

The Brauer group of bielliptic surfaces

Geometry

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I am hiring a postdoc starting July 1st 2021!!!

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The Question



Let X a smooth projective variety over a field k (which will be \mathbb{C} for what it concerns us today). The cohomological Brauer group of X is

$$\operatorname{Br}'(X) := \operatorname{H}^2_{\operatorname{\acute{e}t}}(X, \mathcal{O}^*_X)_{\operatorname{tor}}.$$

Given a morphism $f: X \to Y$ of smooth projective varieties, then, by pulling back classes, we get a group homomorphism

 $f_{\mathsf{Br}}: \mathsf{Br'}(Y) \longrightarrow \mathsf{Br'}(X)$

which we call the Brauer map associated to f.

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which we call the Brauer map associated to f.

Question

What can we say about f_{Br}?

Beauville and Enriques surfaces



Let *S* be a complex Enriques surface and denote by \hat{S} its universal cover. Then \hat{S} is a K3 surface and there is an étale 2-1 morphism

$$\pi: \hat{S} \longrightarrow S.$$

Denote by $\sigma: \hat{S} \to \hat{S}$ the Enriques involution. The Brauer group of an Enriques surfaces is isomomorphic $\mathbb{Z}/2\mathbb{Z}$ so there are only two possible behaviors for the Brauer map associated to π :

Theorem (Beauville 2009)

The Brauer map π_{Br} is trivial if, and only if, there is a line bundle *L* on \hat{S} such that $\sigma^*L = L^{-1}$ and $c_1(L)^2 \equiv 2 \pmod{4}$.

We are going to study the problem for complex bielliptic surfaces. Given a complex bielliptic surface S, then there is always a cyclic cover

$$\pi: \hat{S} \to S$$

where \hat{S} is an abelian surface. Sometimes (more details are coming) there is also a cyclic cover

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Goal

We completely characterize the behavior of the Brauer map associated to these morphisms.

We= this is a joint work with E. Ferrari, M. Vodrup (with an appendix by me and J. Bergström).

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Derived Categories?



- Beauville work was used by Addington and Wray to study (the non existence of) twisted Fourier–Mukai partners of Enriques surfaces.
- Vodrup is using this work to do a similar investigation for bielliptic surfaces.











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Section 1 Bielliptic Surfaces

The Definition



Definition

A bielliptic surface is a surface S with irregularity

$$q(\mathcal{S}) := h^1(X, \mathcal{O}_{\mathcal{S}}) = 1$$

and numerically trivial canonical divisor class.

- The canonical bundle is torsion, but not trivial (we are working on the complex numbers!)
- They are always constructed as quotients of a product of two elliptic curves by a finite group action.

Example



Let *A* and *B* two elliptic curves and let $G := \mathbb{Z}/2\mathbb{Z}$. Choose τ a point of order two in *A* and consider the involution

$$\sigma: \mathbf{A} \times \mathbf{B} \longrightarrow \mathbf{A} \times \mathbf{B}$$

defined by $(a, b) \mapsto (a + \tau, -b)$. Then the surface $S := A \times B / \langle \sigma \rangle$ is bielliptic.

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defined by $(a, b) \mapsto (a + \tau, -b)$. Then the surface $S := A \times B / < \sigma >$ is bielliptic. The surface *S* admits two elliptic fibrations



with general fibers isomorphic to A and B respectively.

Bagnera- de Franchis



Туре	G	Order of $\omega_{\mathcal{S}}$ in $\operatorname{Pic}(\mathcal{S})$	$H^2(S,\mathbb{Z})_{\mathrm{tor}}$
1	$\mathbb{Z}/2\mathbb{Z}$	2	$\mathbb{Z}/2\mathbb{Z} imes \mathbb{Z}/2\mathbb{Z}$
2	$\mathbb{Z}/2\mathbb{Z} imes\mathbb{Z}/2\mathbb{Z}$	2	$\mathbb{Z}/2\mathbb{Z}$
3	$\mathbb{Z}/4\mathbb{Z}$	4	$\mathbb{Z}/2\mathbb{Z}$
4	$\mathbb{Z}/4\mathbb{Z} imes\mathbb{Z}/2\mathbb{Z}$	4	0
5	$\mathbb{Z}/3\mathbb{Z}$	3	$\mathbb{Z}/3\mathbb{Z}$
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Fun Facts

 The Brauer group of a bilelliptic surface is non canonically isomorphic to H²(S, Z)_{tor} so we will disregard types 4, 6 and 7.

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Fun Facts

- The Brauer group of a bilelliptic surface is non canonically isomorphic to H²(S, Z)_{tor} so we will disregard types 4, 6 and 7.
- To construct types 3 and 5 we cannot choose freely the elliptic curve *B*: for type 3 we have j(B) = 1728, and for type 5 we have j(B) = 0.

Canonical Covers



Let *S* be a bielliptic surface and denote by *n* the order of its canonical bundle. Then ω_S induces an étale cyclic cover $\pi_S \colon \hat{S} \to S$, called the *canonical cover of S*.

If we let $\lambda_S := |G|/|(\omega_S)|$, we have that $G \simeq \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/\lambda_S\mathbb{Z}$, and X is the abelian surface sitting as an intermediate quotient



where $H \simeq \mathbb{Z}/\lambda_S \mathbb{Z}$.

Bielliptic covers (after Nuer)



When *G* is not cyclic or when *G* is cyclic, of non prime order number, then the bielliptic surface *S* admits a cyclic cover $\tilde{\pi} : \tilde{S} \to S$, where \tilde{S} is another bielliptic surface.

Example

- If S is a bielliptic surface of type 3, then the canonical bundle has order 4. In addition the canonical cover Ŝ of S is a product of elliptic curves, that is X ≃ A × B. By taking the cover associated with ω_S^{⊗2} we get Ŝ which is a bielliptic surface of type 1.
- Suppose that S is a bielliptic surface of type 2, so the group G is isomorphic to the product Z/2Z × Z/2Z. Then we obtain S from A × B by taking the quotient with respect to (x, y) ↦ (x + τ, -y). Thus S̃ is a again bielliptic surface of type 1.



Section 2 The results

Those easily stated...



Theorem (Ferrari, :-), Vodrup)

Let *S* be a bielliptic surface which admits a bielliptic cover $\tilde{\pi} : \tilde{S} \to S$. If *S* is of type 2, then $\tilde{\pi}_{Br} : Br(S) \to Br(\tilde{S})$ is trivial.

 ${f O}$ If *S* is of type 3, then $ilde{\pi}_{\mathsf{Br}} : \mathsf{Br}(S) o \mathsf{Br}(ilde{S})$ is injective.

Theorem (Ferrari, :-), Vodrup)

Let $S := A \times B/G$ be a bielliptic surface and denote by $\pi : \hat{S} \to S$ its canonical cover. If the elliptic curves *A* and *B* are not isogenous, then the Brauer map π_{Br} is trivial.

• If S is of type 2, then $\tilde{\pi}_{Br} : Br(S) \to Br(\tilde{S})$ is trivial.

2 If S is of type 3, then $\tilde{\pi}_{Br} : Br(S) \to Br(\tilde{S})$ is injective.

... and those not



The description of the behavior of the Brauer map associated to the canonical cover in the isogeny case is far from being neat.

Type 1 bielliptic surfaces:

There are two main subcases:

- When B (and so A) does not have complex multiplication. Here we will see has the map can be non injective, but it is never trivial.
- When *B* (and so *A*) has complex multiplication. Here the map can be trivial.

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- When *B* (and so *A*) has complex multiplication. Here the map can be trivial.

Notation

Recall that in this case *G* is cyclic of order 2 acting by translation by a point τ on *A* and by $-id_B$ on *B*. We identify the dual of *A* with *A* using the isomorphism associated to $\mathcal{O}_A(0_A)$ and we denote by P_{τ} the topologically trivial line bundle on *A* associated to τ .

A whiteboard



Non CM case



If *A* and *B* are isogenous and do not have complex multiplication, then Hom(B, A) is a free \mathbb{Z} -module of rank 1.

Theorem

The Brauer map associated to the canonical cover is not injective if, and only if, for a (and hence all) choice of a generator $\phi : B \to A$ we have that $\phi^* P_{\tau}$ is trivial. In this case the kernel of the Brauer map will be isomorphic to $\mathbb{Z}/2\mathbb{Z}$.





If *A* and *B* are isogenous and do not have complex multiplication, then Hom(B, A) is a free \mathbb{Z} -module of rank 2.

Theorem

The Brauer map associated to the canonical cover is not injective if, and only if, for a (and hence all) choice of a generators ϕ_1 , $\phi_2 : B \to A$ we have that one of the following line bundles is trivial

$$\phi_1^* P_{\tau}, \quad \phi_2^* P_{\tau}, \quad (\phi_1 + \phi_2)^* P_{\tau} \tag{1}$$

In addition the Brauer map is trivial if, and only if, two (and hence all) line bundles in are trivial. (1)





• Suppose that $A \simeq B$. If A does not have complex multiplication, then we can take $\psi = \pm 1_A$. In particular we have that $\psi^* P_{\tau}$ is never trivial and the Brauer map is injective.

Examples



- Suppose that $A \simeq B$. If A does not have complex multiplication, then we can take $\psi = \pm 1_A$. In particular we have that $\psi^* P_{\tau}$ is never trivial and the Brauer map is injective.
- Suppose again that $A \simeq B$ and that the *j*-invariant of *A* is 1728. Then $\operatorname{End}(A) \simeq \mathbb{Z}[i]$ and the multiplication by *i* induces an automorphism ω of *A* of order 4, and we can take 1_A and ω as generators of $\operatorname{End}(A)$. Suppose that P_{τ} is a fixed point of the dual automorphism ω^* (For example we can identify *A* with its dual and ω^* with ω and take $\tau = (\frac{1}{2}, \frac{1}{2}) + \Lambda$, where $\Lambda = <1, i > A \simeq \mathbb{C}/\Lambda$). Then $(1_A + \omega)^* P_{\tau}$ is zero and the Brauer map is not injective (and is neither trivial!!!)

Examples II



• We can also use a similar argument to construct uncountably many Type 1 byelliptic surfaces with non injective Brauer map. Let *B* any elliptic curve without complex multiplication and chose θ a point of over 2 on *B*. Let $A := B/\theta$ and $\psi : B \rightarrow A$ the quotient map. This is a degree 2 isogeny, so it is primitive and hence generating. If τ is the only point of order 2 in Ker ψ^* , then we have that the data A, τ , *B* uniquely identify a Type 1 bielliptic surface which has a non injective Brauer map.

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- Let now $A \simeq \mathbb{C}/\mathbb{Z}[2i]$ and let τ the point $(0, i) + \mathbb{Z}[2i]$. The elliptic curve $B := A / < \tau >$ has *j*-invariant 1728 and Hom(B, A) is generated by the isogenies $\psi_1 := \varphi_2$ and $\psi_2 := \varphi_2 \circ \lambda_B$, where $\varphi_2 : B \rightarrow A$ denotes the isogeny induced by multiplication by 2. Observe that

 $\varphi_2^*(\boldsymbol{P}_{\tau}) \simeq \varphi_2^*(\mathcal{O}_{\mathcal{A}}(\tau - \boldsymbol{p}_0) \simeq \mathcal{O}_{\mathcal{A}}(\varphi_2(\tau) - \varphi(\boldsymbol{p}_0)) \simeq \mathcal{O}_{\mathcal{B}}$

Thus we have that $\psi_1^* P_\tau \simeq \psi_2^* P_\tau \simeq \mathcal{O}_B$ and the Brauer map is trivial.

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The "Moduli" Picture



Type 1 bielliptic surfaces are constructed by choosing two elliptic curves A and B and a 2-torsion point on A. Thus the moduli space has dimension 2. In order to have a non injective Brauer map one can choose freely the elliptic curve B, but has only finitely many possibilities for A and the 2-torsion point. Thus we obtain a 1-dimensional family. On the other hand only countably many type 1 bielliptic surfaces can have a trivial Brauer map to their canonical cover. In fact, to obtain a trivial Brauer map one has to choose the elliptic curve B among those having complex multiplication.

The other types



- **Type 2:** These surfaces are constructed by choosing two elliptic curves *A* and *B* and two 2-torsion points, one on *A* and one on *B*. Hences the moduli space has dimension 1. Similarly to what happens in the previous case, in order to have a trivial Brauer map only the choice of the curve *B* can be made freely, while *A* must be taken among finitely many possibilities.
- **Type 3:** These surfaces are constructed by choosing one elliptic curve *A* and a 4-torsion point on it. Therefore the moduli space has dimension 1. In order to have a non injective (and hence trivial) Brauer map, *A* must be isogenous to the curve with *j*-invariant 1728. Thus there are only finitely many such surfaces.
- **Type 5:** These surfaces are constructed by choosing one elliptic curve *A* and a 3-torsion point on it. We deduce that the moduli space has dimension 1. In order to have a non injective (and hence trivial) Brauer map, *A* must be isogenous to the curve with *j*-invariant 0. Thus, as in the previous case, there are only finitely many such surfaces.



Section 3 How did we do it?

Beauville's Strategy



Let $\pi : X \to Y$ be a finite locally free morphism of projective varieties of degree *n*. To it we can associate a group homomorphism

 $\operatorname{Nm}_{\pi}: \operatorname{Pic}(X) \to \operatorname{Pic}(Y)$

called the norm homomorphism associated to $\boldsymbol{\pi}$

Proposition (Beauville 2009)

Let $\pi: X \to S$ be an étale cyclic covering of smooth projective varieties. Let σ be a generator of the Galois group of π , Nm $_{\pi}$: Pic(X) \to Pic(S) be the norm map and π_{Br} : Br(S) \to Br(X) be the pullback. Then we have a canonical isomorphism

 $\operatorname{Ker}(\pi_{\operatorname{Br}}) \simeq \operatorname{Ker} \operatorname{Nm}_{\pi}/(1 - \sigma^*) \operatorname{Pic}(X).$

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So we studied the quotient in the RHS in the various cases.





We first find numerical conditions for a line bundle to be in the kernel of the norm map, and we get V a subspace of the NS upstair.

Example

- If upstair and downstair we have the same Picard rank, then $L \in \text{Ker Nm iff } L \equiv 0.$
- If the group G is cyclic and π : A × B → S is the canonical cover, then Nm_π(L) is trivial iff the numerical class of L is in Hom(A, B).





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- 2 We study $V/(1 \sigma^*)V$ and get "numerical candidates".

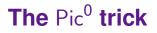




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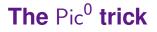
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- 2 We study $V/(1 \sigma^*)V$ and get "numerical candidates".
- Study the norm of line bundles equivalent to the numerical candidates.





How to construct elements in the kernel of the norm

Let $\pi : X \to Y$ be an étale morphism of degree *n* and suppose that there is a line bundle *L* on *X* such that $Nm_{\pi}(L) \in Pic^{0}(Y)$. Then there is an element $\alpha \in Pic^{0}(X)$ such that $Nm_{\pi}(L \otimes \alpha)$ is trivial.

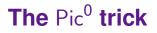




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• By Norm-calculus we have that $Nm_{\pi}(\pi^*M) = M^{\otimes n}$;

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Proof:

- By Norm-calculus we have that $Nm_{\pi}(\pi^*M) = M^{\otimes n}$;
- Solution Pic⁰(*Y*) is a divisible group so we can find $\beta \in \text{Pic}^{0}(Y)$ such that $\beta^{\otimes n} \simeq \text{Nm}_{\pi}(L)^{-1}$;
- then we have

$$\operatorname{Nm}_{\pi}(L \otimes \pi^* \beta) \simeq \operatorname{Nm}_{\pi}(L) \otimes \beta^{\otimes n} \simeq \mathcal{O}_Y.$$





Let *S* be a Type 2 or 3 bielliptic surfaces, then there is a Type 1 bielliptic surface \tilde{S} and an involution $\tilde{\sigma}$ such that $S \simeq \tilde{S}/\tilde{\sigma}$.

Bielliptic covers



Let *S* be a Type 2 or 3 bielliptic surfaces, then there is a Type 1 bielliptic surface \tilde{S} and an involution $\tilde{\sigma}$ such that $S \simeq \tilde{S}/\tilde{\sigma}$.

• The fibration $g : \tilde{S} \to \mathbb{P}^1$ has four multiple fibers all of multiplicity 2 which we will denote by D_1, \ldots, D_4 . Let $\tau_{ij} := \mathcal{O}_{\tilde{S}}(D_i - D_j)$.

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 $H^{2}(\tilde{S},\mathbb{Z})_{tor} = \{0, [\tau_{1j}]\}_{j\neq 1}\}$

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We have that

$$H^{2}(\tilde{S},\mathbb{Z})_{tor} = \{0, [\tau_{1j}]\}_{j\neq 1}\}$$

Type 2

If *S* is of type 2, then by seeing how the involution acts on the *D*_i's we get that $Nm(\tau_{13}) \in Pic^{0}(S)$, but $\tau_{13} \otimes \alpha \notin Im(1 - \tilde{\sigma}^{*})$ for every $\alpha \in Pic^{0}(\tilde{S})$. We conclude by the Pic⁰-trick.

Bielliptic Cover of Type 3



Why they are different

The deep reason for the different behavior of the Brauer map in the two cases is how the $\tilde{\sigma}$ acts on the D_i 's, which affect the computation of the norm map.

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Lemma

Let *n* and *m* be two integers. Then the norm of the line bundle $\tau_{13}^{\otimes n} \otimes \tau_{14}^{\otimes m}$ is zero if and only if *n* and *m* have the same parity. In addition we have that $Nm(\tau_{13}^{\otimes n} \otimes \tau_{14}^{\otimes m})$ is not in $Pic^0(S)$ if *n* and *m* are not congruent modulo 2.

Preposition

If *L* is in the Kernel of the norm map, then *L* is in $\operatorname{Pic}^{\tau}(\tilde{S})$.

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Preposition

If *L* is in the Kernel of the norm map, then *L* is in $\operatorname{Pic}^{\tau}(\tilde{S})$.

(Depends from the fact that \tilde{S} and S have the same Picard rank).

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Thank you for your attention!

