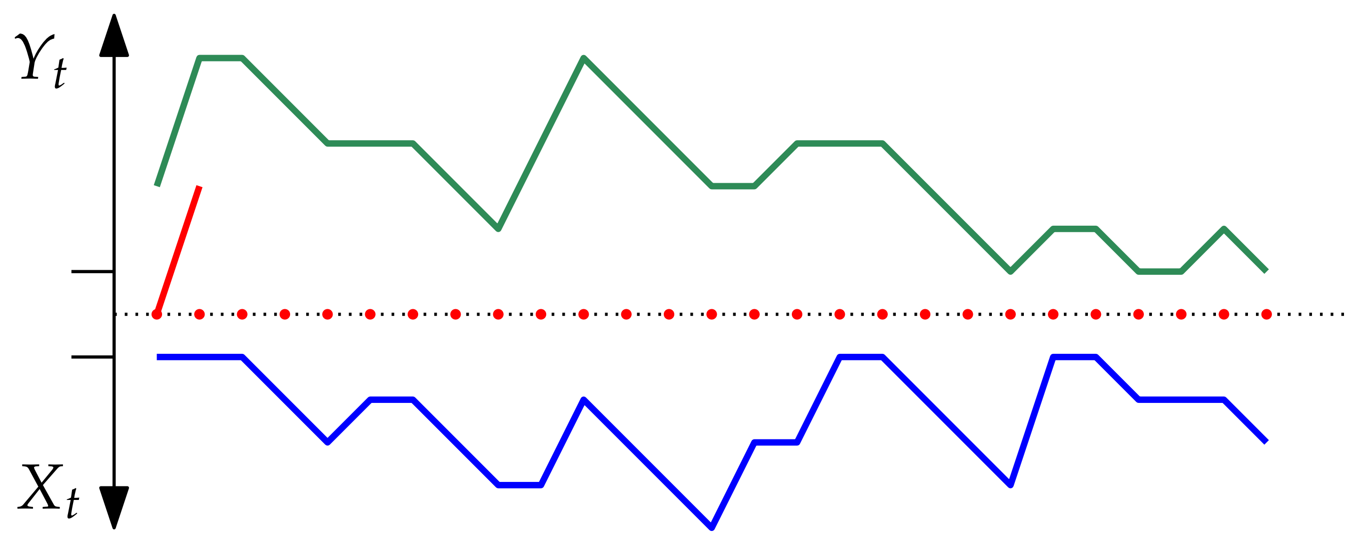

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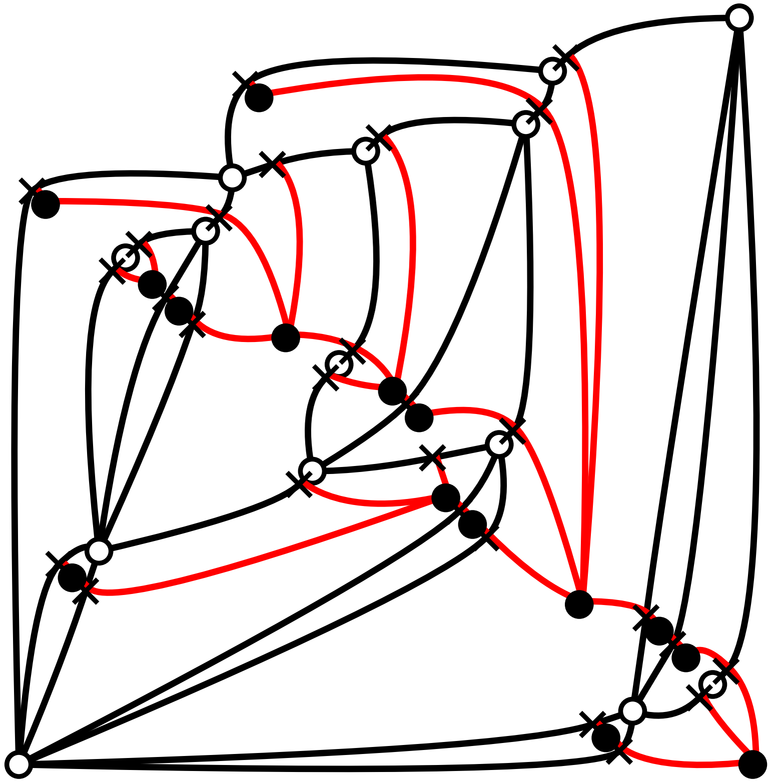

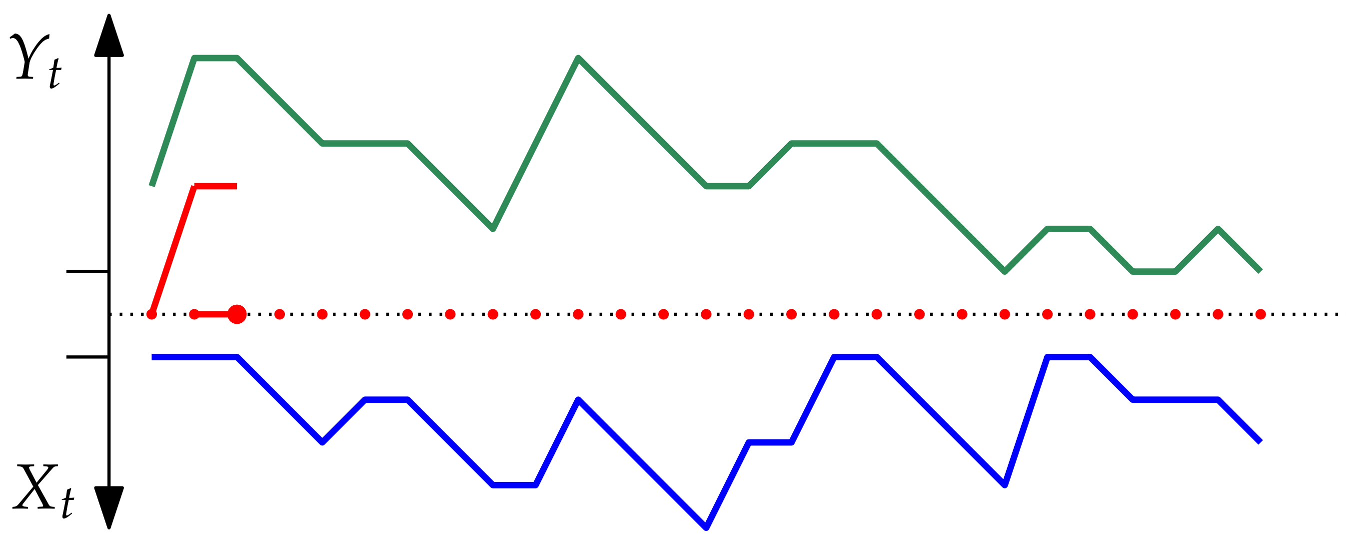

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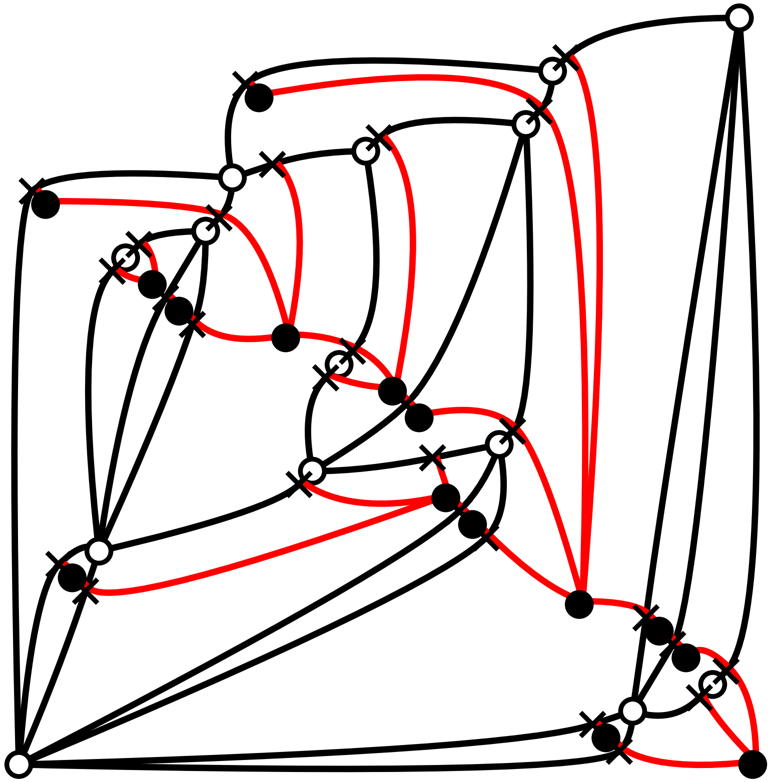

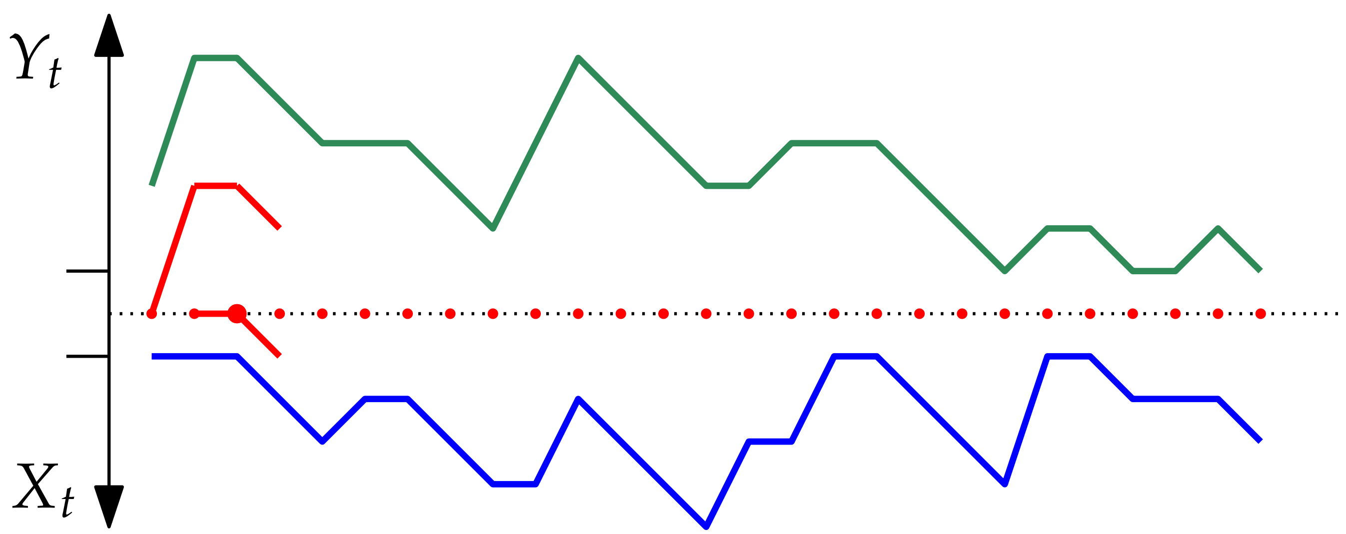

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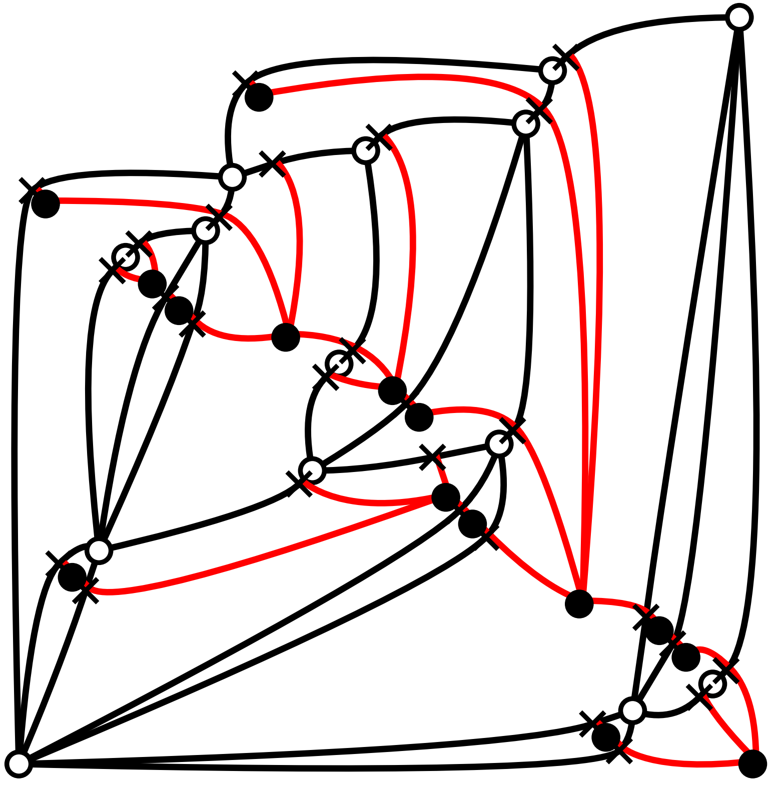

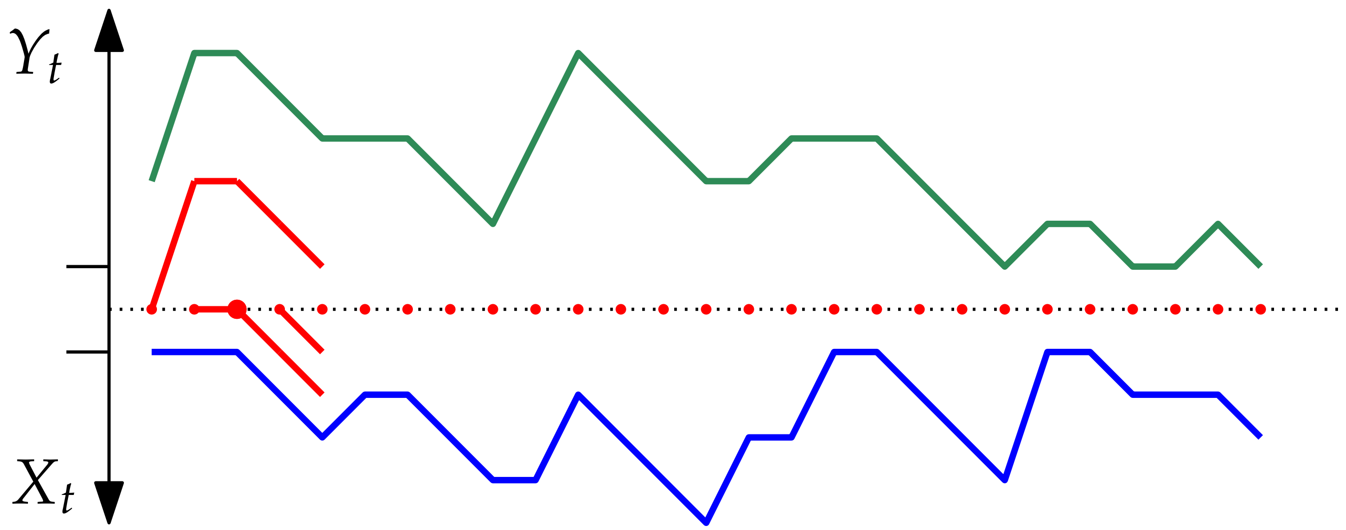

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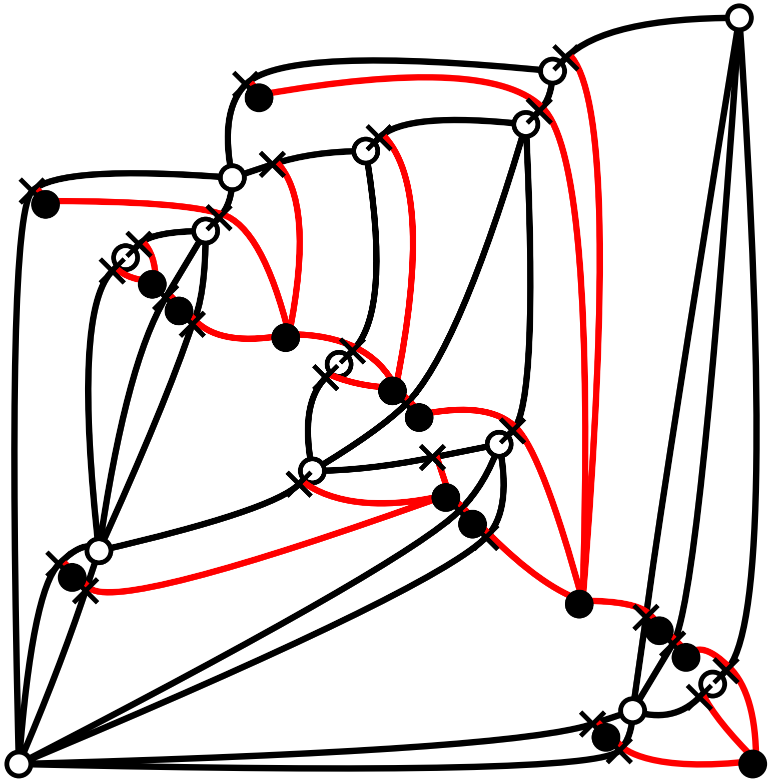

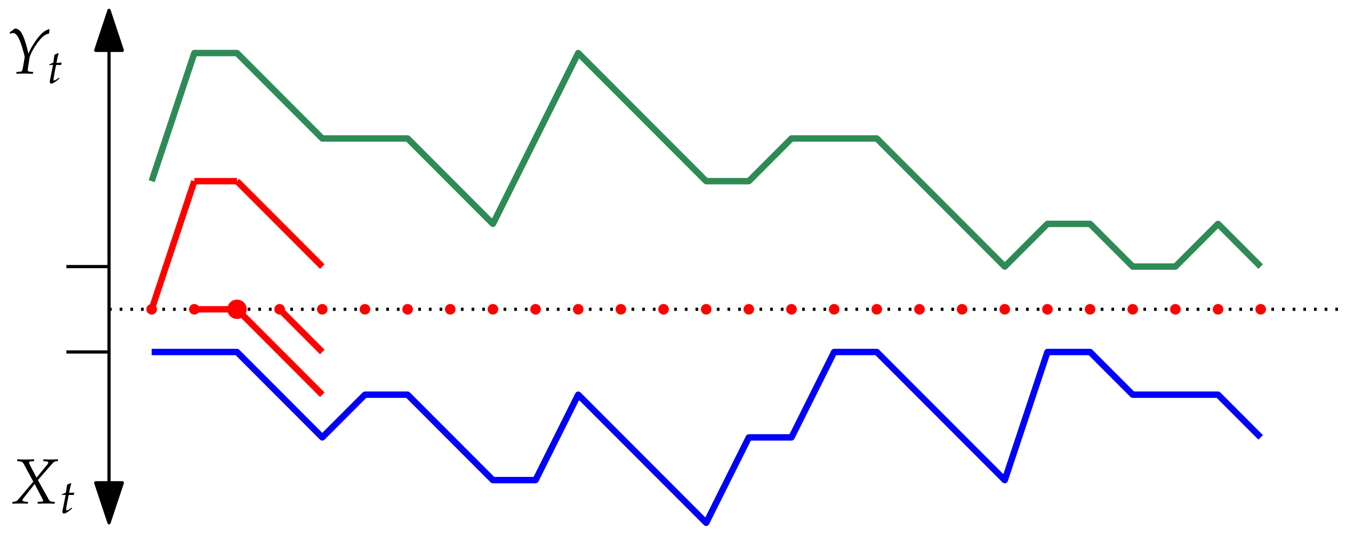

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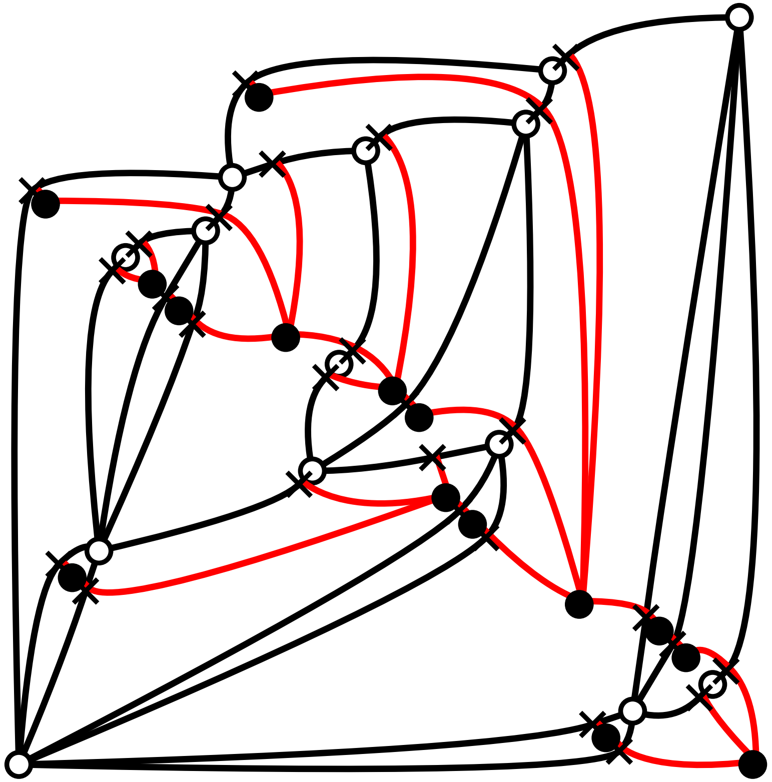

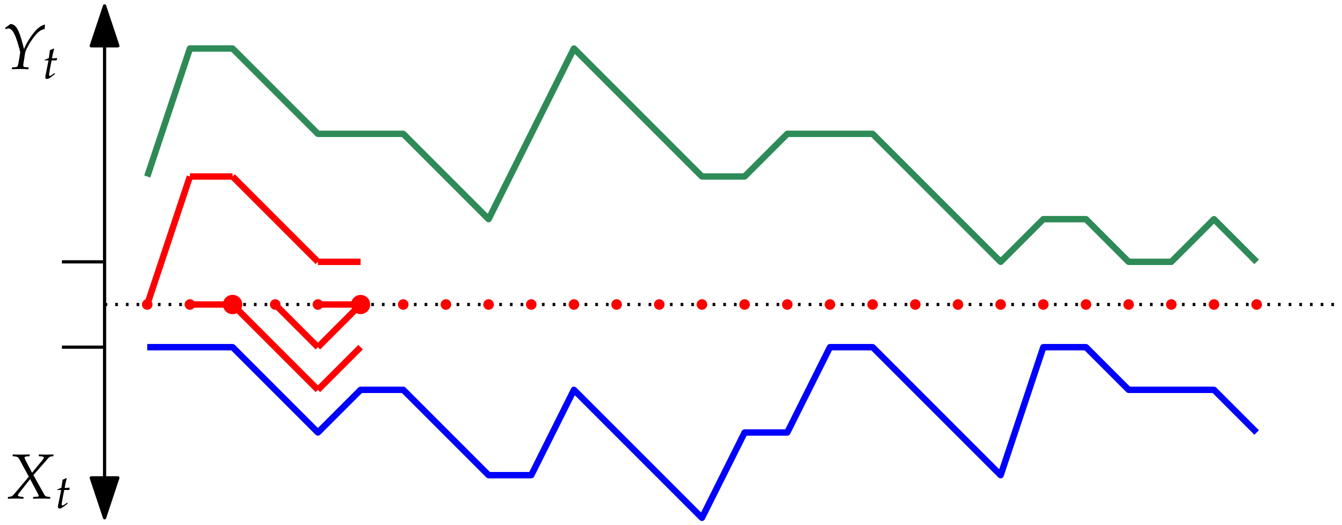

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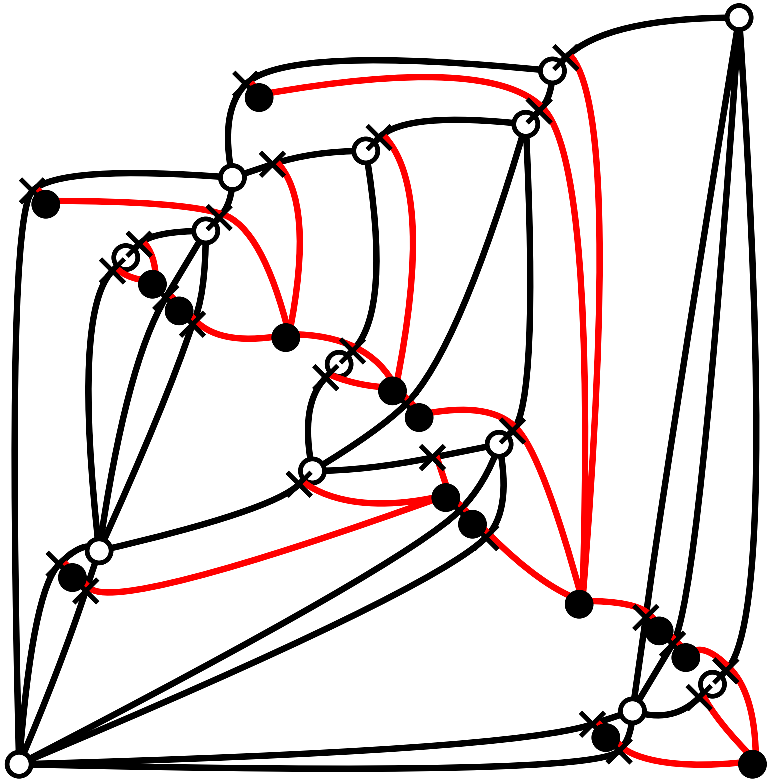

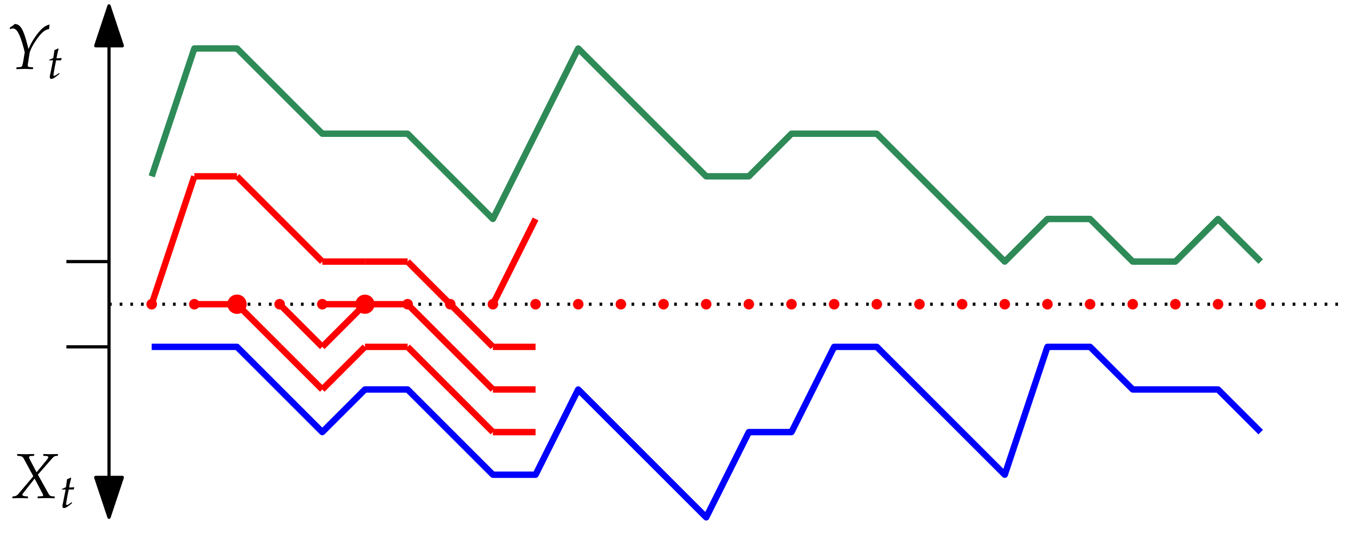

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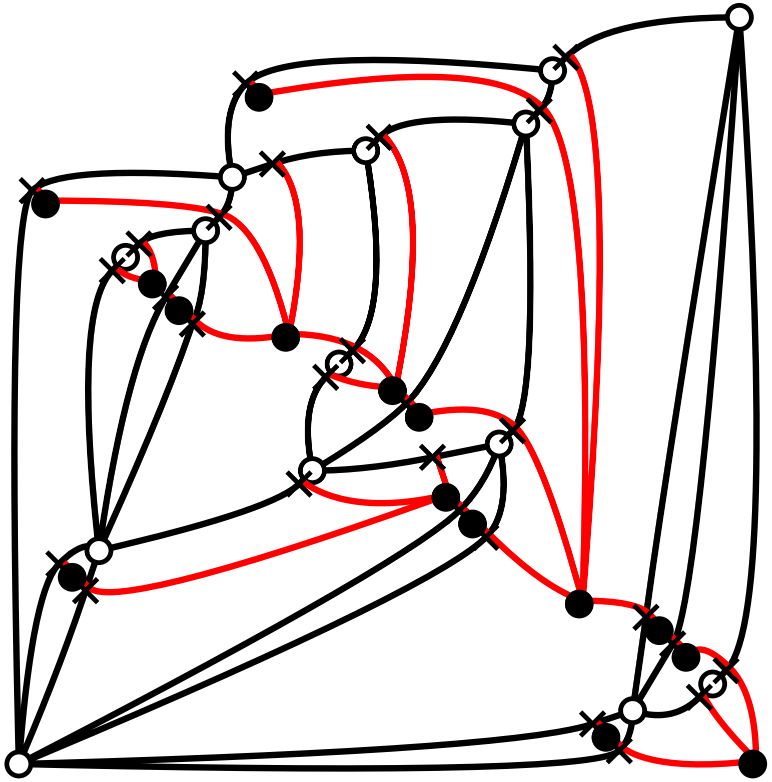

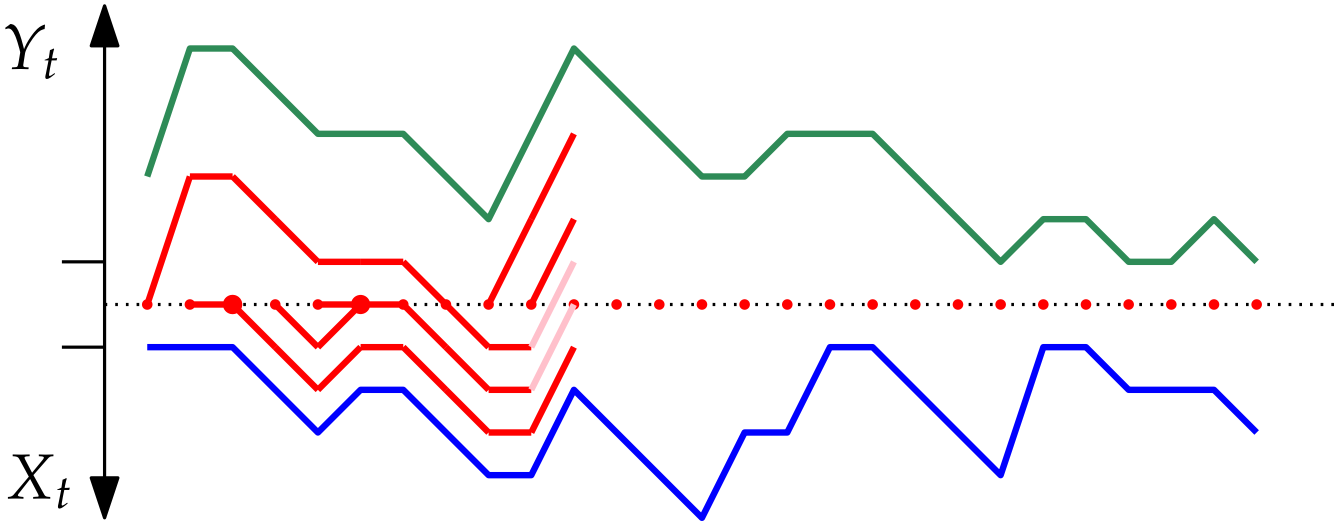

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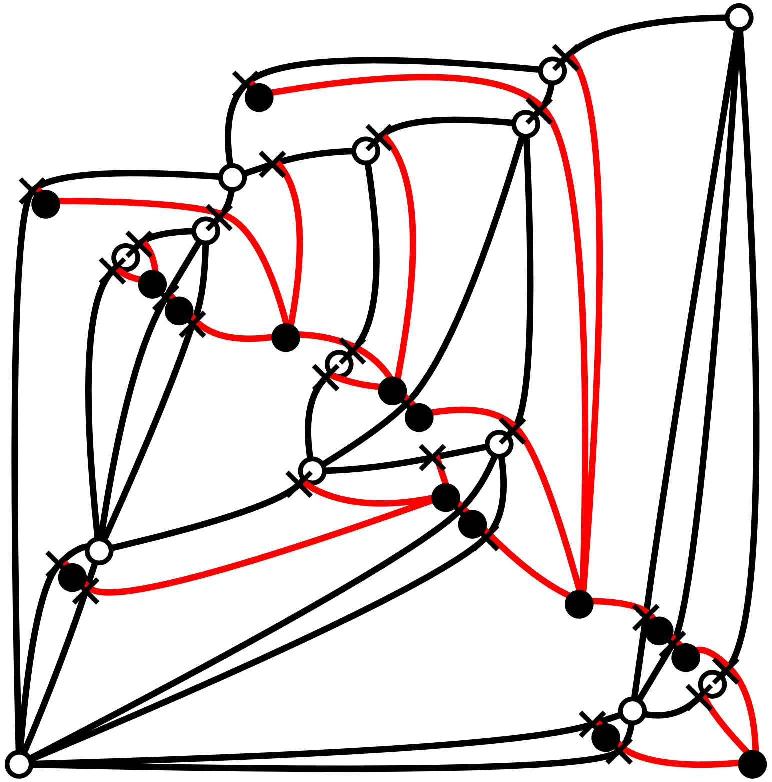

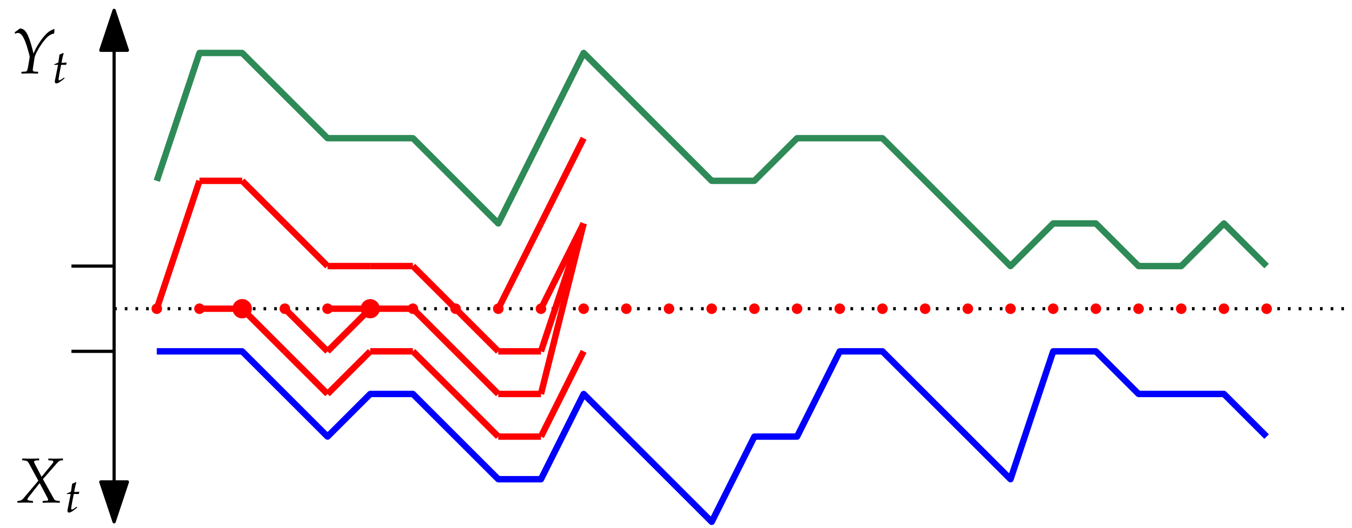

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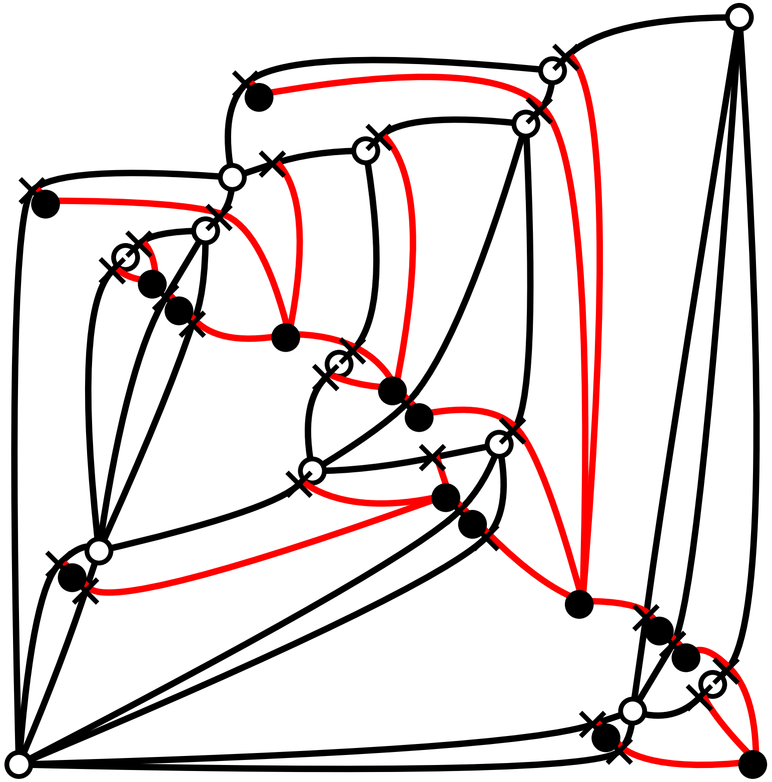

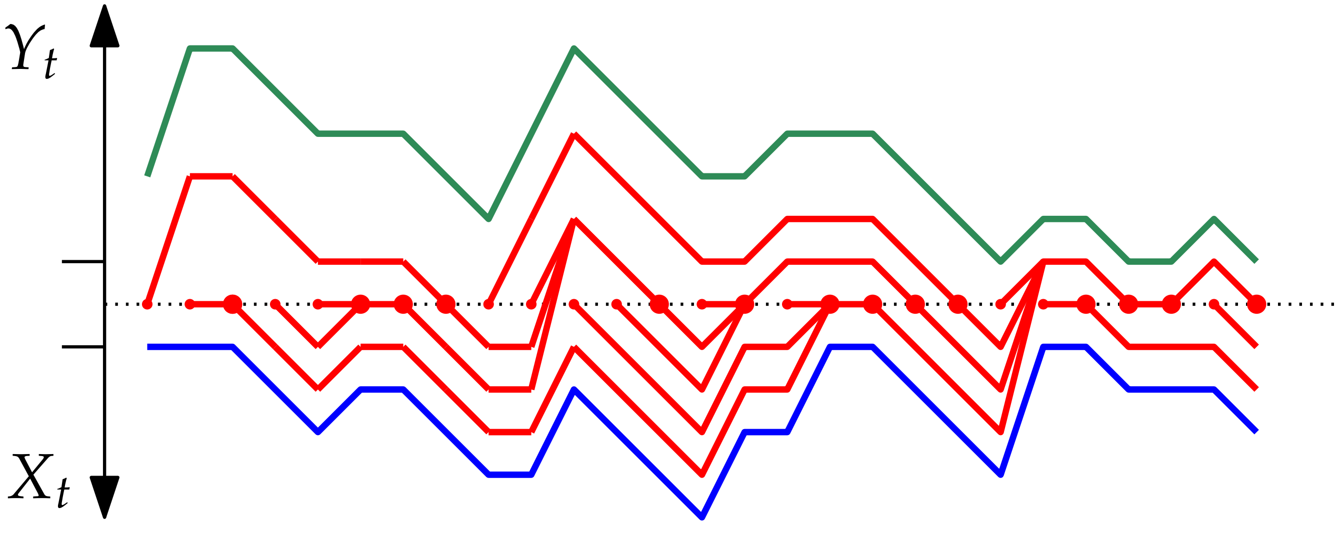

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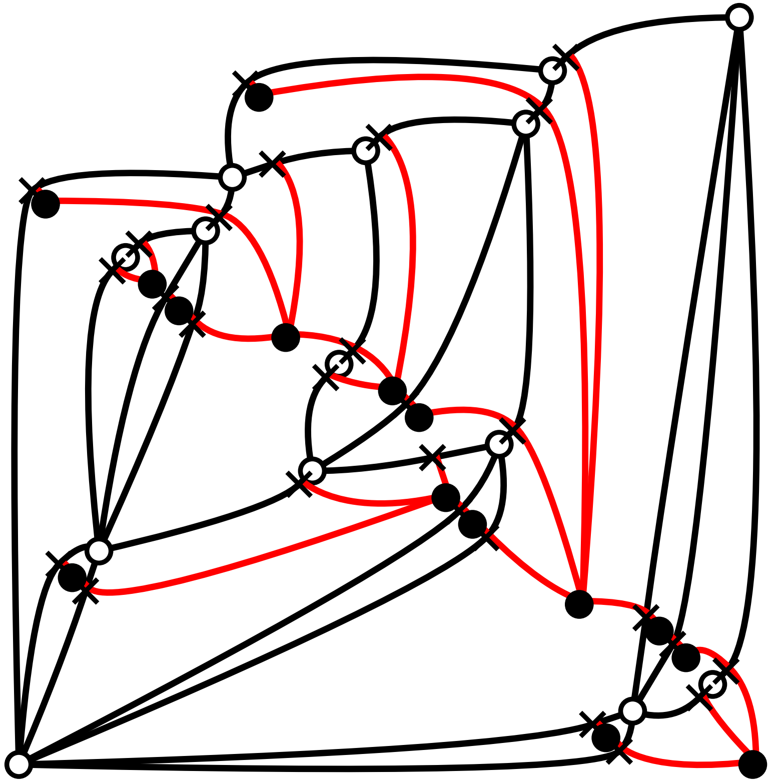
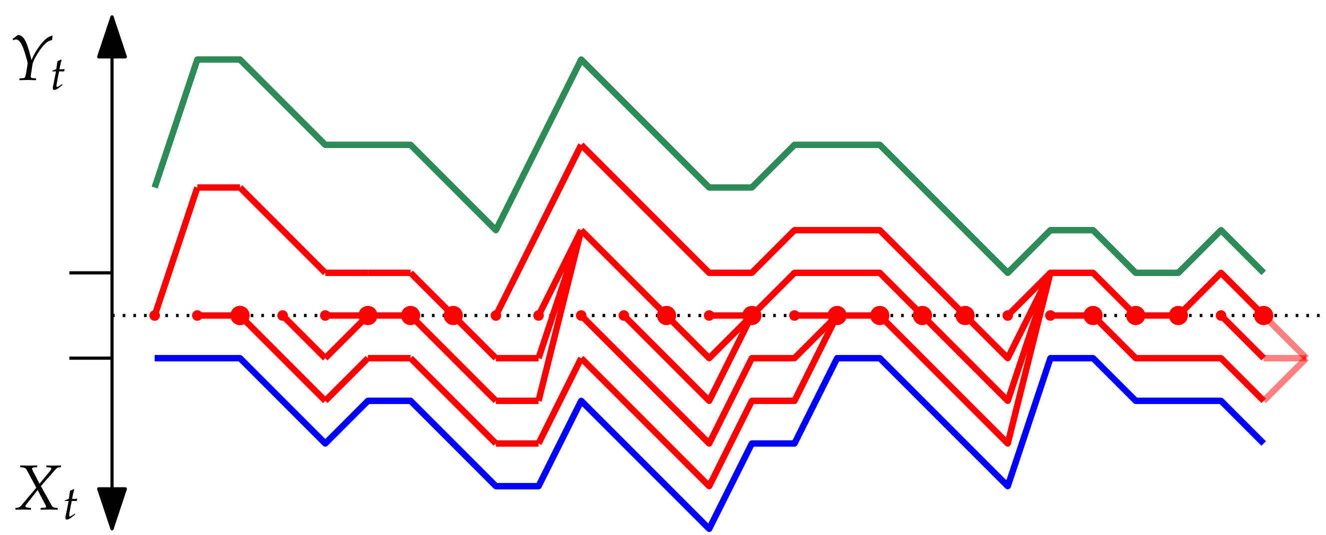


We construct a coalescent process $Z = (Z^{(j)}(i))_{1 \le j \le i \le n}$ driven by (X, Y). The branching structure of the trajectories is that of $T(m^*)$, but edges are visited in the order given by $T(m)$. Comparing the orders given by visit times and by the contour exploration allows to recover the permutation.



Scaling limits of coalescent-walk processes

Theorem (Kenyon, Miller,Sheffield,Wilson) Let (X_n, Y_n) be the coding walk of a uniform bipolar orientation of size n. Then $\frac{1}{\sqrt{2n}}(X_n(n\cdot), Y_n(n\cdot))$ converges to a pair of Brownian excursions with cross-correlation $-1/2$. This is *peanosphere convergence* of bipolar-oriented maps to SLE-decorated LQG.

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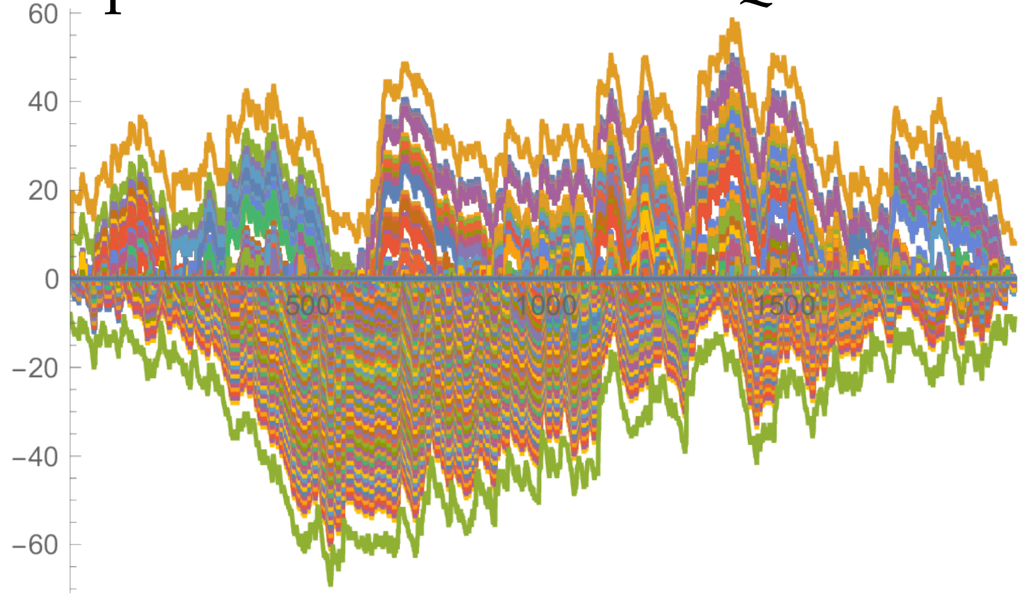

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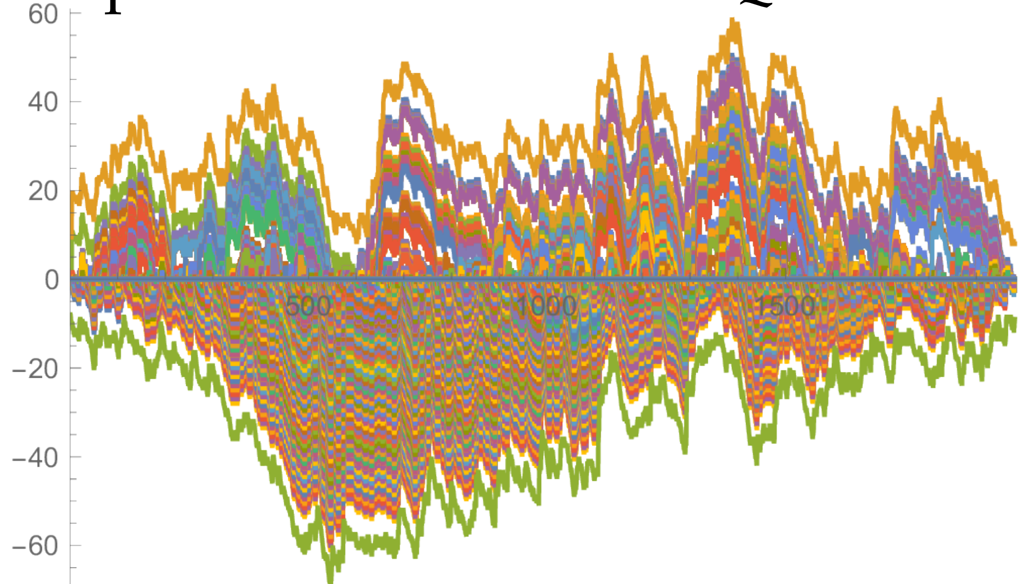

Theorem (Prokaj, Cinlar, Hajri, Karakus) Let (X, Y) be a pair of standard Brownian motions with cross-correlation coefficient $\rho \in [-1, 1)$. Then the *perturbed Tanaka's equation*

$$dZ(t) = \mathbf{1}_{\{Z(t)>0\}} dY(t) - \mathbf{1}_{\{Z(t)\leq 0\}} dX(t), t \geq 0$$

has strong solutions.

Scaling limit of coalescent-walk processes

Let (X,Y) be a Brownian excursion of correlation $-1/2$ in the quarter-plane. For every $u \in [0,1]$, let $Z^{(u)}$ solve the perturbed Tanaka's SDE with the same noise (X,Y), starting at time u. In other words,

$$\begin{cases} dZ^{(u)}(t) = \mathbf{1}_{\{Z^{(u)}(t)>0\}} dY(t) - \mathbf{1}_{\{Z^{(u)}(t)\leq 0\}} dX(t), t \geq u, \\ Z^{(u)}(u) = 0, \end{cases}$$

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Main lemma. Jointly with $\frac{1}{\sqrt{2n}}(X_n(n\cdot), Y_n(n\cdot)) \to (X, Y)$, we have that $\frac{1}{\sqrt{2n}}(Z_n^{(\lfloor nu \rfloor)}(n\cdot) \to Z^{(u)}$.

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Set $\phi(t) = \mathrm{Leb}\{s \in [0,1] : s \prec t\}$ and $\mu_B = (\mathrm{Id}, \phi)_* \mathrm{Leb} = P(X, Y)$.

Scaling limits of bipolar orientations

Let (X_n, Y_n) be the walks coding, respectively, the map m_n and its dual m_n^*. Let $(X, Y$ be a Brownian excursion in the quadrant of correlation $-1/2$. Consider the map $s : \mathcal{C}([0,1], \mathbb{R}^2) \to \mathcal{C}([0,1], \mathbb{R}^2)$ defined by $s(f, g) = (g(1-\cdot), f(1-\cdot))$. Consider also the map $R : \mathcal{M} \to \mathcal{M}$ that rotates a permuton by an angle $-\pi/2$,

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Theorem (Borga,M.) There exist two measurable maps $r : \mathcal{C}([0,1], \mathbb{R}^2_{\geq 0}) \to \mathcal{C}([0,1], \mathbb{R}^2_{\geq 0})$ and $P : \mathcal{C}([0,1], \mathbb{R}^2_{\geq 0}) \to \mathcal{M}$ such that we have the convergence in distribution

$$(X_n, Y_n, X_n^*, Y_n^*, \mu_{\sigma_n}) \to (X, Y, X^*, Y^*, \mu_B),$$

where $(X^*, Y^*) = r(X, Y)$, and $\mu_B = P(X, Y)$. Moreover, we have the following equalities that hold at almost every point of $\mathcal{C}([0,1], \mathbb{R}^2_{\geq 0})$,

$$r^2 = s, \quad r^4 = \mathrm{Id}, \quad P \circ r = R \circ P.$$

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The convergence of the first four marginals is an extension of a result of Gwynne,Holden,Sun that deals with infinite-volume bipolar triangulations.

Perspectives

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We expect the correlation parameter ρ to vary, and might lose symmetry at the origin, as in the study of Schnyder woods by Li-Sun-Watson.