Title, Speaker

New results on a variational inequality formulation of Lavrentiev regularization for nonlinear monotone ill-posed problems

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Joint work with Bernd Hofmann (Chemnitz), see JOTA (2019).

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1. Variational Lavrentiev regularization

1.1 Preparations

Definition

Let $\mathcal{H}$ Hilbert space, $F : \mathcal{H} \to \mathcal{H}$ nonlinear, in general. The operator F is *monotone* on a set $\mathcal{M} \subseteq \mathcal{H}$, if

$$\langle\!\langle Fu - Fv, u - v \rangle\!\rangle \geq 0 \quad \text{for each} \quad u, v \in \mathcal{M}.$$

Example

Consider classical integration and the Abel operators (both linear!)

$$(Vu)(x) = \int_0^x u(y)\, dy,$$

$$(V^\alpha u)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-y)^{-(1-\alpha)} u(y)\, dy \quad (0 < \alpha < 1)$$

for $0 \leq x \leq 1$. Both are monotone on $\mathcal{H} = L^2(0,1)$.

Example (ODE parameter estimation

Consider operator $F : L^2(0,1) \to L^2(0,1)$ given by $Fu = f$, where

$$f'(x) + u(x)f(x) = 0,\ 0 \le x \le 1, \quad f(0) = -c_0,$$

with $c_0 > 0$ given. F is monotone on $\mathcal{M} = \{\, u \in L^2(0,1) \mid u \ge 0 \,\}$.

Example (Autoconvolution)

Consider $F : L^2(0,1) \to L^2(0,1)$ given by

$$(Fu)(x) = \int_0^x u(x-y)u(y)dy, \quad 0 \le x \le 1.$$

F is monotone on

$$\mathcal{M} = \{\, u \in L^2(0,1) \mid u', u'' \in L^2(0,1), u \ge 0, u' \le 0, u'' \ge 0 \,\}.$$

Assumptions

Let

- $\mathcal{H}$ Hilbert space, $\mathcal{M} \subseteq \mathcal{H}$ closed convex.
- $F : \mathcal{H} \to \mathcal{H}$ monotone on $\mathcal{M} \subseteq \mathcal{H}$, i.e.,

$$\langle\!\langle Fu - Fv, u - v \rangle\!\rangle \geq 0 \quad \text{for each} \quad u, v \in \mathcal{M}.$$

- $u_* \in \mathcal{M}, \quad f \in \mathcal{H}, \quad Fu_* = f, \quad f^\delta \in \mathcal{H}, \quad \|f^\delta - f\| \leq \delta.$
- In general, F^{-1} not continuous at f.
- Determine u_*.

Classical Lavrentiev regularization

Consider for $0 < \gamma =$ regularization parameter

$$(F + \gamma I) u_\gamma^\delta = f^\delta$$

with $u_\gamma^\delta \in \mathcal{M}$.

Remark

Recall Tikhonov regularization for linear case:

$$(A^* A + \gamma I) u_\gamma^\delta \; = \; A^* f^\delta,$$

where $A : \mathcal{H}_1 \to \mathcal{H}_2$ linear, bounded, with $\mathcal{H}_1, \mathcal{H}_2$ Hilbert spaces.

Thus: Lavrentiev regularization = Tikhonov regularization without normalization in linear case.

Remark

Note that equation $(F + \gamma I)u_\gamma^\delta = f^\delta$ is solvable on $\mathcal{M}$ in special cases only, e.g.,

- $\mathcal{M} = \mathcal{H}$, and F continuous, or
- $\mathcal{M} =$ closed ball of sufficiently large radius.

In applications, frequently $\mathcal{H} = L^2(0,1), \mathcal{M} = \{\, u \in L^2(0,1) \mid u \geq 0 \,\}$, however.

Notation

For $F : \mathcal{H} \to \mathcal{H}$ we use the notation

$$F_\gamma = F + \gamma I : \mathcal{H} \to \mathcal{H} \quad \text{for} \quad \gamma > 0.$$

Variational Lavrentiev regularization (RVI)

For $\gamma =$ regularization parameter > 0, determine $u_\gamma^\delta \in \mathcal{M}$ with

$$\langle\!\langle F_\gamma u_\gamma^\delta - f^\delta, u - u_\gamma^\delta \rangle\!\rangle \geq 0 \quad \text{for each} \quad u \in \mathcal{M}. \qquad \text{(RVI)}$$

Proposition

Let $\mathcal{P}_\mathcal{M} =$ metric projection onto $\mathcal{M}, u_\gamma^\delta \in \mathcal{M}$. Then:

$$u_\gamma^\delta \text{ solves RVI} \Longleftrightarrow u_\gamma^\delta = \mathcal{P}_\mathcal{M}(u_\gamma^\delta - \mu(F_\gamma u_\gamma^\delta - f^\delta))$$

with $\mu > 0$.

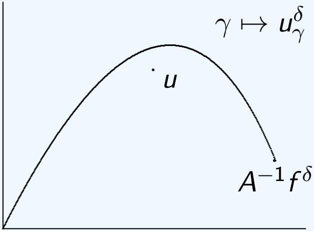
Theorem

If $F : \mathcal{H} \to \mathcal{H}$ continuous + bounded then RVI has a (unique) solution $u_\gamma^\delta \in \mathcal{M}$.

Some new convergence rates

Trajectory in $\mathcal{H}$; semiconvergence

Variational Lavrentiev regularization $\rightsquigarrow$ $\{u_\gamma^\delta\}_{\gamma \ge 0} \subset \mathcal{H}$



Lemma

For $u_\gamma = u_\gamma^0$, i.e., variational Lavrentiev regularization with exact data, we have

$$\|u_\gamma^\delta - u_*\| \le \|u_\gamma - u_*\| + \frac{\delta}{\gamma} \quad \text{for} \quad \gamma > 0.$$

Proof

Simple decomposition:

$$\|u_\gamma^\delta - u_*\| \le \|u_\gamma - u_*\| + \|u_\gamma^\delta - u_\gamma\|,$$
$$\|u_\gamma^\delta - u_\gamma\| \le \frac{\delta}{\gamma}.$$

So $e_\gamma = u_\gamma - u_*$ has to be estimated!

Note: For Tikhonov method, estimation of $\|u_\gamma^\delta - u_\gamma\|$ possible under additional smoothness of solution, see Scherzer (1993).

Convergence:

Theorem

Let F be continuous and $u_* =$ minimum norm solution of (non-regularized) VI. Then:

(a) $e_\gamma = u_\gamma - u_* \to 0$ as $\gamma \to 0$,
(b) $e_\gamma^\delta = u_\gamma^\delta - u_* \to 0$ as $\gamma_\delta \to 0, \frac{\delta}{\gamma_\delta} \to 0$.

Alber, Ryazantseva, Khan, Tichatschke, Tammer, Gwinner, ...

Definition

An operator $F : \mathcal{H} \to \mathcal{H}$ is *cocoercive* on a subset $\mathcal{M} \subseteq \mathcal{H}$ if, for some $\tau > 0$,

$$\langle\!\langle Fu - Fv, u - v \rangle\!\rangle \geq \tau \| Fu - Fv \|^2 \quad \text{for each} \quad u, v \in \mathcal{M}.$$

Example (ODE parameter estimation revisited

Consider again $F : L^2(0,1) \to L^2(0,1)$ given by $Fu = f$, where

$$f'(x) + u(x) f(x) = 0, \; 0 \leq x \leq 1, \quad f(0) = -c_0,$$

with $c_0 > 0$ given. For any $\theta > 0$, the operator F is cocoercive on $\mathcal{M} = \{ u \in L^2(0,1) \mid u \geq \theta \}$.

Remark

1. Obviously, cocoercive $\Longrightarrow$ monotone
2. Baillon–Hadard theorem: Assume $\mathcal{M}$ open, and F has potential. Then F Lipschitz continuous $\Longrightarrow$ F cocoercive.

Theorem

Suppose that

- F is cocoercive on $\mathcal{M}$,
- F is Fréchet differentiable on $\mathcal{H}$, with

$$\|F'(u) - F'(v)\| \le L\|u - v\| \quad \text{for each} \quad u, v \in \mathcal{M},$$

  with some $L \ge 0$,
- $u_* = F'(u_*)^* z$ for some $z \in \mathcal{H}$, with $\|z\|$ small enough.

Then

$$\|e_\gamma\| = \mathcal{O}(\gamma^{1/2}), \qquad \|F(u_\gamma) - f\| = \mathcal{O}(\gamma) \quad \text{as} \quad \gamma \to 0.$$

Corollary

Under the conditions of the theorem we have, with $\gamma_\delta = c\delta^{2/3}$,

$$\|u_\gamma^\delta - u\| = \mathcal{O}(\delta^{1/3}) \qquad \text{as} \quad \delta \to 0.$$

Remark

1. Parts of proof of Theorem are similar to related results in Scherzer/Engl/Kunisch (1993) on Tikhonov regularization.
2. Liu/Nashed (1998) for RVI: $\|e_\gamma\| = \mathcal{O}(\gamma^{1/3})$ only (under more general assumptions, however, eg., possible set perturbations).
3. Adjoint source condition not well suited for Lavrentiev regularazation. For linear problems, see Plato/Hofmann/Mathé (2018).
4. Results improvable in linear case under standard source conditions.
5. Similar rates for classical Lavrentiev regularization: Hofmann/Kaltenbacher/Resmerita (2016).

3 Variational Lavrentiev regularization with initial guess

Method

Let initial guess $\overline{u} \in \mathcal{H}$ be fixed. For $\gamma > 0$, determine $u_\gamma^\delta \in \mathcal{M}$ with

$$\langle\!\langle F u_\gamma^\delta + \alpha(u_\gamma^\delta - \overline{u}) - f^\delta, u - u_\gamma^\delta \rangle\!\rangle \geq 0 \quad \text{for each} \quad u \in \mathcal{M}. \qquad (*)$$

Let $u_\gamma = u_\gamma^0$ be obtained by $(*)$, with exact data $f^\delta = f$.

Theorem

Let F be cocoercive on $\mathcal{M}$, and F' be Lipschitz continuous on $\mathcal{M}$. If

$$u_* \in \mathcal{M}, \quad F u_* = f, \quad u_* - \overline{u} = F'(u_*)^* z, \quad \varrho := \|z\|,$$

holds for some $z \in \mathcal{H}$ and $\varrho L < 2$, then

$$\|u_\gamma - u_*\| = \mathcal{O}(\gamma^{1/2}) \text{ as } \gamma \to 0, \quad \|u_{\gamma_\delta}^\delta - u_*\| = \mathcal{O}(\delta^{1/3}) \text{ as } \delta \to 0,$$

for any a priori parameter choice $\gamma_\delta \sim \delta^{2/3}$.

4 Numerical results

ODE parameter estimation revisited

Consider again $F : L^2(0,1) \to L^2(0,1)$ given by $Fu = f$, where

$$f'(x) + u(x)f(x) = 0, \ 0 \le x \le 1, \quad f(0) = -1,$$

This means

$$(Fu)(x) = -\exp\Big(-\int_0^x u(y)\,dy \Big), \quad 0 \le x \le 1. \qquad (1)$$

For any $\theta > 0$, the operator F is cocoercive on $\mathcal{M} = \{ u \in L^2(0,1) \mid u \ge \theta \}$.

Example 1

Consider first $Fu = f$, with

$$f(x) = -\exp(-\tfrac{a}{2}x^2 - bx) \quad \text{for} \quad 0 \le x \le 1,$$

with $a = b = \frac{1}{2}$. Exact solution:

$$u_*(x) = ax + b \quad \text{for} \quad 0 \le x \le 1.$$

Note that $u_* \ge b$.
The modified adjoint source condition is satisfied for initial guess

$$\overline{u} \equiv u_*(1).$$

Consider modified variational Lavrentiev with initial guess $\overline{u}$ and parameter $\gamma_\delta = \delta^{2/3}$.

Numerical results for Example 1:

δ	$100 \cdot \delta / \|f\|$	$\|u_{\gamma_\delta}^{\delta} - u_*\|$	$\|u_{\gamma_\delta}^{\delta} - u_*\| \ / \delta^{1/3}$
$1.0 \cdot 10^{-2}$	$1.33 \cdot 10^{0}$	$9.87 \cdot 10^{-2}$	0.46
$5.0 \cdot 10^{-3}$	$6.66 \cdot 10^{-1}$	$8.23 \cdot 10^{-2}$	0.48
$2.5 \cdot 10^{-3}$	$3.33 \cdot 10^{-1}$	$6.72 \cdot 10^{-2}$	0.50
$1.2 \cdot 10^{-3}$	$1.67 \cdot 10^{-1}$	$5.42 \cdot 10^{-2}$	0.50
$6.2 \cdot 10^{-4}$	$8.33 \cdot 10^{-2}$	$4.17 \cdot 10^{-2}$	0.49
$3.1 \cdot 10^{-4}$	$4.16 \cdot 10^{-2}$	$3.26 \cdot 10^{-2}$	0.48
$1.6 \cdot 10^{-4}$	$2.08 \cdot 10^{-2}$	$3.26 \cdot 10^{-2}$	0.61
$7.8 \cdot 10^{-5}$	$1.04 \cdot 10^{-2}$	$2.72 \cdot 10^{-2}$	0.64
$3.9 \cdot 10^{-5}$	$5.21 \cdot 10^{-3}$	$2.53 \cdot 10^{-2}$	0.75

Table: Numerical results for first example

Example 2

Now consider $Fu = f$, with

$$f(x) = -\exp\left(\tfrac{a}{\pi}(\cos \pi x - 1) - bx\right)$$

with $a = \frac{1}{4}$, $b = \frac{1}{3}$. Exact solution:

$$u_*(x) = a \sin \pi x + b \quad \text{for} \quad 0 \le x \le 1.$$

Note that $u_* \ge b$.
The modified adjoint source condition is satisfied for initial guess

$$\overline{u} \equiv u_*(1).$$

Consider modified variational Lavrentiev with initial guess $\overline{u}$ and parameter $\gamma_\delta = \delta^{2/3}$.

Numerical results for Example 2:

δ	$100 \cdot \delta / \lVert f \rVert$	$\lVert u_{\gamma_\delta}^\delta - u_* \rVert$	$\lVert u_{\gamma_\delta}^\delta - u_* \rVert \; / \delta^{1/3}$
$1.0 \cdot 10^{-2}$	$1.25 \cdot 10^{0}$	$7.00 \cdot 10^{-2}$	0.32
$5.0 \cdot 10^{-3}$	$6.25 \cdot 10^{-1}$	$4.66 \cdot 10^{-2}$	0.27
$2.5 \cdot 10^{-3}$	$3.12 \cdot 10^{-1}$	$3.87 \cdot 10^{-2}$	0.29
$1.2 \cdot 10^{-3}$	$1.56 \cdot 10^{-1}$	$3.01 \cdot 10^{-2}$	0.28
$6.2 \cdot 10^{-4}$	$7.81 \cdot 10^{-2}$	$2.22 \cdot 10^{-2}$	0.26
$3.1 \cdot 10^{-4}$	$3.90 \cdot 10^{-2}$	$1.60 \cdot 10^{-2}$	0.24
$1.6 \cdot 10^{-4}$	$1.95 \cdot 10^{-2}$	$1.08 \cdot 10^{-2}$	0.20
$7.8 \cdot 10^{-5}$	$9.76 \cdot 10^{-3}$	$7.54 \cdot 10^{-3}$	0.18
$3.9 \cdot 10^{-5}$	$4.88 \cdot 10^{-3}$	$4.70 \cdot 10^{-3}$	0.14

Table: Numerical results for second example

5 Conclusions

Conclusions

- More examples.
- Other source conditions.
- Get rid of cocoercivess.
- Many thanks for invitation.
- Many thanks to Zuhair.
- Many thanks for your attention.