Igor, Feb. 16:

The BIRS workshop is coming up in a month ... We were wondering if you might be interested in giving a survey talk in your favorite part of enumerative combinatorics, which we feel would be beneficial to the audien

C.K., Feb. 17:

Rightaway, I would not have an idea. Do you have a wish?

Igor, Feb. 17:

Yes, we were thinking of the "advanced determina calculus" type survey.

C.K., Feb. 18:

Okay, I can do that.

Advanced Determinant calculus

Christian Krattenthaler

Universität Wien

Advanced determinant calculus, Sém. Lothar. Combin. 42 (1999), Art. B42q, 67 pp.

Advanced determinant calculus: a complement, Linear Alg. Appl. 411 (2005), 64-166.

Methods for the evaluation of determinants

"Method 0": row/column operations

"Method 1": take out as many factors as possible until something polynomial remains; match with one of the lemmas in ADC I

Method 2: LU-factorisation

Method 3: condensation

Method 4: identification of factors

Gert Almkvist:

I have a sequence of determinants.
I must show that they are non-zero.
Here are the first few:

k	$\det(A(k))$
1	$2^{91}3^{8}5^{7}7^{2}$
2	$-2^{523}3^{52}5^{17}7^{14}11^{4}13^{3}$
3	$2^{1367}3^{177}5^{41}7^{25}11^{20}13^{19}17^{5}19^{4}23^{2}$
4	$-2^{3231}3^{167}5^{83}7^{53}11^{28}13^{27}17^{25}19^{8}23^{6}29^{3}31^{2}$
5	$2^{5399}3^{290}5^{345}7^{93}11^{41}13^{37}17^{33}19^{32}23^{10}29^{7}31^{6}37^{3}$

Can you help?

An expansion due to Bill Gosper

$$\pi = \sum_{n=0}^{\infty} \frac{50n-6}{\binom{3n}{n} 2^n} .$$

An expansion due to Fabrice Bellard

$$\pi = \frac{1}{740025}\left(\sum_{n=1}^{\infty} \frac{3P(n)}{\binom{7n}{2n} 2^{n-1}} - 20379280\right),$$

where

$$\begin{aligned} P(n) = &-885673181n^5 + 3125347237n^4 \\ &-2942969225n^3 + 1031962795n^2 \\ &-196882274n + 10996648 . \end{aligned}$$

Gert Almkvist and Joakim Petersson

Are there more expansions of the type

$$\pi = \sum_{n=0}^{\infty} \frac{S(n)}{\binom{mn}{pn} a^n},$$

where $S(n)$ is a polynomial in n?

Choose some m, p, a,
go to the computer, compute

$$p(k) = \sum_{n=0}^{\infty} \frac{n^k}{\binom{mn}{pn} a^n}$$

to many, many digits for $k = 0, 1, 2, \ldots$,
put

$$\pi, p(0), p(1), p(2), \ldots$$

into the LLL-algorithm, and see if you get an integral linear combination of $\pi, p(0), p(1), p(2), \ldots$.

m	p	a	$\deg(S)$	
3	1	2	1	Gosper
7	2	2	5	Bellard
8	4	-4	4	
10	4	4	8	
12	4	-4	8	
16	8	16	8	
24	12	-64	12	
32	16	256	16	
40	20	-4^5	20	
48	24	4^6	24	
56	28	-4^7	28	
64	32	4^8	32	
72	36	-4^9	36	
80	40	4^{10}	40	

$$\pi = \frac{1}{r} \sum_{n=0}^{\infty} \frac{S(n)}{\binom{16n}{8n} 16^n},$$

$$r = 3^6 5^3 7^2 11^2 13^2$$

$$\begin{aligned} S(n) = {} & -869897157255 - 3524219363487888n \\ & + 112466777263118189n^2 - 1242789726208374386n^3 \\ & + 6693196178751930680n^4 - 19768094496651298112n^5 \\ & + 32808347163463348736n^6 - 28892659596072587264n^7 \\ & + 10530503748472012800n^8. \end{aligned}$$

$$\pi = \frac{1}{r} \sum_{n=0}^{\infty} \frac{S(n)}{\binom{32n}{16n} 256^n},$$

$$r = 2^3 3^{10} 5^6 7^3 11 \cdot 13^2 17^2 19^2 23^2 29^2 31^2$$

$$
\begin{aligned}
S(n) = &- 206211188475634747908570928087 5\\
&+ 1505491740302839023753569717261882091900n\\
&- 1124011494040876582138393867162119752919 75n^2\\
&+ 32578816551942682891818555772622584067411 0002n^3\\
&- 516773095108906305006078985994630362679612 80n^4\\
&+ 517337977987354819322786909541179043148522 720n^5\\
&- 352639649432956071875808639284125815239024 5120n^6\\
&+ 171145766235995166227501216110074805943799 363584n^7\\
&- 607394166132282199408865396581459044020680 29440n^8\\
&+ 159935882563435860391195903248596461569183 580160n^9\\
&- 313951952615028230229958218839819183812205 608960n^{10}\\
&+ 457341091673257198565533286493831205566468 325376n^{11}\\
&- 486846784774707448105420279985074159657397 780480n^{12}\\
&+ 367314505118245777241612044490633887668208 926720n^{13}\\
&- 185647326591648164598342857319777582801297 080320n^{14}\\
&+ 562246880357070156879991289943246904184673 40288n^{15}\\
&- 768725577881655778607397779514936040861204 4800n^{16}.
\end{aligned}
$$

This suggests that there is a formul

$$\pi = \sum_{n=0}^{\infty} \frac{S(n)}{\binom{8kn}{4kn}(-4)^{kn}}$$

for any $k = 1, 2, \ldots$

$= \pi.$ $\square$

How does one prove such identities?

Let us consider Gosper's formula:

$$\pi = \sum_{n=0}^{\infty} \frac{50n-6}{\binom{3n}{n} 2^n} .$$

The beta integral evaluation gives

$$\frac{1}{\binom{3n}{n}} = (3n+1) \int_0^1 x^{2n} (1-x)^n \, dx .$$

Hence:

$$\sum_{n=0}^{\infty} \frac{50n-6}{\binom{3n}{n} 2^n} = \int_0^1 \sum_{n=0}^{\infty} (50n-6)(3n+1) \left(\frac{x^2(1-x)}{2}\right)^n dx$$

We have

$$\sum_{n=0}^{\infty} (50n-6)(3n+1)\, y^n = \frac{2(56y^2+97y-3)}{(1-y)^3} .$$

Thus, if substituted, we obtain

$$= 8 \int_0^1 \frac{28x^6 - 56x^5 + 28x^4 - 97x^3 + 97x^2 - 6}{(x^3-x^2+2)^3} \, dx$$

$$= \left[\frac{4x(x-1)(x^3-28x^2+9x+8)}{(x^3-x^2+2)^2} + 4\arctan(x-1) \right]_0^1$$

$$= \pi .$$

$\square$

The case $\displaystyle\sum_{n=0}^{\infty} \frac{S(n)}{\binom{8kn}{4kn}(-4)^{kn}}$, $k=1,2,\dots$

The beta integral evaluation gives

$$\frac{1}{\binom{8kn}{4kn}} = (8kn+1)\int_0^1 x^{4kn}(1-x)^{4kn}\,dx.$$

Hence, if $S(n)$ has degree d,

$$\sum_{n=0}^{\infty} \frac{S(n)}{\binom{8kn}{4kn}(-4)^{kn}} = \int_0^1 \sum_{n=0}^{\infty} S(n)(8kn+1)\left(\frac{x^{4k}(1-x)^{4k}}{(-4)^k}\right)^n dx$$

$$= \int_0^1 \frac{P(x)}{\left(x^{4k}(1-x)^{4k}-(-4)^k\right)^{d+2}}\,dx.$$

Let $Q(x) := x^{4k}(1-x)^{4k}-(-4)^k$. Perhaps:

$$\int \frac{P(x)}{Q(x)^{d+2}}\,dx = \frac{R(x)}{Q(x)^{d+1}} + 2\arctan(x) + 2\arctan(x-1),$$

for some polynomial $R(x)$ with $R(0)=R(1)=0$.

Then the original sum would indeed equal π.

The last equality is equivalent to

$$\frac{P}{Q^{d+2}} = \frac{R'}{Q^{d+1}} - (d+1)\frac{Q'R}{Q^{d+2}} + 2\left(\frac{1}{x^2+1} + \frac{1}{x^2-2x+2}\right)$$

or

$$QR' - (d+1)Q'R = P - 2Q^{d+2}\left(\frac{1}{x^2+1} + \frac{1}{x^2-2x+2}\right).$$

In our examples, we observed that

$$R(x) = (2x-1)\check{R}(x(1-x))$$

for a polynomial $\check{R}$.

So, let us make the substitution

$$t = x(1-x).$$

Then, after some simplification, the above differential equation becomes

$$-(1-4t)Q\frac{d\check{R}}{dt} + (2Q + 4k(4k+1)(1-4t)t^{4k-1})\check{R}$$
$$-P + 2(3-2t)\frac{Q^{4k+2}}{t^2-2t+2} = 0,$$

where $Q(t) = t^{4k} - (-4)^k$.

Now, writing $N(k) = 4k(4k+1)$, we make the Ansatz

$$\check{R}(t) = \sum_{j=1}^{N(k)-1} a(j)\, t^j$$

$$S(n) = \sum_{j=0}^{4k} a(N(k)+j)\, n^j$$

(recall: $S(n)$ defines $P(t)$).

Comparing coefficients of powers of t on both sides of the last equation, we get a system of $N(k)+4k$ linear equations for the unknowns $a(1), a(2), \dots, a(N(k)+4k)$.

Hence: If the determinant of this system is nonzero, then there does indeed exist a representation

$$\pi = \sum_{n=0}^{\infty} \frac{S(n)}{\binom{8kn}{4kn}(-4)^{kn}}.$$

Now make an Ansatz

$$P(x) = \sum_{j=1}^{L} a(j)\, x^j, \quad R(x) = \sum_{j=L+1}^{M} a(j)\, x^j$$

(with appropriate L and M),
differentiate with respect to x,
and compare coefficients of powers of x.
If this is done appropriately, then we get a system of $16k^2$ linear equations in $16k^2$ unknowns.

The determinant


0 . . . . . 0* 0 . . . . 0* 0 . . . . 0* . . . . 0 . . . . 0*
4k-1
4k-1
4k-1
4k-1
4k-1
4k-1
F_1 0 0 0
G_1 F_2 0
0 G_2 F_3
0 0 G_3
G_{4k-1} F_{4k}
0 G_{4k}
4k 4k 4k 4k

$16k^2 \times 16k^2$

where the ℓ-th non-zero entry in the first row is

$$(-1)^{\ell-1}(-4)^{(\ell+1)k}\, 8k(4k+1)\prod_{i=1}^{4k-\ell}(4ik-1)\prod_{i=1}^{\ell-1}(4ik+1),$$

where

$$F_\ell = \begin{pmatrix} f_1(4(\ell-1)k+1) & f_0(4(\ell-1)k+2) & & & 0 \\ & f_1(4(\ell-1)k+2) & f_0(4(\ell-1)k+3) & & \\ & & \ddots & \ddots & \\ 0 & & & f_1(4\ell k-1) & f_0(4\ell k) \end{pmatrix}$$

and

$$G_\ell = \begin{pmatrix} g_1(4(\ell-1)k+1) & g_0(4(\ell-1)k+2) & & & 0 \\ & g_1(4(\ell-1)k+2) & g_0(4(\ell-1)k+3) & & \\ & & \ddots & \ddots & \\ 0 & & & g_1(4\ell k-1) & g_0(4\ell k) \end{pmatrix}$$

with

$$f_0(j) = j(-4)^k$$

$$f_1(j) = -(4j+2)(-4)^k$$

$$g_0(j) = (N(k) - j \quad)$$

$$g_1(j) = -(4N(k) - (4j+2) \quad)$$

$$N(k) = 4k(4k+1)$$

```
In[1]:= A[k_, i_, j_] := Module[{Var},
    Var = {Floor[(i - 2) / (4 k - 1)],
      Floor[(j - 1) / (4 k)],
      Mod[i - 2, 4 k - 1],
      Mod[j - 1, 4 k]};
    If[i == 1, If[Mod[j, 4 k] === 0,
      a[k, j], 0],
     If[Var[[1]] - Var[[2]] == 0,
      Switch[Var[[3]] - Var[[4]],
       0, f1[k, i, j], -1,
       f0[k, i, j], _, 0],
      If[Var[[1]] - Var[[2]] == 1,
       Switch[Var[[3]] - Var[[4]],
        0, g1[k, i, j], -1,
        g0[k, i, j], _, 0], 0]]]]

In[2]:= A[k_] := Table[A[k, i, j],
    {i, 1, 16 k^2}, {j, 1, 16 k^2}]

In[3]:= f0[k_, i_, j_] := j (-4)^k
        f1[k_, i_, j_] :=
         -(2 + 4 j) (-4)^k
        g0[k_, i_, j_] := (4 k (4 k + 1) - j)
        g1[k_, i_, j_] :=
         (-4 * 4 k (4 k + 1) + 2 + 4 j)
```

```
In[7]:= Det[A[2]]

Out[7]= -601576375580370166777074138695:
        851819603114251897156894671 2:
        220413667478103830277423172 5:
        971306459064075121023092662 2:
        79814015195545600000000000
```

```
In[8]:= FactorInteger[%]

Out[8]= {{-1, 1}, {2, 325},
         {3, 39}, {5, 11},
         {7, 11}, {11, 3}, {13, 2}}
```

In fact, we can prove:

$$\det(A(k)) = (-1)^k\, 2^{32k^3+24k^2+2k-1}\, k^{8k^2+2k}$$
$$\times \left((4k+1)!\right)^{4k} \frac{(8k)!}{(4k)!} \prod_{j=1}^{4k} \frac{(2j)!}{j!} .$$

Thus:

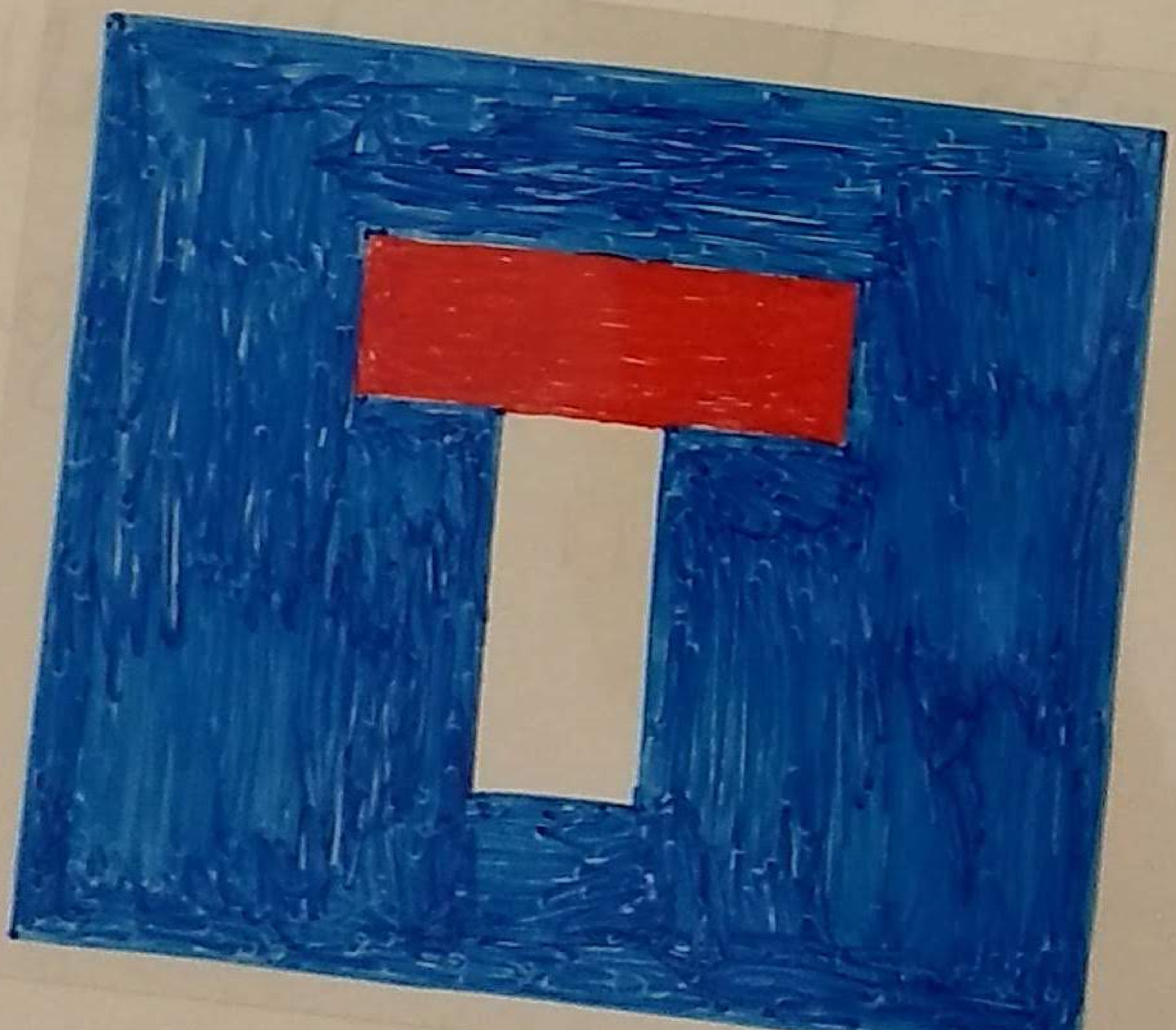
Theorem. For all $k \geq 1$ there is a formula

$$\pi = \sum_{n=0}^{\infty} \frac{S_k(n)}{\binom{8kn}{4kn} (-4)^{kn}} ,$$

where $S_k(n)$ is a polynomial in n of degree $4k$ with rational coefficients. The polynomial $S_k(n)$ can be found by solving the previously described system of linear equations.

How to evaluate this determinant?

"**Method 0**": Do row and column operations until the determinant reduces to something managable.



"Method 1": Take out as many factors as possible until something polynomial remains. Then match with lemmas in Section 2 of ADC I.

Example.

$$\det_{1\le i,j\le n}\left(\binom{s+i-1}{t+j-1}\right)$$

$$=\det_{1\le i,j\le n}\left(\frac{(s+i-1)!}{(t+j-1)!\,(s-t+i-j)!}\right)$$

$$=\prod_{i=1}^{n}\frac{(s+i-1)!}{(t+i-1)!\,(s-t+i-1)!}$$

$$\times\det_{1\le i,j\le n}\Big(\underbrace{(s-t+i-j+1)(s-t+i-j+2)\cdots(s-t+i-1)}_{p_j(i)}\Big)$$

Proposition 1 in ADC I.

Let $X_1, X_2, \dots, X_n$ be indeterminates. If $p_1, p_2, \dots, p_n$ are polynomials of the form $p_j(x) = a_j x^{j-1} + \text{lower terms}$, then

$$\det_{1\le i,j\le n}\left(p_j(X_i)\right) = a_1 a_2 \cdots a_n \prod_{1\le i<j\le n}(X_j - X_i).$$

the guess by proving the corresponding identities.

Example.

$$\det_{1\le i,j\le n}\left(\frac{x_j^{\,i}\,(b/x_j;q)_i}{(ax_j;q)_i}\right)$$

$$=\prod_{j=1}^{n}\frac{(1-b/x_j)\cdot x_j}{(ax_j;q)_n}\det_{1\le i,j\le n}\Big(\Big(1-\frac{qb}{x_j}\Big)\Big(1-\frac{q^2b}{x_j}\Big)\cdots\Big(1-\frac{q^{i-1}b}{x_j}\Big)$$

$$\cdot\, x_j^{\,i-1}(1-aq^ix_j)(1-aq^{i+1}x_j)\cdots(1-aq^nx_j)\Big)$$

$$=\prod_{j=1}^{n}\frac{(x_j-b)}{(ax_j;q)_n}\det_{1\le i,j\le n}\Big((x_j-qb)(x_j-q^2b)\cdots(x_j-q^{i-1}b)$$

$$\cdot(1-aq^ix_j)(1-aq^{i+1}x_j)\cdots(1-aq^nx_j)\Big).$$

Lemma 3 in ADC I.

Let $X_1, X_2, \dots, X_n, A_2, \dots, A_n, B_2, \dots, B_n$ be indeterminates. Then

$$\det_{1 \le i,j \le n} \Big((X_j + A_n)(X_j + A_{n-1}) \cdots (X_j + A_{i+1}) \cdot (X_j + B_i)(X_j + B_{i-1}) \cdots (X_j + B_2) \Big)$$

$$= \prod_{1 \le i < j \le n} (X_i - X_j) \prod_{2 \le i \le j \le n} (B_i - A_j).$$

the guess by proving the corresponding identities.

Method 2: LU-factorization

Suppose we are given a family of matrices $A(1), A(2), A(3), \ldots$, of which we want to compute the determinants.

Suppose further that we can write

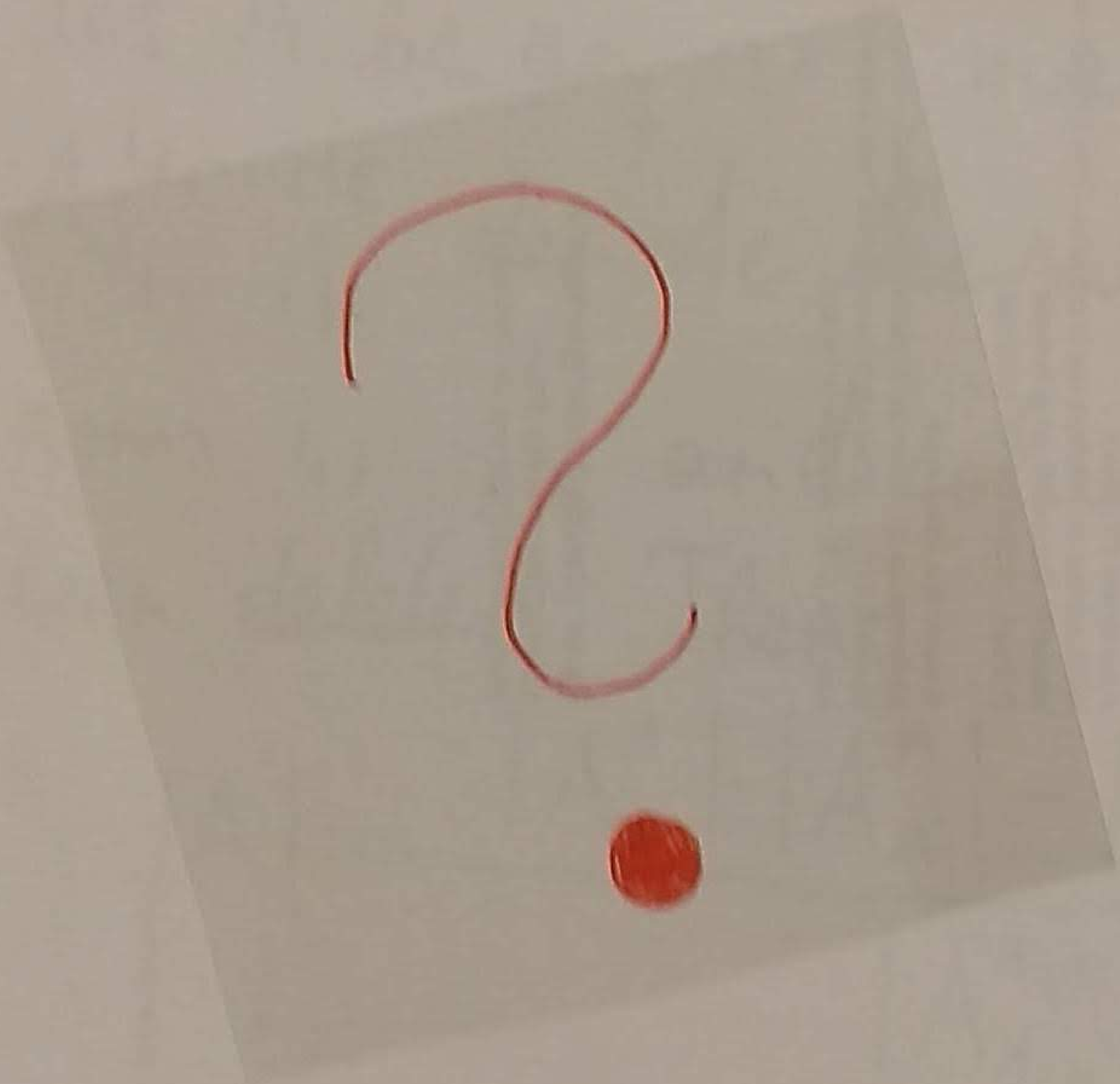
$$A(k) \cdot U(k) = L(k),$$

where $U(k)$ is an upper triangular matrix with 1s on the diagonal, and where $L(k)$ is a lower triangular matrix.

Then $\det(A(k))$ = product of diagonal entries of $L(k)$.

But how do we find $U(k)$ and $L(k)$?

We go to the computer, crank out $U(k)$ and $L(k)$ for $k = 1, 2, 3, \ldots$, until we are able to make a guess. Afterwards we prove the guess by proving the corresponding identities.



Method 3: Condensation

This is based on a determinant identity due to Jacobi.
Let A be an $n \times n$ matrix. Let $A^{i_1, \dots, i_l}_{j_1, \dots, j_l}$ denote the submatrix of A where rows $i_1, \dots, i_l$ and columns $j_1, \dots, j_l$ have been deleted. Then:

$$|A| = \frac{|A^1_1| \cdot |A^n_n| - |A^1_n| \cdot |A^n_1|}{|A^{1,n}_{1,n}|}.$$

If we consider a family of matrices $A(1), A(2), \dots$, and if all the consecutive minors of $A(n)$ belong to the same family, then this allows to give an inductive proof of a conjectured determinant evaluation for $A(n)$.

This works for

$$M_n(b,c) = \left(\binom{b+c}{b-i+j} \right)_{1 \le i,j \le n},$$

because

$$M_n(b,c)_1^1 = M_{n-1}(b,c),$$

$$M_n(b,c)_n^n = M_{n-1}(b,c),$$

$$M_n(b,c)_n^1 = M_{n-1}(b-1,c+1),$$

$$M_n(b,c)_1^n = M_{n-1}(b+1,c-1),$$

$$M_n(b,c)_{1,n}^{1,n} = M_{n-2}(b,c).$$

Method 4: Identification of factors

A short proof of the Vandermonde determinant evaluation

$$\det_{1\le i,j\le n}\left(X(i)^{j-1}\right) = \prod_{1\le i<j\le n}\left(X(j)-X(i)\right)$$

Proof. If $X(i_1)=X(i_2)$ for $i_1\neq i_2$ then the determinant vanishes. Hence,

$$\prod_{1\le i<j\le n}\left(X(j)-X(i)\right) \text{ divides } \det\left(X(i)^{j-1}\right)$$

as a polynomial in $X(1), X(2), \dots, X(n)$.
The degree of the product is $\binom{n}{2}$.
The degree of the determinant is $\le \binom{n}{2}$.
Hence,

$$\det\left(X(i)^{j-1}\right) = \text{const.} \times \prod_{1\le i<j\le n}\left(X(j)-X(i)\right).$$

One can compute the constant by comparing coefficients of $X(1)^0 X(2)^1 X(3)^2 \cdots X(n)^{n-1}$ on both sides.

$\square$

1) identification of factors
2) comparison of degrees
3) evaluation of the constant

with

$$f_0(j) = j(-4)^k$$

$$f_1(j) = -(4j+2)(-4)^k$$

$$g_0(j) = (X \quad -j \quad)$$

$$g_1(j) = -(4X \quad -(4j+2) \quad)$$

```
In[9]:=  f0[k_, i_, j_] := j (-4) ^k
         f1[k_, i_, j_] :=
          -(2 + 4 j) (-4) ^k
         g0[k_, i_, j_] := (X - j)
         g1[k_, i_, j_] := (-4 * X + 2 + 4 j)

In[13]:= Factor[Det[A[2]]]

Out[13]= -14063996084748823231549105259⋱
           86578515918369681041517636 1⋱
           1783762359972003840000000
          (-64 + X) (-48 + X) (-40 + X)^2
          (-32 + X)^3 (-24 + X)^4 (-16 + X)^5
          (-8 + X)^6 X^7 (965307869429760 0 -
            9160006576373 76 X +
            36130368757760 X^2 -
            758218948608 X^3 + 8928558848
             X^4 - 55938432 X^5 + 145673 X^6)
```

with

$$f_0(j)=\left(N(k)+j-X\right)(-4)^k$$

$$f_1(j)=-\left(4N(k)+4j+2-4X\right)(-4)^k$$

$$g_0(j)=\left(X-j\right)$$

$$g_1(j)=-\left(4X-(4j+2)\right)$$

```
In[14]:= f0[k_, i_, j_] :=
           (4 k (4 k + 1) - X + j) (-4)^k
         f1[k_, i_, j_] :=
          -(4*4 k (4 k + 1) - 4 X + 2 + 4 j)
            (-4)^k
         g0[k_, i_, j_] := (X - j)
         g1[k_, i_, j_] :=
          (-4*X + 2 + 4 j)

In[18]:= Factor[Det[A[2]]]

Out[18]= -2967779753976246799013698097\
           9441210445413476349407084111\
           5536519612475477031747227179\
           0417634937439881166252558632\
           61667419750400000000000
          (-141 + 2 X)
          (-139 + 2 X) (-137 + 2 X)
          (-135 + 2 X) (-133 + 2 X)
          (-131 + 2 X) (-129 + 2 X)
```

with

$f_0(j) = (N(k) + j \quad - X_2 \quad)(-4)^k$

$f_1(j) = -(\,4N(k) + 4j + 2 \quad - 4X_1 \quad)(-4)^k$

$g_0(j) = (X_2 \quad - j \quad)$

$g_1(j) = -(4X_1 \quad -(4j+2) \quad)$

In[19]:=
```
f0[k_, i_, j_] :=
  (4 k (4 k + 1) - X[2] + j) (-4) ^k
f1[k_, i_, j_] :=
  - (4 * 4 k (4 k + 1) - 4 X[1] + 2 + 4 j)
   (-4) ^k
g0[k_, i_, j_] := (X[2] - j)
g1[k_, i_, j_] :=
  (-4 * X[1] + 2 + 4 j)
```

In[23]:=
```
Factor[Det[A[1]]]
```

Out[23]=
$$324259173170675712000 0 (-37 + 2 X[1]) (-35 + 2 X[1]) (-33 + 2 X[1]) (1 + 2 X[1] - 2 X[2])^3 (3 + 2 X[1] - 2 X[2])^2 (5 + 2 X[1] - 2 X[2])$$

with

$$f_0(j) = \left((N(k)+j)Y - X_2\right)(-4)^k$$

$$f_1(j) = -\left((4N(k)+4j+2)Y - 4X_1\right)(-4)^k$$

$$g_0(j) = \left(X_2 - jY\right)$$

$$g_1(j) = -\left(4X_1 - (4j+2)Y\right)$$

```
In[24]:= f0[k_, i_, j_] :=
          (4 k (4 k + 1) * y - X[2] + j * y)
            (-4) ^k
         f1[k_, i_, j_] :=
          - (4 * 4 k (4 k + 1) * y - 4 X[1] +
               (2 + 4 j) * y) (-4) ^k
         g0[k_, i_, j_] := (X[2] - j * y)
         g1[k_, i_, j_] :=
          (-4 * X[1] + (2 + 4 j) * y)

In[28]:= Factor[Det[A[1]]]

Out[28]= -3242591731706757120000
          y^6 (33 y - 2 X[1])
          (35 y - 2 X[1]) (37 y - 2 X[1])
          (y + 2 X[1] - 2 X[2])^3
          (3 y + 2 X[1] - 2 X[2])^2
          (5 y + 2 X[1] - 2 X[2])
```

with

$$f_0(j) = \left((N(k)+j)Y_\ell - X_{2,\ell}\right)(-4)^k$$

$$f_1(j) = -\left((4N(k)+4j+2)Y_\ell - 4X_{1,\ell}\right)(-4)^k$$

$$g_0(j) = \left(X_{2,\ell} - jY_\ell\right)$$

$$g_1(j) = -\left(4X_{1,\ell} - (4j+2)Y_\ell\right)$$

```
In[34]:= f0[k_, i_, j_] :=
          (4 k (4 k + 1) * Y[i] -
             X[2, i] + j * Y[i]) (-4) ^k
         f1[k_, i_, j_] :=
          - (4 * 4 k (4 k + 1) * Y[i] -
              4 X[1, i] +
              (2 + 4 j) * Y[i]) (-4) ^k
         g0[k_, i_, j_] :=
          (X[2, i] - j * Y[i])
         g1[k_, i_, j_] :=
          (-4 * X[1, i] + (2 + 4 j) * Y[i])

In[38]:= Factor[Det[A[1]]]

Out[38]= 32425917317067571200000
          (2 X[1, 1] - 33 Y[1]) Y[1]
          (2 X[1, 1] - 2 X[2, 1] + Y[1])
          (2 X[1, 2] - 35 Y[2]) Y[2]
          (2 X[1, 2] - 2 X[2, 2] + Y[2])
          (-2 X[2, 2] Y[1] +
            2 X[1, 1] Y[2] + 3 Y[1] Y[2])
         (2 X[1, 3] - 37 Y[3]) Y[3]
         (2 X[1, 3] - 2 X[2, 3] + Y[3])
         (-2 X[2, 3] Y[1] + 2 X[1, 1] Y[3] +
           5 Y[1] Y[3]) (-2 X[2, 3] Y[2] +
           2 X[1, 2] Y[3] + 3 Y[2] Y[3])
```

Apparently:

$$\det(A(k)) = (-1)^{k-1}\, 4^{2k(4k^2+7k+2)}\, k^{2k(4k+1)}$$

$$\times \prod_{i=1}^{4k} (i+1)_{4k-i+1} \prod_{\alpha=1}^{4k-1} \left(2X_{1,\alpha} - (32k^2+2\alpha-1)Y_\alpha\right)$$

$$\times \prod_{1 \le a \le b \le 4k-1} \left(2X_{2,b}Y_a - 2X_{1,a}Y_b - (2b-2a+1)Y_aY_b\right).$$

The special case that we need to prove our theorem is $X_{1,\ell} = X_{2,\ell} = N(k)$ and $Y_\ell = 1$.

Sketch of proof.

1) For each factor of the (conjectured) result, we find a linear combination of the rows which vanishes if the factor vanishes.

For example: The factor $\left(2X_{1,1}-(32k^2+2-1)Y_1\right)$

We have: If $X_{1,1}=\frac{32k^2+2a-1}{2}Y_1$, then

$$\frac{2\left(X_{2,4k-1}-(N(k)-1)Y_{4k-1}\right)}{(-4)^{k(4k+1)+1}(16k^2+1)\prod_{\ell=1}^{4k-1}(4\ell k+1)}\cdot(\text{row }1)$$

$$+\sum_{s=0}^{4k}\sum_{t=0}^{4k-a-1}\Bigg(\frac{(-1)^{s(k-1)}}{4^{sk}}\prod_{\ell=0}^{s-1}\frac{4k-1+4\ell k}{16k^2+1-4\ell k}$$

$$\cdot 2^t\prod_{\ell=4k-t}^{4k-1}\frac{2X_{1,\ell}-(32k^2+2\ell-1)Y_\ell}{X_{2,\ell-1}-(16k^2+\ell-1)Y_{\ell-1}}$$

$$\cdot\left(\text{row }(16k^2-(4k-1)s-t)\right)\Bigg)=0.$$

2) The total degree in the $X_{1,e}$'s, $X_{2,e}$'s, Y_e's of the product is $16k^2-1$.
The degree of the determinant is $\le 16k^2-1$.
Hence,

$$\det = \text{const} \times \text{product}.$$

3) Evaluation of the constant.
Compare coefficients of

$$X_{1,1}^{4k} X_{1,2}^{4k-1} \cdots X_{1,4k-1}^{2} Y_1^1 Y_2^2 \cdots Y_{4k-1}^{4k-1}.$$

As it turns out, the constant is equal to a determinant of the same form,

but with

$$f_0(j) = (N(k) + j)(-4)^k$$

$$f_1(j) = 4(-4)^k$$

$$g_0(j) = -j$$

$$g_1(j) = -4$$

but with

$$f_0(j) = (Z_\ell + j)(-4)^k$$

$$f_1(j) = 4(-4)^k X_\ell$$

$$g_0(j) = -j$$

$$g_1(j) = -4 X_\ell$$

but with

$$f_0(j) = (Z_\ell + j)(-4)^k$$

$$f_1(j) = 4(-4)^k X_\ell$$

$$g_0(j) = -j$$

$$g_1(j) = -4X_\ell$$

The computer says that apparently

$$\det A^{\mathrm{const}}(k) = (-1)^{k-1}\, 2^{16k^2+20k^2+14k-1}$$

$$\times\, k^{4k}\,(4k+1)! \prod_{a=1}^{4k-1}\left(X_a^{4k+1-a}\prod_{b=0}^{a-1}(Z_a - 4bk)\right).$$

Okay. Let us apply identification of factors again.

1) ✓ 2) ✓ 3) The constant:

is again a determinant of the same form with

$$f_0(j) = (-4)^k$$

$$f_1(j) = 4(-4)^k$$

$$g_0(j) = 0$$

$$g_1(j) = -4$$

is again a determinant of the same form with

$$f_0(j) = (-4)^k$$

$$f_1(j) = 4(-4)^k$$

$$g_0(j) = 0$$

$$g_1(j) = -4$$

For this one "Method 0" (row and column manipulations) works!

Hence:

Theorem. We have

$$\det\left(A(k)^{\text{general}}\right)$$

$$= (-1)^{k-1} 4^{2k(4k^2+7k+2)} k^{2k(4k+1)}$$

$$\times \prod_{i=1}^{4k} (i+1)_{4k-i+1} \prod_{a=1}^{4k-1} \left(2X_{1,a} - (32k^2+2a-1)Y_a\right)$$

$$\times \prod_{1\le a\le b\le 4k-1} \left(2X_{2,b}Y_a - 2X_{1,a}Y_b - (2b-2a+1)Y_aY_b\right).$$

Corollary. We have

$$\det\left(A(k)^{\text{special}}\right) = (-1)^k 2^{32k^3+24k^2+2k-1}$$

$$\times k^{8k^2+2k} \left((4k+1)!\right)^{4k} \frac{(8k)!}{(4k)!} \prod_{j=1}^{4k} \frac{(2j)!}{j!}$$

$$\neq 0 \;!$$

Theorem. For all $k \geq 1$ there is a formula

$$\pi = \sum_{n=0}^{\infty} \frac{S_k(n)}{\binom{8kn}{4kn}(-4)^{kn}},$$

where $S_k(n)$ is a polynomial in n of degree $4k$ with rational coefficients.

$\square$