Stable determination of polygonal and polyhedral interfaces from boundary measurements

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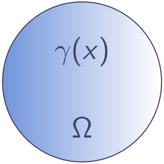
Calderón problem - Electrical Impedance Tomography



Recover $\gamma > 0$ defined in Ω from boundary values of solutions of the equation

$$\operatorname{div}(\gamma \nabla u) = 0 \quad \text{in } \Omega.$$

Boundary measurements are encoded in the Dirichlet to Neumann map:

$$\begin{aligned} \Lambda_\gamma : H^{1/2}(\partial\Omega) &\to H^{-1/2}(\partial\Omega) \\ f &\to \gamma \frac{\partial u}{\partial \nu} \end{aligned}$$

where u solves

$$\begin{cases} \operatorname{div}(\gamma \nabla u) = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}$$

The forward map is the nonlinear map

$$F : \gamma \in L^\infty(\Omega) \to \Lambda_\gamma \in \mathcal{L}\left(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\right)$$

We are interested in

1. **Uniqueness** of the solution: injectivity of F.

$$F(\gamma_0) = F(\gamma_1) \overset{?}{\Rightarrow} \gamma_0 = \gamma_1$$

   Are the measurements enough to distinguish between two different coefficients?

2. **Stability**: continuity of F^{-1}.

$$\|\gamma_0 - \gamma_1\| \overset{?}{\leq} \omega\left(\|F(\gamma_0) - F(\gamma_1)\|\right)$$

   for $\omega(t) \to 0$ as $t \to 0$.

Uniqueness for Calderón problem

$$F(\gamma_0) = F(\gamma_1) \stackrel{?}{\Rightarrow} \gamma_0 = \gamma_1$$

Isotropic conductivities

- $n \geq 2$, γ_0, γ_1 piecewise analytic Kohn Vogelius (1984, 1985)
- $n \geq 3$ and γ_0, $\gamma_1 \in C^2(\overline{\Omega})$ Sylvester Uhlmann (1987), $W^{1,\infty}(\Omega)$ Caro Rogers (2016), $W^{1,3}(\Omega)$ Haberman (2015)
- $n = 2$ and γ_0, $\gamma_1 \in W^{2,p}(\Omega)$ Nachman (1995), Brown Uhlmann (1997) , γ_0, $\gamma_1 \in L^\infty(\Omega)$ Astala Paivairinta (2006)

Anisotropic conductivities

- **Nonuniqueness** Kohn-Vogelius (1984) counterexample. When γ is a *matrix* function (anisotropic materials), it is impossible to determine γ uniquely.

Instability of Calderón problem

Stability of the inverse problem $\iff$ **Continuity** of F^{-1}

$$\|\gamma_0 - \gamma_1\| = \|F^{-1}(\Lambda_{\gamma_0}) - F^{-1}(\Lambda_{\gamma_1})\| \overset{?}{\leq} \omega\left(\|\Lambda_{\gamma_0} - \Lambda_{\gamma_1}\|_\star\right)$$

Positivity and boundness of conductivity are not enough to guarantee stability, Example of instability Alessandrini (1988)

Stability in ill-posed problems can be restored **conditionally** to a-priori bounds on the unknowns that guarantee compactness

Tikhonov (1943)

$F : K \subset X \to Y$, where X, Y are Banach spaces, K compact set and F injective and continuous operator. Then,

$$(F_{|K})^{-1} : F(K) \to K$$

is continuous.

Conditional stability for Calderón problem

- Alessandrini (1988) $n \geq 3$, $\|\gamma\|_{W^{2,\infty}(\Omega)} \leq E$
- Barcelo, Faraco, Ruiz (2007) $n = 2$, $\|\gamma\|_{C^{\alpha}(\overline{\Omega})} \leq E$.

$$\|\gamma_0 - \gamma_1\|_{L^\infty(\Omega)} \leq \omega(\|\Lambda_{\gamma_0} - \Lambda_{\gamma_1}\|_\star).$$

- Clop, Faraco, Ruiz (2010) $n = 2$, $\|\gamma\|_{W^{\alpha,p}(\Omega)} \leq E$, $\alpha > 0$

$$\|\gamma_0 - \gamma_1\|_{L^2(\Omega)} \leq \omega(\|\Lambda_{\gamma_0} - \Lambda_{\gamma_1}\|_\star)$$

- Caro, García, Reyes (2013) $n \geq 3$, $\gamma \in C^{1,\epsilon}(\overline{\Omega}) \leq E$

$$\|\gamma_0 - \gamma_1\|_{C^{0,\delta}(\Omega)} \leq \omega(\|\Lambda_{\gamma_0} - \Lambda_{\gamma_1}\|_\star)$$

In all these results $\omega(t) = C|\log t|^{-\eta}$.

A limit to stability with information on regularity

Mandache (2001) has proved that logarithmic stability is the best possible stability using as a-priori assumption of the form

$$\|\gamma\|_{C^k(\overline{\Omega})} \leq E, \quad \forall k = 0, 1, 2....$$

Strategy:

Look for **a-priori assumptions** on conductivity

- physically relevant
- give rise to better stability (**Lipschitz stability**)

Reduce the number on unknowns

Assume $F : K \subset L^\infty \to \mathcal{L}$, K subset of a finite dimensional space
unknown conductivity depends on finitely many parameters
For example

$$\gamma(x) = \sum_{j=1}^{N} \gamma_j \chi_{D_j}(x).$$

- Reasonable for most applications, e.g. medical imaging (different tissues), geophysical prospection (different rocks, layers of the earth), nondestructive testing of materials (composite materials)
- Finite element discretization for effective reconstruction

Strategy of the proof

$F : \gamma \in K \subset L^\infty \to \Lambda_\gamma \in \mathcal{L}$, where K denotes compact subset of a finite dimensional space.

Steps to prove Lipschitz stability estimate:

1. prove that F is injective (uniqueness);
2. prove that F is differentiable and evaluate the Frechét derivative DF;
3. prove that DF is bounded from below.

The bound from below of the derivative DF gives a bound from above of the constant C in the stability estimate

$$\|\gamma_0 - \gamma_1\| \leq C \|\Lambda_{\gamma_0} - \Lambda_{\gamma_1}\|_\star$$

Main tools

(I) **Unique continuation property**

$$\operatorname{div}(\gamma \nabla u) = 0 \quad \text{in} \quad \Omega$$

$$\textbf{(UCP)} \quad u = 0 \text{ in } B_r(x_0) \subset \Omega \Rightarrow u = 0 \ \text{ in } \Omega$$

$$\textbf{(QUCP)} \quad \|u\|_{L^2(B_r(x_0))} \le \epsilon, \ \|u\|_{L^2(\Omega)} \le E \Rightarrow \|u\|_{L^2(G)} \le C\epsilon^{\delta}$$

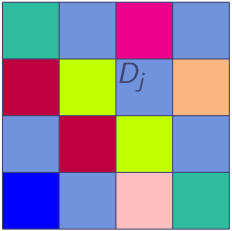
$B_r(x_0) \subset G \subset \Omega$ and C and $\delta \in (0,1)$ depend on $d(G, \partial\Omega)$.

(II) **Regularity estimates** for solutions of equations and systems with discontinuous coefficients

(III) **Singular solutions** for equations and systems with discontinuous coefficients. Asymptotic behaviour near discontinuity interfaces

Parameter identification

$$\gamma = \sum_{j=1}^{N} \gamma^j \chi_{D_j}, \quad \bigcup_{j=1}^{N} \overline{D}_j = \Omega \subset \mathbb{R}^n$$



known domains D_j

unknown parameters γ^j

Lipschitz stability estimates

$$\sum_{j=1}^{N} |\gamma_0^j - \gamma_1^j| \le C \|\Lambda_{\gamma_0} - \Lambda_{\gamma_1}\|_\star$$

Alessandrini, Vessella (2005)

Differentiability of F

$F : K \subset L^\infty \to \mathcal{L}$ is differentiable: for given f and $g \in H^{1/2}(\partial\Omega)$

$$F(\gamma)(f,g) = < \Lambda_\gamma f, g >$$

and for any direction h

$$< DF_\gamma[h]f, g > = \frac{d}{dt} < \Lambda_{\gamma+th} f, g >_{|_{t=0}} = \int_\Omega h \nabla u \nabla v,$$

where u and v are solutions to

$$\left\{ \begin{array}{rcl} \mathrm{div}(\gamma\nabla u) & = & 0 \text{ in } \Omega, \\ u & = & f \text{ on } \partial\Omega. \end{array} \right. \qquad \left\{ \begin{array}{rcl} \mathrm{div}(\gamma\nabla v) & = & 0 \text{ in } \Omega, \\ v & = & g \text{ on } \partial\Omega. \end{array} \right.$$

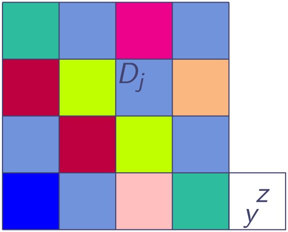
Injectivity of DF

$$\int_\Omega h \nabla u \nabla v \quad \forall u, v \Rightarrow h = 0$$

Take

$$u(x) = G(x, y), \quad v(x) = G(x, z)$$

with G **Green function** in an extended domain.



Other PDEs and systems

- Complex valued coefficients γ

  Beretta, F. (2011)

- Elasticity

  Beretta, F.,Vessella (2014)

  Beretta, F., Morassi, Rosset, Vessella (2014)

  Beretta, de Hoop, F., Vessella, Zhai (2017)

- Anisotropic equation

  Gaburro, Sincich (2015)

- Helmholtz equation $\Delta u + qu = 0$

  Beretta, de Hoop, Qiu (2013)

  Beretta, de Hoop, Faucher, Scherzer (2016)

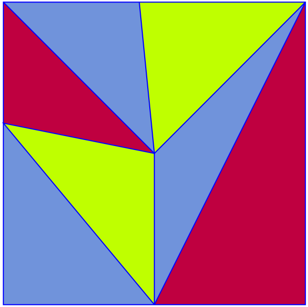
- Piecewise linear coefficients

  Alessandrini, de Hoop, Gaburro, Sincich (2016 and 2017)

Interface identification

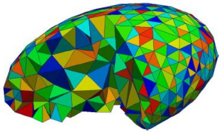
$$\gamma = \sum_{j=1}^{N} \gamma_j \chi_{D_j}, \quad \bigcup_{j=1}^{N} \overline{D}_j = \Omega$$

"known" parameters γ_j
unknown polygonal or polihedral domains D_j



Helmholtz equation

$$\Delta u + \omega^2 q u = 0 \quad \text{in } \Omega \quad \text{for } q = \sum_{j=1}^{N} q_j \chi_{T_j}$$



"known" parameters q_j, unknown domains T_j.

Theorem

Let $q_0 = \sum_{j=1}^{N} q_0^j \chi_{T_j^0}$, $q_1 = \sum_{j=1}^{M} q_1^j \chi_{T_j^1}$, with $\{T_j^k\}$ regular partitions of tetrahedra, q_k^j in a given set of finite, "distinguished" positive values, and ω small, there exist ϵ_0 and C_0, such that if $\|\Lambda_{q_0} - \Lambda_{q_1}\|_\star \leq \epsilon_0$ then

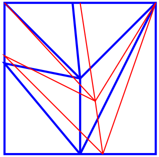
$$N = M, \quad q_0^j = q_1^j \quad \text{and} \quad d_{\mathcal{H}}(T_j^0, T_j^1) \leq C_0 \|\Lambda_{q_0} - \Lambda_{q_1}\|_\star$$

Beretta, F., De Hoop, Vessella (2015)

Derivative of the DN map with respect to movements of vertices

$$q_0 = \sum_{j=1}^{N} q_0^j \chi_{T_j^0} \quad q_t = \sum_{j=1}^{N} q_0^j \chi_{T_j^t}$$

$P_{j,i}^0$ (vertex of T_j^0) $\Rightarrow P_{j,i}^0 + tV_{i,j}$ (vertex of T_j^t)



$$\frac{d}{dt} < \Lambda_{q_t} f, g >_{|_{t=0}} = \omega^2 \sum_{j=1}^{N} q_0^j \int_{\partial T_j^0} u_0 v_0 \left(\Phi_j^V \cdot n_j \right) d\sigma$$

Where u_0 and v_0 are solutions to

$$\left\{ \begin{array}{rcl} \Delta u_0 + \omega^2 q_0 u_0 & = & 0 \text{ in } \Omega, \\ u_0 & = & f \text{ on } \partial\Omega. \end{array} \right. \qquad \left\{ \begin{array}{rcl} \Delta v_0 + \omega^2 q_0 v_0 & = & 0 \text{ in } \Omega, \\ v_0 & = & g \text{ on } \partial\Omega. \end{array} \right.$$

Φ_j^V is an piecewise linear function such that $\Phi_j^V(P_{i,j}^0) = V_{i,j}$

Calderón problem

Difficulty:
Solutions to the conductivity equation are less regular: the gradient jumps on the boundary of the inclusion and might become **singular** at vertices and edges of the partition.

Particular case:

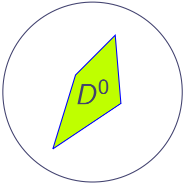
$$\gamma(x) = \chi_{\Omega\setminus D}(x) + k\chi_D(x),\ D \subset \Omega$$

D smooth **logarithmic stability** Alessandrini-Di Cristo (2005)
If D is a polygon then we show that **Lipschitz stability** holds

Simplified 2-dimensional geometry

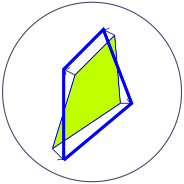
- Let D^0 be a non degenerate polygon with vertices P_j^0, $j = 1, \ldots, N$ and

$$\gamma_0(x) = 1 + (k-1)\chi_{D^0}(x)$$



- Let D^t be the polygon with vertices $P_j^t = P_j^0 + tV_j$ and

$$\gamma_t(x) = 1 + (k-1)\chi_{D^t}(x).$$



- For $f, g \in H^{1/2}(\partial\Omega)$ consider

$$< F(\gamma_t)f, g > = < \Lambda_{\gamma_t} f, g >$$

- Compute $\frac{d}{dt} F(\gamma_t)_{|t=0} = \frac{d}{dt}\Lambda_{\gamma_t}{}_{|t=0}$

Boundary representation of the derivative

$$\begin{cases} \operatorname{div}(\gamma_0 \nabla u_0) &= 0 \text{ in } \Omega, \\ u_0 &= f \text{ on } \partial\Omega. \end{cases} \qquad \begin{cases} \operatorname{div}(\gamma_0 \nabla v_0) &= 0 \text{ in } \Omega, \\ v_0 &= g \text{ on } \partial\Omega. \end{cases}$$

$$u_0^e = u_0|_{\Omega \setminus D^0} \text{ and } v_0^e = v_0|_{\Omega \setminus D^0}$$

$$< \frac{d}{dt} F(\gamma_t) f, g >_{|_{t=0}} = (k-1) \int_{\partial D^0} (M_0 \nabla u_0^e \cdot \nabla v_0^e)(\Phi_0^{\vec{V}} \cdot n_0)$$

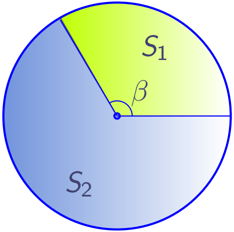
with $M_0 = \tau_0 \otimes \tau_0 + \frac{1}{k} n_0 \otimes n_0$ where τ_0 and n_0 are the tangent and outer normal directions on ∂D^0 and $\Phi_0^{\vec{V}}$ is a piecewise affine map such that

$$\Phi_0^{\vec{V}}(P_j^0) = V_j, \text{ for } j = 1, 2, \ldots, N$$

Beretta-F.-Vessella (2017)

Singularity at vertices

$$\mathrm{div}\left(\left(1+(k-1)\chi_{S_1}\right)\nabla u\right)=0 \text{ in } B_R(0)$$



Set

$$u_j = u_{|_{S_j}}.$$

There exist $\omega > 1/2$ and C depending only on k, R and β such that

$$|\nabla u_j(x,y)| \leq C\|u\|_{L^2(B_R(0))}(x^2+y^2)^{\frac{\omega-1}{2}}$$

Bellout, Friedman, Isakov (1992)

Inclusion determination

Let D^1 and D^2 be **polygonal** conductivity inclusions in $\Omega \subset \mathbb{R}^2$.

$$dist(D^j, \partial\Omega) \geq r_0, \text{ length(sides}(D^j)) \geq r_0, \text{ internal angles} \geq \alpha_0.$$

Let $k \neq 1$ and define $\gamma_j(x) = \chi_{\Omega \setminus D^j}(x) + k\chi_{D^j}(x)$

Lipschitz stability

There exists ϵ_0 and C such that

$$\|\Lambda_{\gamma_0} - \Lambda_{\gamma_1}\|_\star \leq \epsilon_0$$

then D^1 and D^2 have the same number of vertices $\{P_j^1\}_1^N$ and $\{P_j^2\}_1^N$ respectively and

$$d(P_j^1, P_j^2) \leq C\|\Lambda_{\gamma_0} - \Lambda_{\gamma_1}\|_\star \quad \forall j = 1, \ldots, N$$

Beretta-F.-Vessella (submitted)

No convexity assumption.

Main step in the proof

Bound from below is the quantitative counterpart of **injectivity** of the derivative.

$$\frac{d}{dt}\Lambda_{\gamma_t}\Big|_{t=0} = 0 \quad \Rightarrow \quad V = 0,$$

that is:

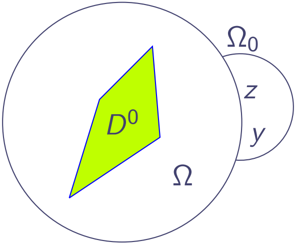
$$\int_{\partial D^0} (M_0 \nabla u_0^e \cdot \nabla v_0^e)(\Phi_0^{\vec{V}} \cdot n_0) = 0$$

for every u_0, v_0 such that $\mathrm{div}\,(\gamma_0 \nabla u_0) = \mathrm{div}\,(\gamma_0 \nabla v_0) = 0$ in Ω. implies

$$V = 0.$$

Choose $u_0(x) = G_0(x, y)$ and $v_0(x) = G_0(x, z)$, where G_0 is the Green's function in a larger domain $\Omega_0 \supset \Omega$
The function

$$S(y, z) = \int_{\partial D^0} (M_0 \nabla G_0^e(\cdot, y) \cdot \nabla G_0^e(\cdot, z))(\Phi_0^{\vec{V}} \cdot n_0)$$



- is **harmonic** with respect to both y and z in $\Omega_0 \setminus D^0$,
- is **zero** for $y, z \in \Omega_0 \setminus \Omega$,
- $M_0 \nabla G_0^e(\cdot, y) \cdot \nabla G_0^e(\cdot, z)$ **diverges** to ∞ as y, z tends to ∂D_0
- $\Rightarrow \Phi_0^{\vec{V}} \cdot n_0 = 0 \Rightarrow V = 0.$

Shape derivative

Minimize with respect to all possible affine motions of the polygon D^0

$$\mathcal{J}(D^t) = \frac{1}{2}\int_{\partial\Omega}(u_t - u_{meas})^2$$

for

$$\begin{cases} \operatorname{div}(\gamma_t \nabla u_t) & = & 0 \text{ in } \Omega, \\ \gamma_t \frac{\partial u_t}{\partial \nu} = g \text{ on } \partial\Omega, & & \int_{\partial\Omega} u_t = 0. \end{cases}$$

Theorem

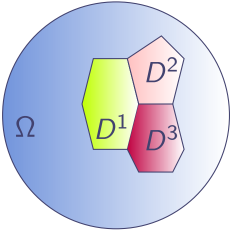
$$\lim_{t\to 0} \frac{\mathcal{J}(D^t) - \mathcal{J}(D^0)}{t} = (k-1)\int_{\partial D^0} (M_0 \nabla u_0^e \cdot \nabla w_0^e)(\Phi_0^{\vec{V}} \cdot n_0) d\sigma$$

where w_0 is the solution to

$$\begin{cases} \operatorname{div}(\gamma_0 \nabla w_0) & = & 0 \textit{ in } \Omega, \\ \gamma_0 \frac{\partial w_0}{\partial \nu} & = & u_0 - u_{meas} \textit{ on } \partial\Omega, \int_{\partial\Omega} w_0 = 0. \end{cases}$$

Beretta, F., Vessella (2017)

Distributed representation of the derivative



$$\gamma_0 = 1 + \sum_{j=1}^{M} (\gamma^j - 1)\chi_{D^j}$$

$$\Phi_t(x) = x + tV(x), \quad D_t^j = \Phi_t(D^j),$$

$$\gamma_t = 1 + \sum_{j=1}^{M} (\gamma^j - 1)\chi_{D_t^j} \quad F(t) = < \Lambda_{\gamma_t} f, g >$$

$$\frac{d}{dt} < \Lambda_{q_t} f, g >_{|_{t=0}} = - \int_{\Omega} \gamma_0 \mathcal{A} \nabla u_0 \nabla v_0 dx$$

where

$$\mathcal{A} = \mathrm{div}(V)I - (\nabla V + \nabla^T V)$$

and $\mathrm{div}(\gamma_0 \nabla u_0) = \mathrm{div}(\gamma_0 \nabla v_0) = 0$ in Ω, $u_0 = f$ and $v_0 = g$ on $\partial\Omega$.

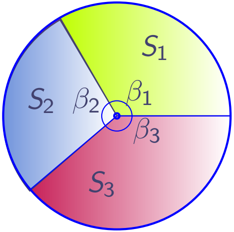
Beretta, Micheletti, Perotto, Santacesaria (2018)

Singularity at vertices with 3 regular sectors

$$\operatorname{div}(\gamma \nabla u) = 0 \text{ in } B_R(0)$$

with

$\overline{\beta} \leq \beta_j \leq \pi - \overline{\beta}$ and $\overline{\gamma} \leq \gamma_j \leq \overline{\gamma}^{-1}$ for $j = 1, 2, 3$



Set

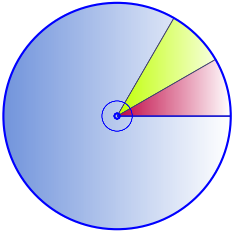
$$u_j = u_{|_{S_j}}.$$

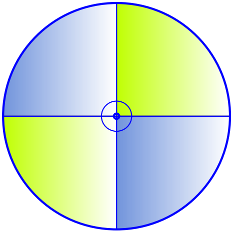
There exist $\omega > 1/2$ and C depending only on $\overline{\gamma}$, R and $\overline{\beta}$ such that

$$|\nabla u_j(x,y)| \leq C \|u\|_{L^2(B_R(0))} (x^2 + y^2)^{\frac{\omega - 1}{2}}$$

Non integrable singularities

3 sectors with $\beta_3 > \pi$





4 regular sectors

Admissible partitions

Let us consider a polygonal inclusion

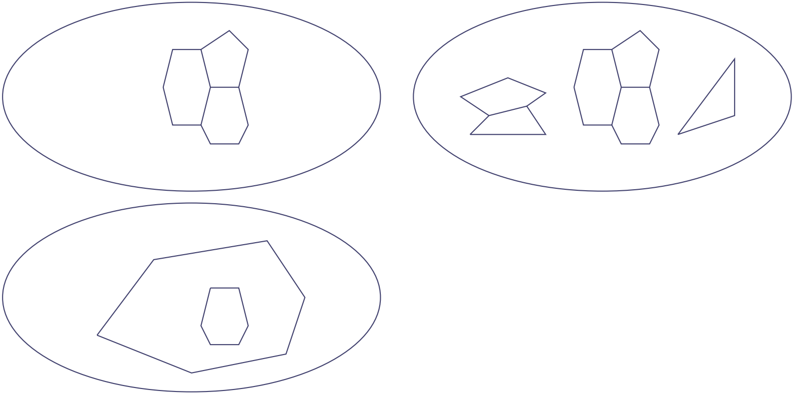
$$\mathcal{P} = \cup_{i=1}^{M} \overline{D}^i,$$

where D^i is an open polygon and $d(\mathcal{P}, \partial\Omega) \geq d_0$. We assume that:

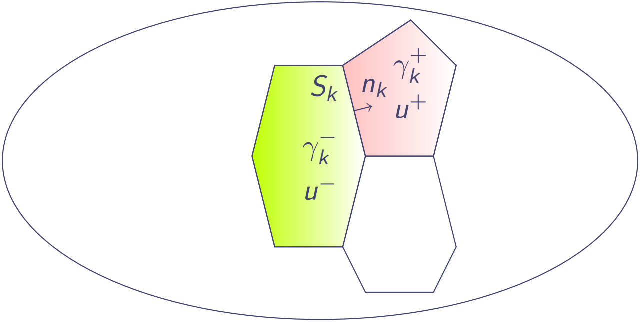
- each vertex does not belong to more the three sides of polygons;
- the length of each side is bigger than d_0
- each polygon D^i contains a disk of radius greater than r_1
- denoting by β_j^k, $k = 1, \ldots, k_j \leq 3$, the angles in the vertex Q_j, we assume there exists $\overline{\beta} \in (0, \pi)$ such that

$$\text{if } k_j = 2, \quad 0 < \overline{\beta} < \beta_j^k < 2\pi - \overline{\beta} \text{ for } k = 1, 2$$
$$\text{if } k_j = 3, \quad 0 < \overline{\beta} < \beta_j^k < \pi - \overline{\beta} \text{ for } k = 1, 2, 3.$$

Admissible configurations



Derivative



$$\frac{d}{dt} < \Lambda_{\gamma_t} f, g >_{|_{t=0}} = \sum_{k=1}^{M_1} \int_{S_k} (\gamma_k^- - \gamma_k^+) M_k \nabla u^+ \nabla v^+ (\Phi_k^V \cdot n_k) d\sigma$$

where $M_k = \tau_k \otimes \tau_k + \frac{\gamma_k^+}{\gamma_k^-} n_k \otimes n_k$ and $\mathrm{div}(\gamma \nabla u) = \mathrm{div}(\gamma \nabla v) = 0$ in Ω, $u = f$ and $v = g$ on $\partial\Omega$.

Work in progress

- Stable determination of a general nondegenerate partition in 2 dimensions;
- Stable determination of a polyhedral inclusion in 3 dimensions;
- Elastic inclusions.