Cyclic sieving and orbit harmonics

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Cyclic Sieving Phenomena

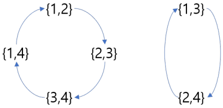
Let $C = \langle c \rangle$ be a cyclic group acting on a finite set X and let $X(q) \in \mathbb{Z}_{\geq 0}[q]$ be a polynomial. We say the triple $(X, C, X(q))$ exhibits CSP if for all $r \geq 0$ and $\omega = exp(2\pi i/|C|)$,

$$|X^{c^r}| = X(\omega^r).$$

For example, let $X = \begin{bmatrix} [4] \\ 2 \end{bmatrix}$, $C = \langle c \rangle$ be a cyclic group of order 4 acting on X as in the figure and let $X(q) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = q^4 + q^3 + 2q^2 + q + 1$. Note that

$$|X^{id}| = X(1) = 6, |X^c| = X(i) = 0, |X^{c^2}| = X(i^2) = 2 \text{ and } |X^{c^3}| = X(i^3) = 0.$$

Thus the triple $(X, C, X(q))$ exhibits CSP.



Orbit harmonics

X: a finite set in $\mathbb{C}^n$ which is closed under $\mathfrak{S}_n \times C$, where

- symmetric group $\mathfrak{S}_n$ acts on $\mathbb{C}^n$ by coordinate permutation
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$$I(X) := \{f \in \mathbb{C}[x_n] : f(x) = 0, \quad \forall x \in X\}.$$

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For any $f \in \mathbb{C}[x_n]$, let $\tau(f)$ be the top degree component of f. The ideal $T(X) \subseteq \mathbb{C}[x_n]$ is defined by

$$T(X) := \langle \tau(f) : f \in I(X), \quad f \neq 0 \rangle.$$

and we have an isomorphism as $\mathfrak{S}_n$-modules,

$$\mathbb{C}[X] \cong \mathbb{C}[x_n]/T(X),$$

as $\mathfrak{S}_n \times C$-module on which C acts on $\mathbb{C}[x_n]/T(X)$ by scaling a root of unity in each variable.

CSP generating theorem

Main theorem (O.–Rhoades 20+)

Let $X \subseteq \mathbb{C}^n$ be a finite set with $\mathfrak{S}_n \times C$ acting on it. For a subgroup $G \subseteq \mathfrak{S}_n$, the triple $(X/G, C, (X/G)(q))$ exhibits CSP where

$$(X/G)(q) = \mathrm{Hilb}((\mathbb{C}[x_n]/T(X))^G; q),$$

where Hilbert series of graded vector space $V = \bigoplus_d V_d$ is defined by

$$\mathrm{Hilb}(V; q) := \sum_{d \geq 0} \dim(V_d) q^d.$$

The key idea of the proof is the isomorphism given in the last slide,

$$\mathbb{C}[X] \cong \mathbb{C}[x_n]/T(X).$$

Since the C action is encoded in the grading in $\mathbb{C}[x_n]/T(X)$, we can calculate the number of fixed points of c^r by trace of the action of c^r, and thus by the root of unity evaluation of the Hilbert series.

Injective functional locus

Proposition

For $n < k$, let $\mathcal{I}_{n,k} = \{(a_1, \ldots, a_n) \in \mathbb{C}^n : \{a_1, \ldots, a_n\} \subseteq \{\omega^1, \ldots, \omega^k\}\}$ be the injective functional locus, where $\omega = \exp(2\pi i/k)$. Then,

$$\mathbb{C}[x_n]/T(\mathcal{I}_{n,k}) = \mathbb{C}[x_n]/\langle h_{k-n+1}(x_n), \ldots, h_k(x_n)\rangle \quad \textit{and}$$

$$\mathrm{grFrob}(\mathbb{C}[x_n]/T(\mathcal{I}_{n,k}); q) = \begin{bmatrix} k \\ n \end{bmatrix}_q \sum_{T \in SYT(n)} q^{maj(T)} \cdot s_{sh(\lambda)},$$

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pf) We claim that $\langle h_{n-k+1}(x_n) \dots h_n(x_n)\rangle_{d>k-n} \subseteq T(\mathcal{I}_{n,k})$. Consider

$$\frac{(1-\omega t)\cdots(1-\omega^k t)}{(1-x_1 t)\cdots(1-x_n t)} = \sum_{d \geq 0} \sum_{a+b=d} (-1)^a \cdot e_a\left(\omega, \omega^2, \dots, \omega^k\right) \cdot h_b\left(x_n\right) t^d.$$

If $(x_1, \dots, x_n) \in \mathcal{I}_{n,k}$, the above gives a polynomial in t of degree $k-n$. Taking coefficient of t^d for $d > k-n$, we have

$$\sum_{a+b=d} (-1)^a \cdot e_a\left(\omega, \omega^2, \dots, \omega^k\right) \cdot h_b\left(x_n\right) \in I(\mathcal{I}_{n,k}) \text{ and } h_d(x_n) \in T(\mathcal{I}_{n,k}).$$

A Cyclic sieving phenomenon

Theorem

The triple $\left(\begin{bmatrix} [k] \\ n \end{bmatrix}, \mathbb{Z}_k, \begin{bmatrix} k \\ n \end{bmatrix}_q \right)$ exhibits CSP, where $\mathbb{Z}_k$ acts by rotating elements of subsets.

Proof) Since $\mathcal{I}_{n,k}/\mathfrak{S}_n = \begin{bmatrix} [k] \\ n \end{bmatrix}$ and

$$\mathrm{Hilb}((\mathbb{C}[x_n]/T(\mathcal{I}_{n,k}))^{\mathfrak{S}_n}; q) = \text{the coefficient of } s_{(n)} \text{ in } \mathrm{grFrob}(V; q),$$

Recall that

$$\mathrm{grFrob}(\mathbb{C}[x_n]/T(\mathcal{I}_{n,k}); q) = \begin{bmatrix} k \\ n \end{bmatrix}_q \sum_{T \in SYT(n)} q^{maj(T)} \cdot s_{sh(T)}.$$

Then the main theorem gives the result.

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- Various combinatorial loci X
  - Functional loci (any functions, surjective functions)
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- Various subgroups $G \subseteq \mathfrak{S}_n$
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- Various complex reflection groups other than $\mathfrak{S}_n$
  e.g. $G(r, 1, n)$, a group of r-colored permutations gives sieving results for twisted rotation. (recovers results of Barcelo–Reiner–Stanton on colored permutations and of Alexandersson–Linusson–Potka on binary words)

Thank you!