Sums of Squares and Quadratic Persistence

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27 May 2019

MOTIVATION: A homogeneous polynomial $f \in S := \mathbb{R}[x_0, x_1, \ldots, x_n]$ is a sum of squares if there exists a positive-semidefinite matrix A such that

$$f = \begin{bmatrix} x_0^j & x_0^j x_1 & \cdots & x_n^j \end{bmatrix} \mathrm{A} \begin{bmatrix} x_0^j \\ x_0^j x_1 \\ \vdots \\ x_n^j \end{bmatrix}.$$

To reduce the search space, replace A by $\mathrm{B}\,\mathrm{B}^{\mathsf{T}}$ where B is a $\binom{n+j}{j} \times r$-matrix and r is the minimal rank of a matrix representation.

PROBLEM: Need an *a priori* bound on r.

Let $X \subseteq \mathbb{P}^n$ be an irreducible real subvariety whose real points $X(\mathbb{R})$ are Zariski dense and let $R := S/I$ be its $\mathbb{Z}$-graded coordinate ring.

A homogeneous element $f \in R_2$ is a **sum of squares** if $f = h_0^2 + h_1^2 + \cdots + h_{r-1}^2$ for some $h_0, h_1, \dots, h_{r-1} \in R_1$. These elements form a convex cone Σ_2.

For all $X \subseteq \mathbb{P}^n$, $\mathrm{py}(X)$ is the smallest $r \in \mathbb{N}$ such that each $f \in \Sigma_2$ is a sum of r squares.

QUESTION: Can we **effectively** bound $\mathrm{py}(X)$?

Syzygetic Invariants

HILBERT (1890): There exists an exact sequence of $\mathbb{Z}$-graded S-modules

$$0 \longleftarrow R \longleftarrow F_0 \longleftarrow F_1 \longleftarrow \cdots \longleftarrow F_n \longleftarrow 0$$

where $F_i := \bigoplus_{j \in \mathbb{N}} S(-j)^{\beta_{i,j}}$ for all $0 \leqslant i \leqslant n$.

The **Betti table** of X is the matrix whose (i, j)-entry is $\beta_{j,i+j}$. It has the form

$_i\backslash^j$	0	1	2	$\cdots$	$\mathrm{a}(X)$	$\mathrm{a}(X)+1$	$\cdots$	$\ell(X)$	$\ell(X)+1$	$\cdots$
0	1	0	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$
1	0	$*$	$*$	$\cdots$	$*$	$*$	$\cdots$	$*$	0	$\cdots$
2	0	0	0	$\cdots$	0	$*$	$\cdots$	$*$	$*$	$\cdots$

Upper Bounds

$$\mathrm{a}(X) := \max\{j : \mathrm{Tor}_k^S(R, \mathbb{R})_{2+k} = 0 \text{ for all } k \leqslant j\}.$$

Theorem (Blekherman–Sinn–Smith–Velasco, 2019):

For all $X \subseteq \mathbb{P}^n$, we have:

- $\binom{\mathrm{py}(X)+1}{2} < \dim(R_2)$.
- $\mathrm{py}(X) \leqslant n+1-\min\{\mathrm{a}(X), \mathrm{codim}(X)\}$.
- $\mathrm{py}(X)$ is at most one more than the dimension of any variety of minimal degree containing X.

Quadratic Persistence

For all $\Gamma \subset X$ with $s := |\Gamma|$, let $\pi_\Gamma : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-s}$ be the linear projection away from $\mathrm{Span}(\Gamma)$.

DEFINITION: For all $X \subset \mathbb{P}^n$, $\mathrm{qp}(X)$ is the smallest $s \in \mathbb{N}$ such that there exists $\Gamma \subset X$ with $s := |\Gamma|$ and the ideal of $\pi_\Gamma(X) \subseteq \mathbb{P}^{n-s}$ contains no quadratic polynomials.

THEOREM: We have $\ell(X) \leqslant \mathrm{qp}(X) \leqslant \mathrm{codim}(X)$ where $\ell(X) := \max\{j : \mathrm{Tor}_j^S(R, \mathbb{R})_{1+j} \neq 0\}$.

Lower Bound

THEOREM: $\mathrm{py}(X) \geqslant n+1-\mathrm{qp}(X) \geqslant 1+\dim(X)$.

COROLLARIES:

- $\mathrm{qp}(X) = \mathrm{codim}(X) \Leftrightarrow \mathrm{py}(X) = 1+\dim(X)$
  $\Leftrightarrow \deg(X) = 1+\mathrm{codim}(X)$.
- If X is arithmetically Cohen–Macaulay, then
  $\mathrm{qp}(X) = \mathrm{codim}(X)-1 \Leftrightarrow \mathrm{py}(X) = 2+\dim(X)$
  $\Leftrightarrow \deg(X) = 2+\mathrm{codim}(X)$ or X is a divisor
  in a variety of minimal degree.