Conic programming : infeasibility certificates and projective geometry

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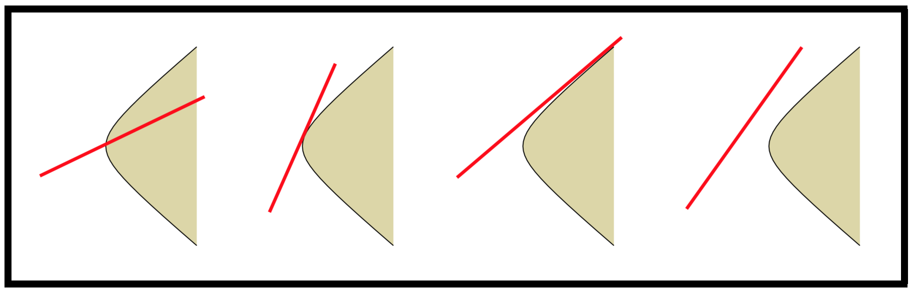

Geometry of Real Polynomials, Convexity and Optimization

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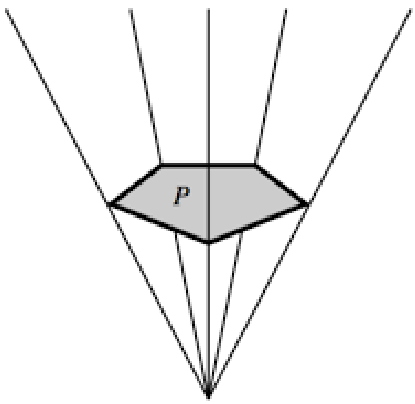
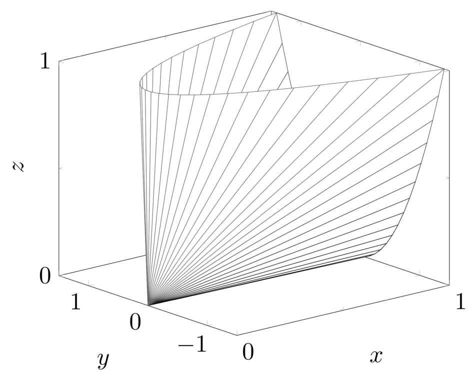
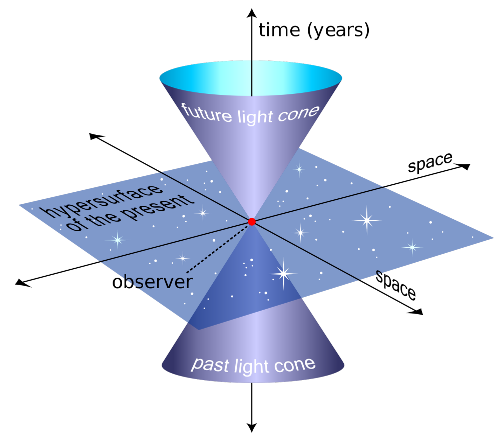
Feasible set in a conic program

$$\mathbf{K} \cap L$$

Intersection of a convex cone $\mathbf{K} \subset V$ such as







with an affine space $L = \{x \in V : \mathscr{A}(x) = b\}$, with

$$\mathscr{A} : V \to W \quad \text{a linear map}$$

between (finite-dimensional) real vector spaces V, W.

Standard duality in CP

Let $(V, V^\vee)$ and $(W, W^\vee)$ be two dual pairs with duality pairings (non-degenerate bilinear maps)

$$\langle \cdot\,, \cdot \rangle_V : V^\vee \times V \to \mathbb{R} \quad \text{and} \quad \langle \cdot\,, \cdot \rangle_W : W^\vee \times W \to \mathbb{R}.$$

Standard primal-dual pair of conic programs

$$\begin{array}{llll} p^* & := & \inf & \langle c, x \rangle_V \\ & & \text{s.t.} & \mathscr{A}(x) = b \\ & & & x \in \mathbf{K} \end{array} \qquad \begin{array}{llll} d^* & := & \sup & \langle b, y \rangle_W \\ & & \text{s.t.} & c - \mathscr{A}^*(y) = s \\ & & & s \in \mathbf{K}^* \end{array}$$

Motivations for studying feasibility in a CP :

- Applications : if a program is infeasible, there is no candidate solution, hence the constraints are too strong

- Necessary/sufficient conditions for having good properties (*e.g.* strong duality) are related to feasibility

Feasibility types

Recall that $L = \{x \in V : \mathscr{A}(x) = b\}$ and suppose that $\mathbf{K} \subset V$ is a closed convex cone with $Int(\mathbf{K}) \neq \emptyset$.

We say the (primal) conic program is

feasible if $\mathbf{K} \cap L \neq \emptyset$ and in particular

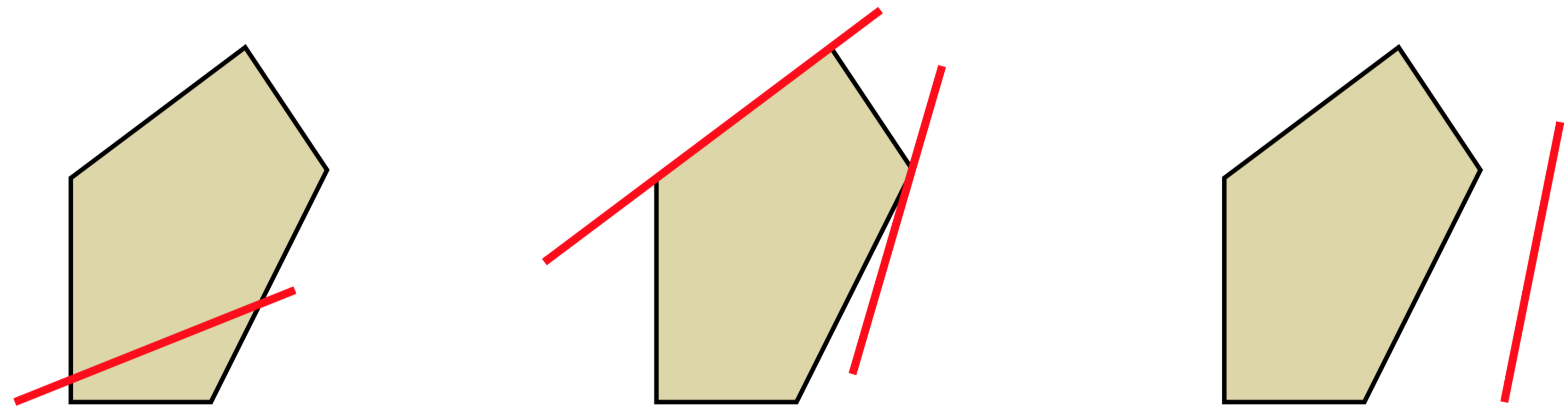
- *strongly feasible* if $Int(\mathbf{K}) \cap L \neq \emptyset$
- *weakly feasible* if feasible but $Int(\mathbf{K}) \cap L = \emptyset$

infeasible if $\mathbf{K} \cap L = \emptyset$ and in particular

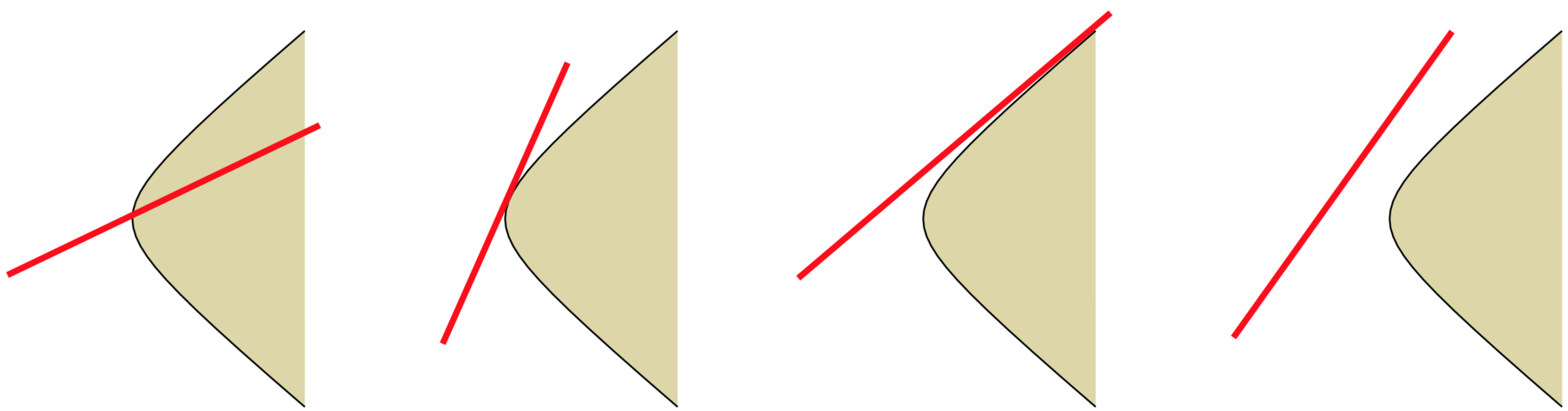
- *strongly infeasible* if $d(\mathbf{K}, L) > 0$
- *weakly infeasible* if infeasible but $d(\mathbf{K}, L) = 0$

General question : can we detect the feasibility type of a CP ?

From linear to non-linear CP



3 types for Linear Programming



4 types for Conic Programming

Example from semidefinite relaxations

Weak infeasibility is quite common and arises for example in the context of SD relaxations for polynomial optimization. Let

$$f^* \;=\; \inf \;\; f(x) \quad \text{s.t.} \;\; f_1(x) \geq 0, \ldots, f_m(x) \geq 0$$

be the standard polynomial optimization problem, and

$$M_r(f_1, \ldots, f_m) := \left\{ \sigma_0 + \sum_i \sigma_i f_i : \sigma_i \, SOS, \deg \sigma_i \leq r - \left\lceil \frac{\deg f_i}{2} \right\rceil \right\}$$

Theorem (Waki, Optim Lett. 2012). There exists $\overline{r} \in \mathbb{N}$ such that for $r \geq \overline{r}$ and $2r > \deg f$ the following holds :

If $f - \lambda \notin M_r(f_1, \ldots, f_m), \forall \lambda \in \mathbb{R}$, *then the* $r-$*th level of the relaxation is weakly infeasible*.

Homogenization of LP

Consider the feasible set in a standard (primal) LP :

$$\begin{array}{lll} (L) & Ax & = b \\ (\mathbf{K}) & x_i & \geq 0, \;\; i = 1, \ldots, n \end{array}$$

Let x_0 be a new variable, and homogenize it to

$$\begin{array}{lll} (\widehat{L}) & Ax & = bx_0 \\ (\widehat{\mathbf{K}}) & x_i & \geq 0, \;\; i = 0, 1, \ldots, n \end{array}$$

This operation *lifts* the positive orthant $\mathbf{K} = \mathbb{R}^n_{\geq}$ to another positive orthant $\widehat{\mathbf{K}} = \mathbb{R}^{n+1}_{\geq} \subset \mathbb{R}^{n+1}$, and remark that

$$\mathbf{K} \approx \widehat{\mathbf{K}} \cap \{x_0 = 1\} \quad \textit{and} \quad L \approx \widehat{L} \cap \{x_0 = 1\}$$

Can we do the same for the general CP ?

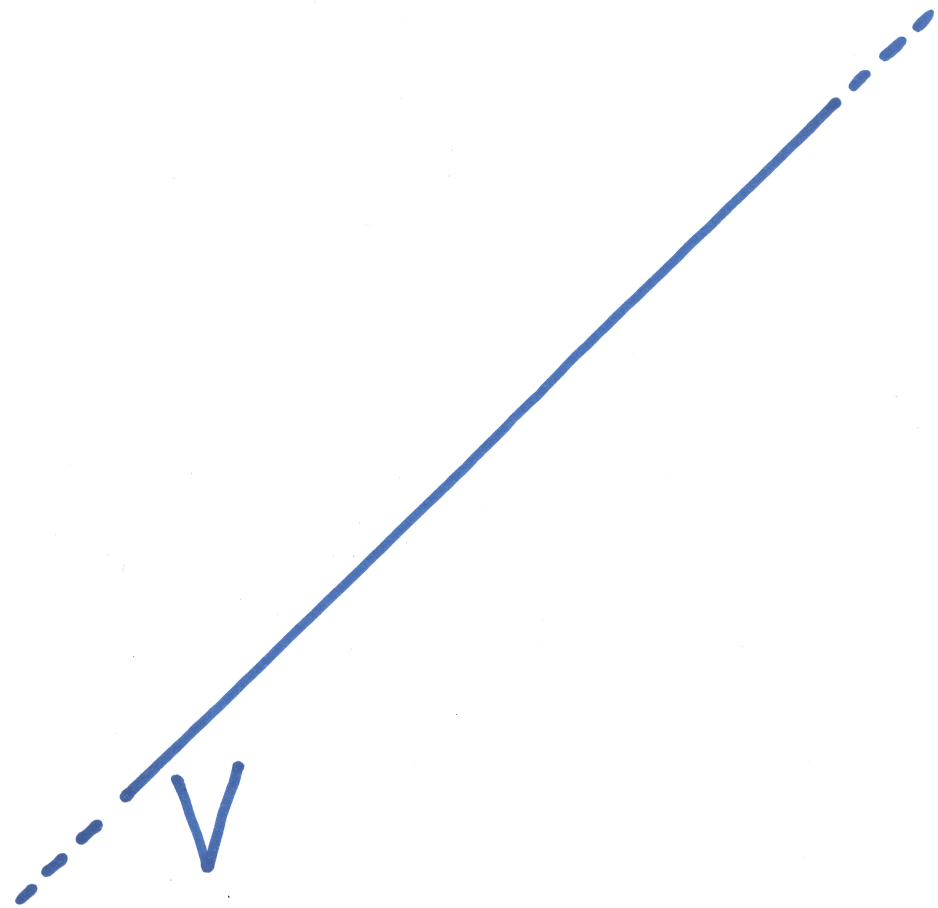
Homogenization of CP : projective viewpoint

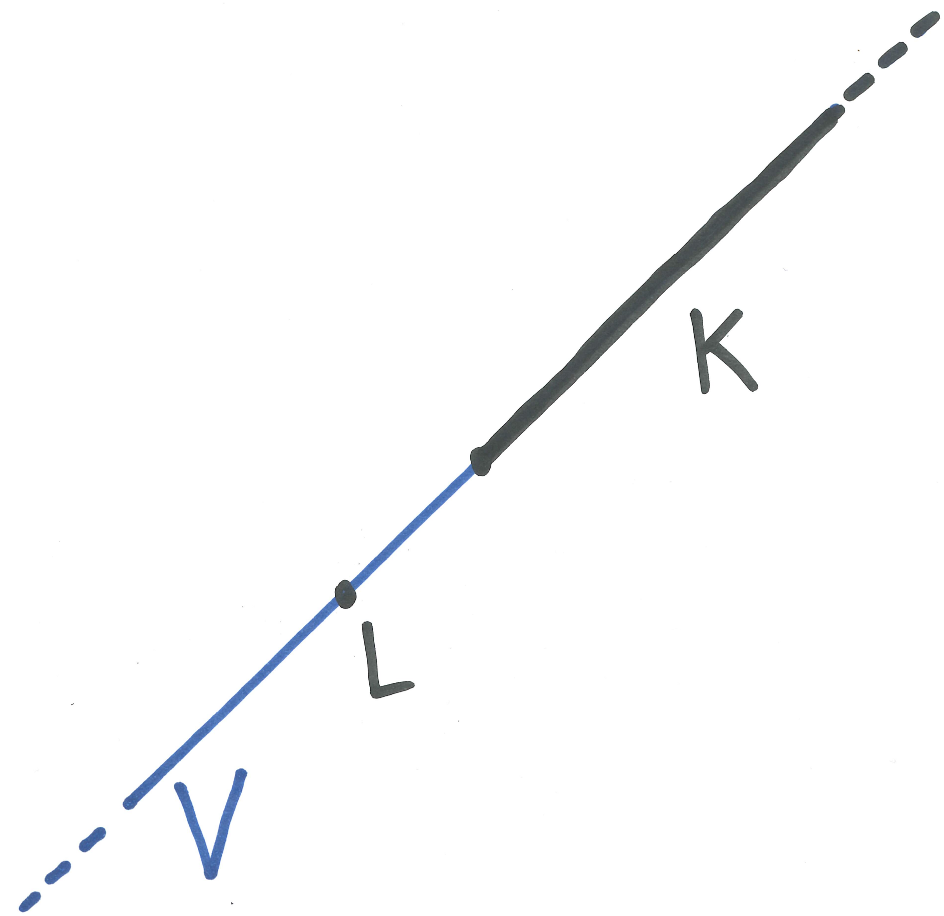
Let U be a finite-dimensional Euclidean space, $\widehat{\mathbf{K}} \subset U$ a regular (closed, pointed, with interior) convex cone.

Let $V \subset U$ be a hyperplane with $0 \notin V$ and set $\mathbf{K} = \widehat{\mathbf{K}} \cap V$. We assume $\mathbf{K}$ is also a cone in V (after appropriate choice of coordinates). Let $L \subset V$ be an affine subspace.

From a projective viewpoint V determines an affine chart in the projective space $\mathbb{P}(U)$ and $\mathbf{K} \subset V$ is the part of the cone $\widehat{\mathbf{K}}$ that we see on this chart. The set $\widehat{\mathbf{K}} \cap lin(V)$ is said to be *at infinity* with respect to V, where $lin(V) = V - v_0$, for some $v_0 \in V$.

Let $\widehat{L}$ be the linear hull of L in U. The idea is to compare the feasibility types of $\mathbf{K} \cap L$ and $\widehat{\mathbf{K}} \cap \widehat{L}$.


V


K
L
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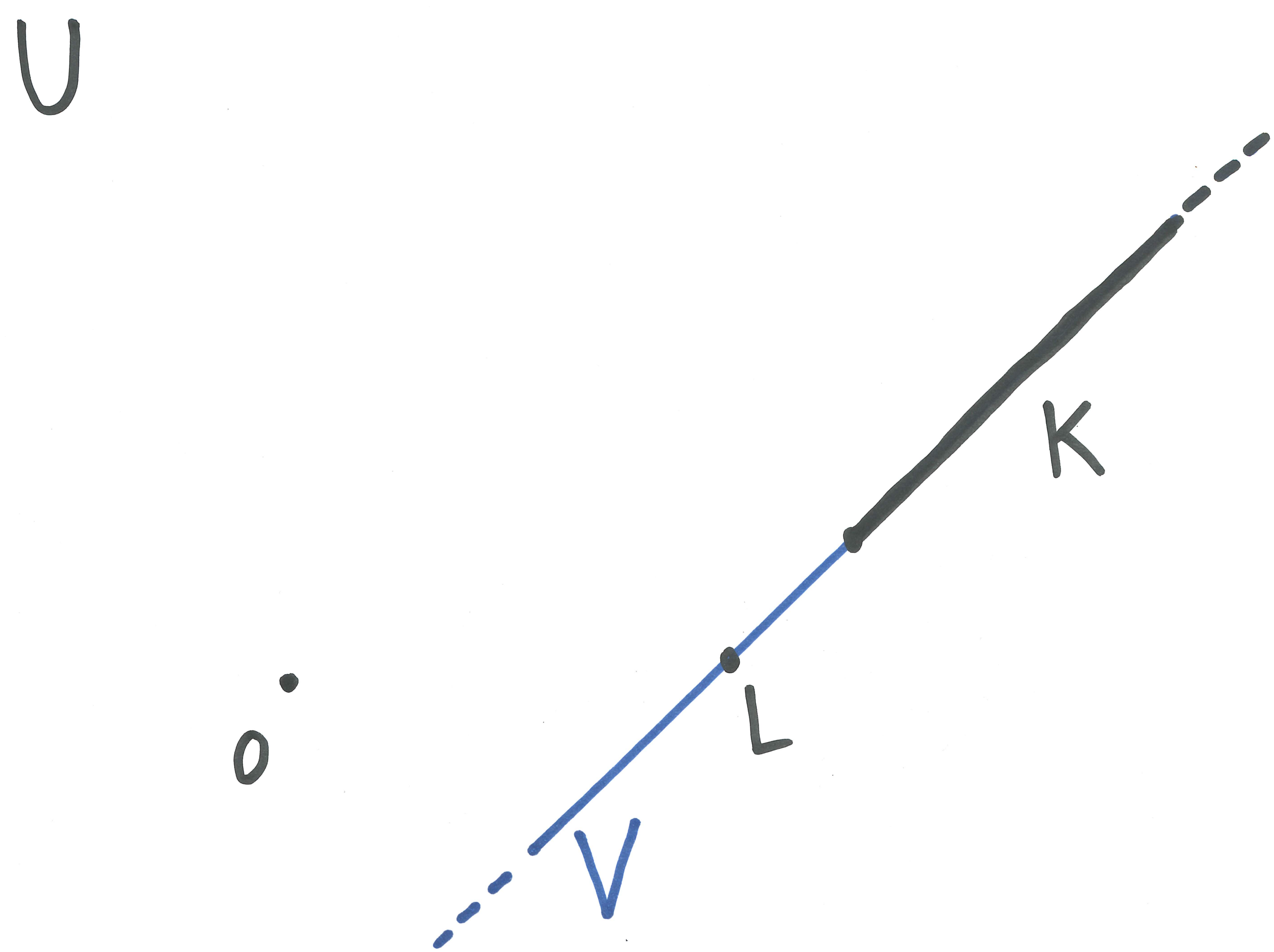
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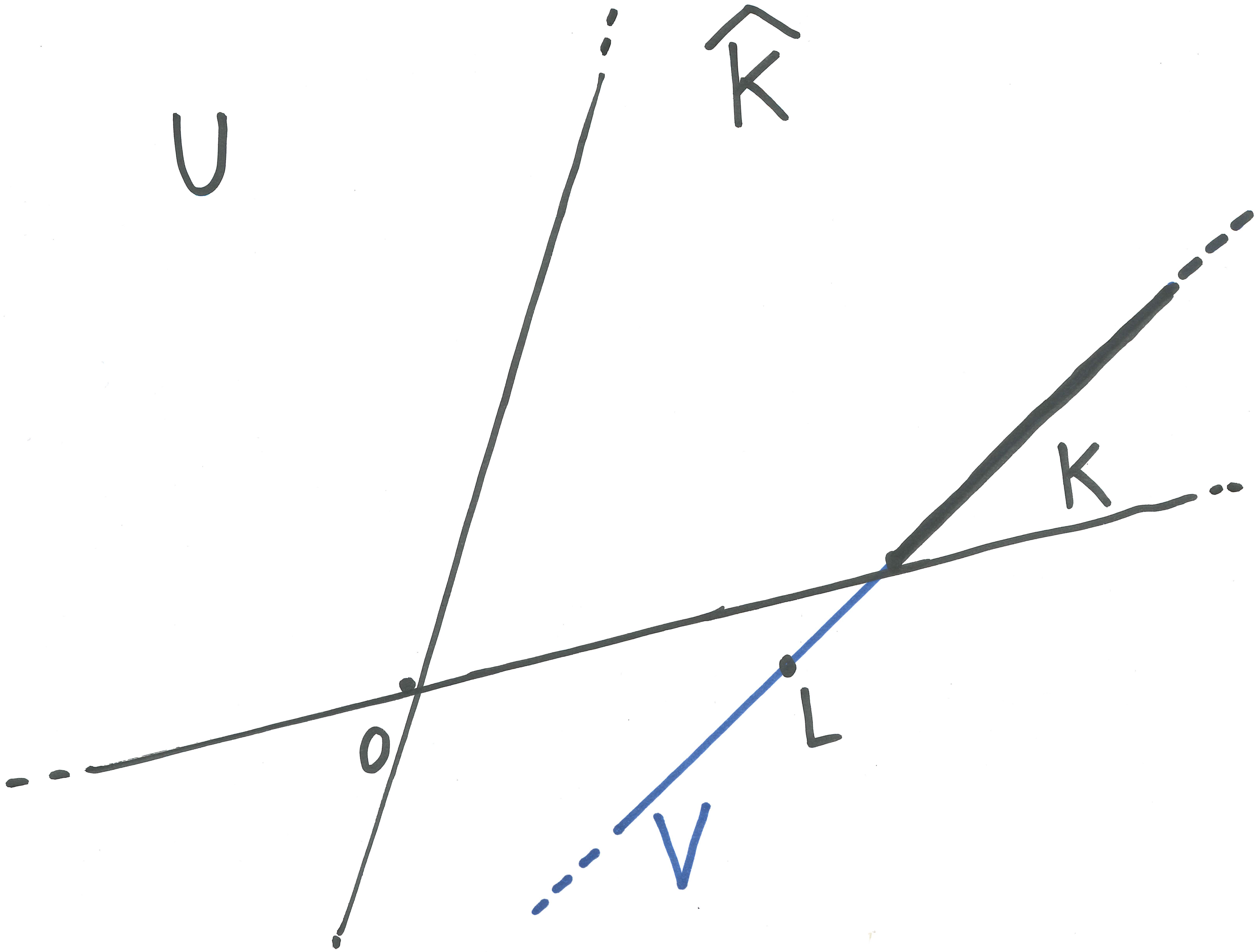
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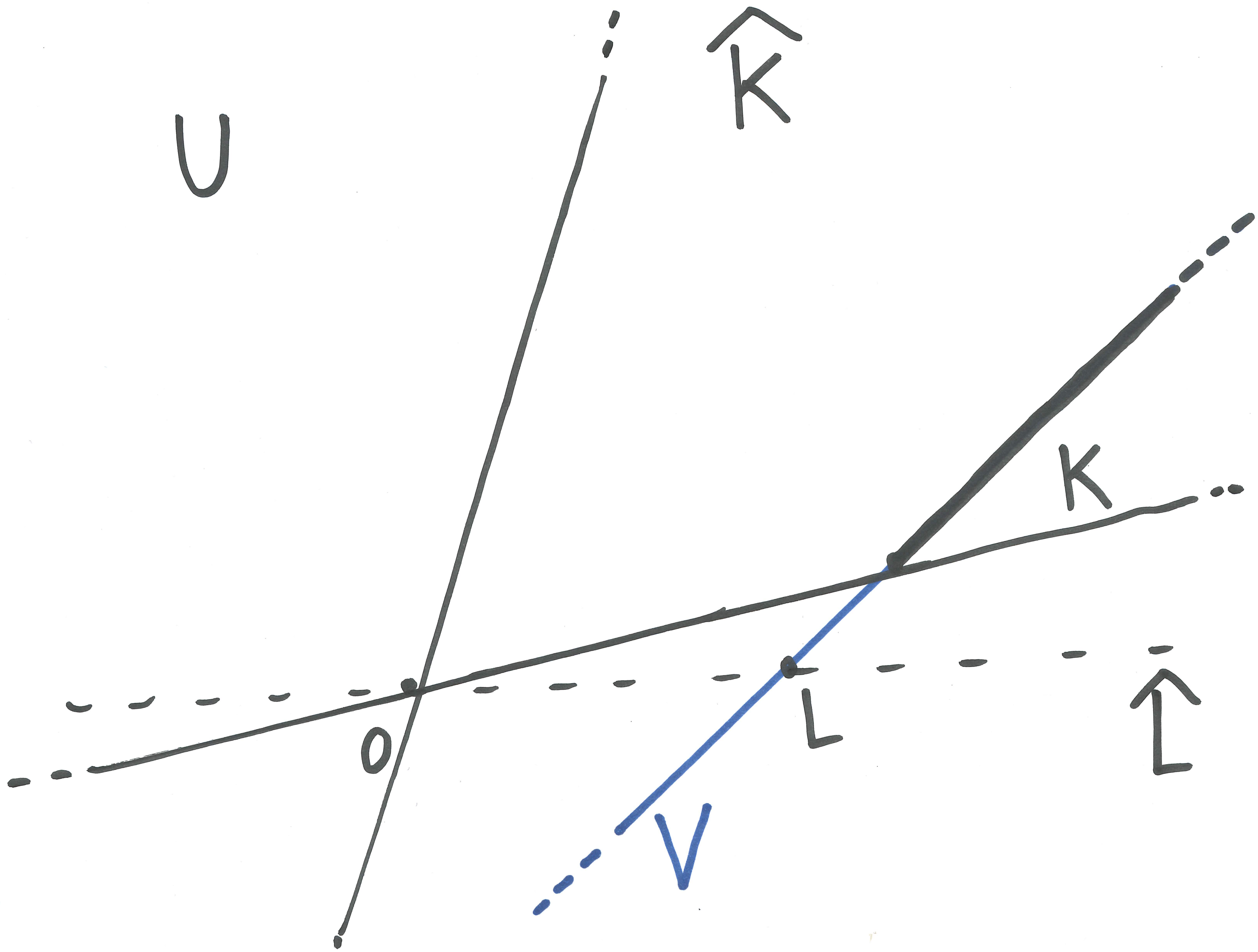

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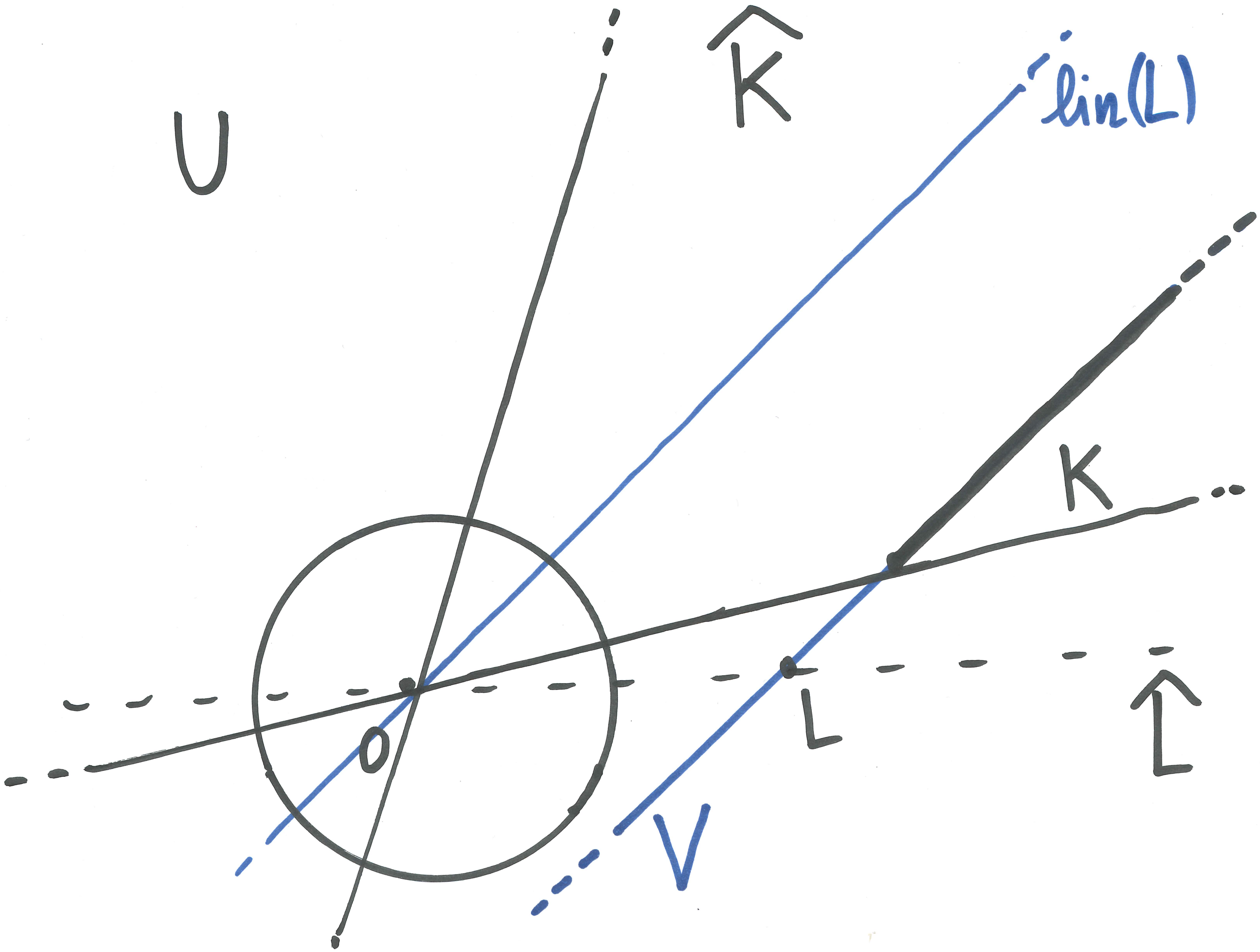

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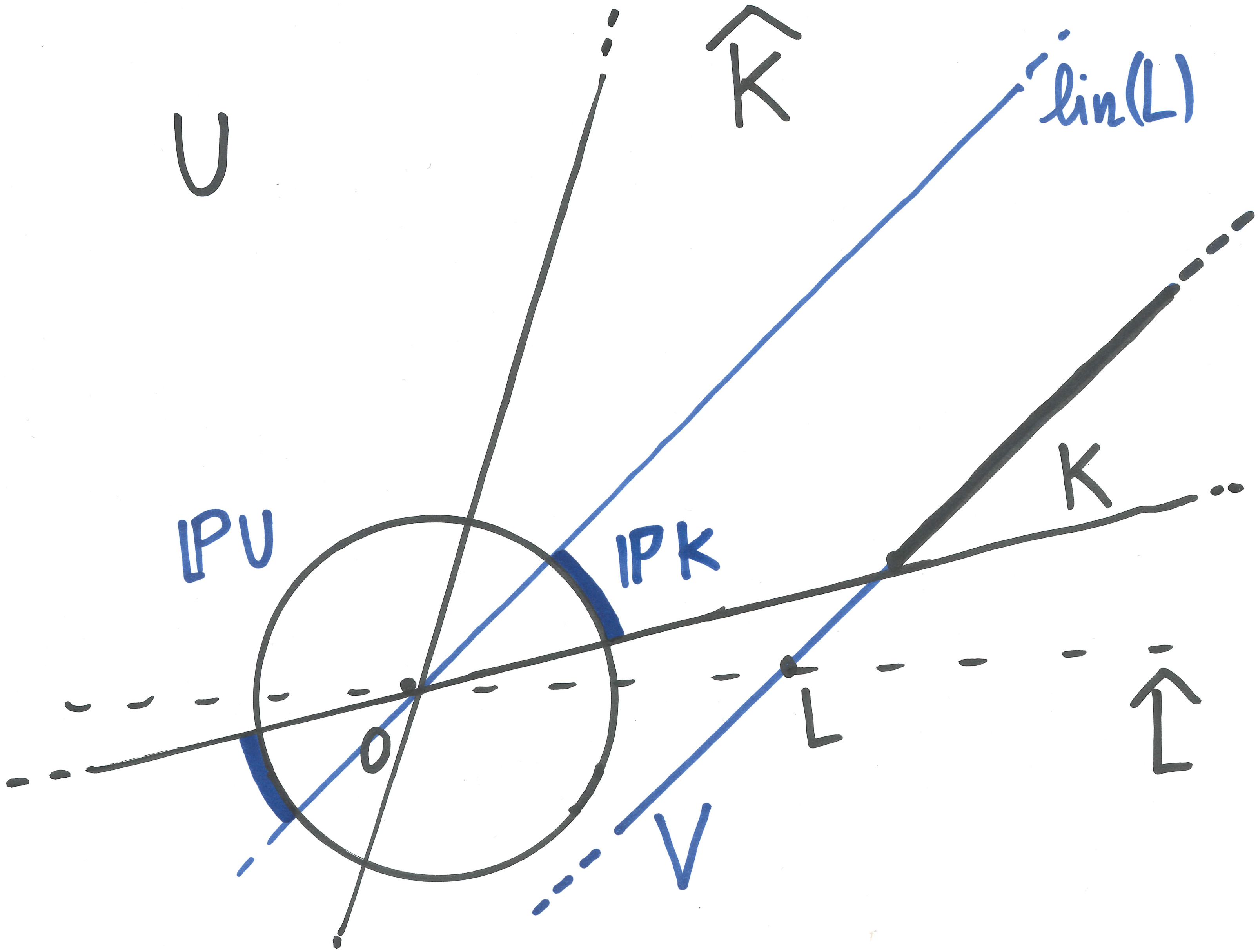

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PK
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L
$\hat{L}$
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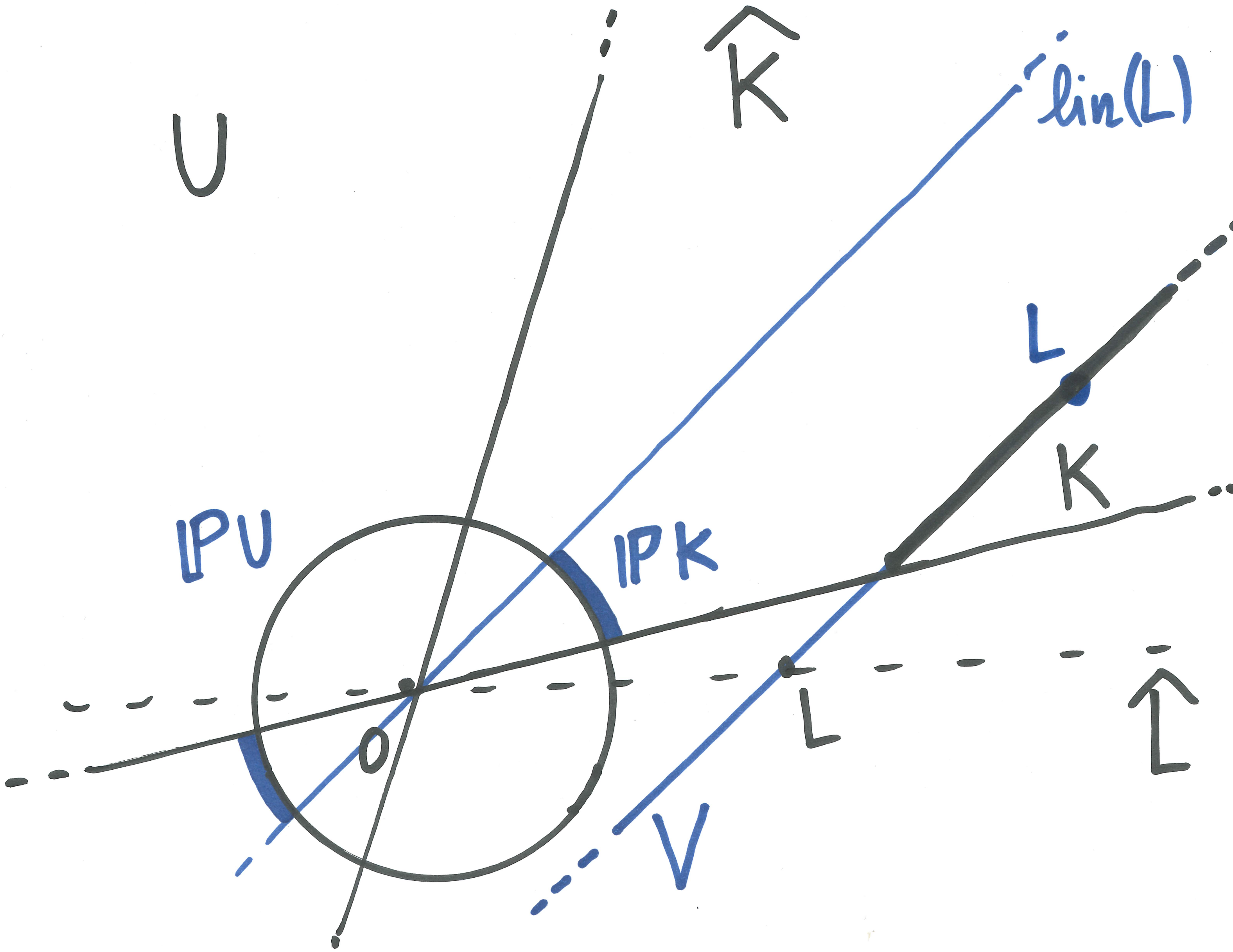

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U
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IPK
0
L
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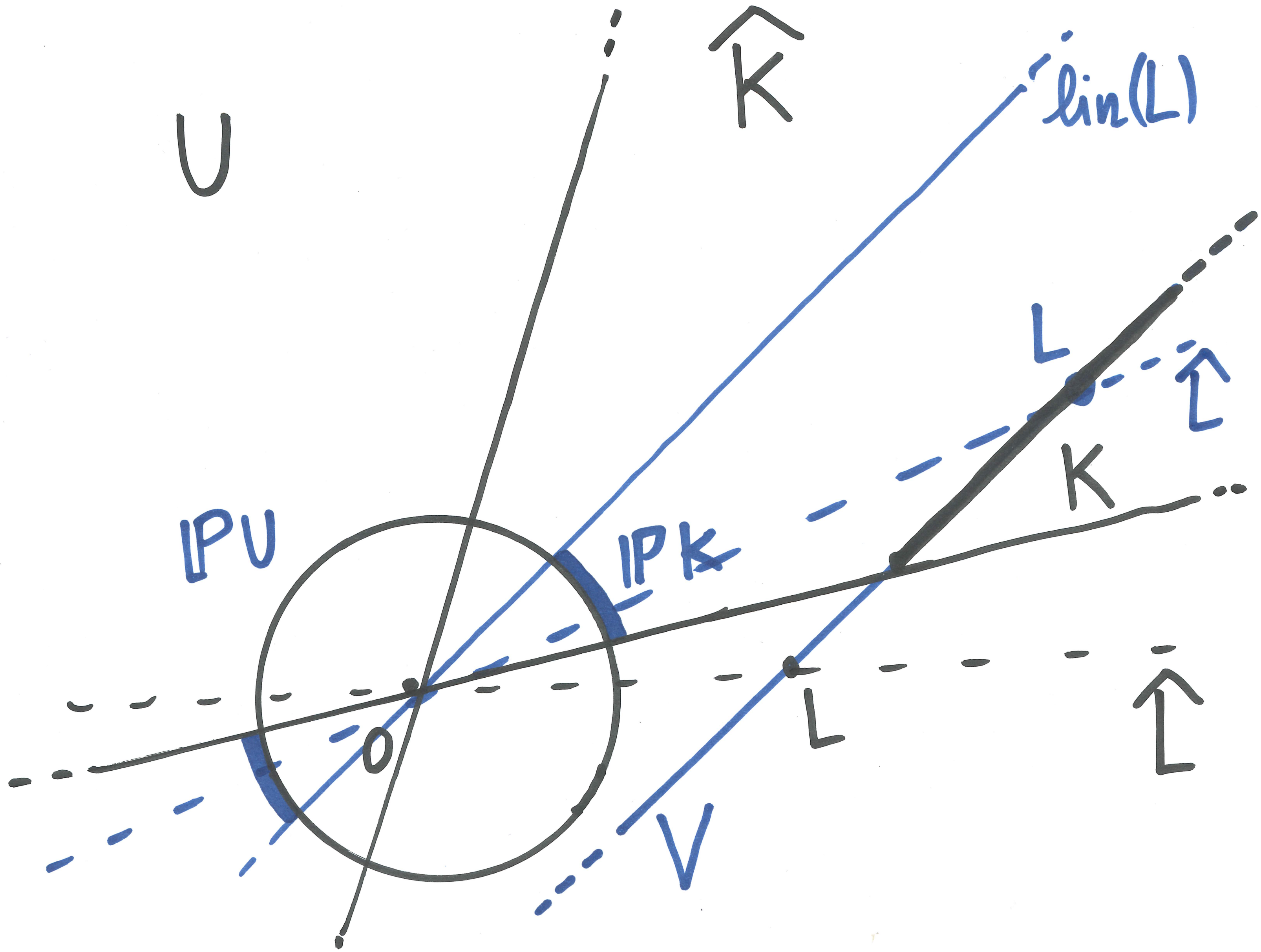

$\widehat{K}$
lin(L)
U
L
$\hat{L}$
K
$\mathbb{P}U$
$\mathbb{P}K$
0
L
$\hat{L}$
V

Comparing feasibility types

These are the implications that hold for the general CP :

Theorem.

- $\mathbf{K} \cap L$ infeasible $\Leftrightarrow \widehat{\mathbf{K}} \cap \hat{L} \subset lin(V)$
- $\mathbf{K} \cap L$ strongly feasible $\Leftrightarrow \widehat{\mathbf{K}} \cap \hat{L}$ strongly feasible
- $\widehat{\mathbf{K}} \cap \hat{L} = \{0\} \Rightarrow \mathbf{K} \cap L$ strongly infeasible

The converse does not hold for the third point, we will need to define a more refined type of strong infeasibility.

A projective facial reduction

Theorem. $\mathbf{K}$ regular, nice* convex cone. Let $L \subset H \subset V$ with H hyperplane, $0 \notin H$. If $\mathbf{K} \cap L = \emptyset$, there exist $\ell_1, \ldots, \ell_k \in \mathbf{K}^*$ with the following properties. Set $F_i = \{x \in \mathbf{K} \colon \ell_i(x) = 0\}$ and $L_i = L_{i-1} \cap span(F_{i-1})$ for $i > 1$ with $L_1 = \widehat{L}$. We have

$$k \leq 1 + \dim(L)$$
$$F_i \supset F_{i+1}$$
$$F_i \supset \mathbf{K} \cap L_i \supset \mathbf{K} \cap \widehat{L}$$
$$F_k \subset lin(V)$$

One deduces $\mathbf{K} \cap L \subset \mathbf{K} \cap \widehat{L} \subset F_k \subset lin(V)$, hence $\mathbf{K} \cap L = \emptyset$.

This yields an alternative proof† that the SDP feasibility problem is in $\text{NP}_{\mathbb{R}} \cap \text{co-NP}_{\mathbb{R}}$ (Blum-Shub-Smale)

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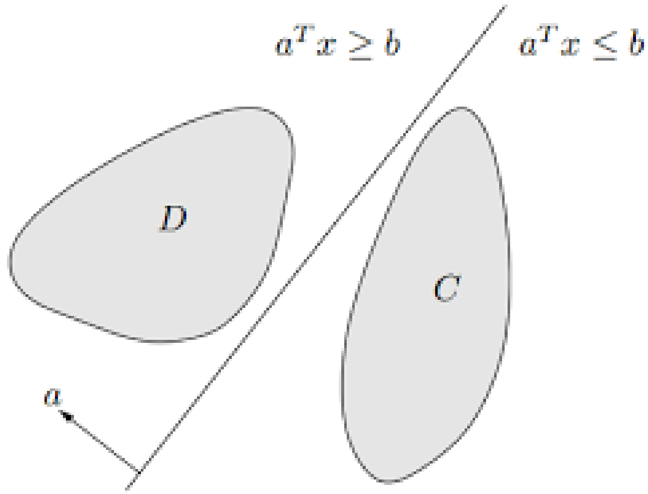
*Pataki : A cone $\mathbf{K}$ is nice if $\mathbf{K}^* + F^{\perp}$ is closed for every face F

†First proof by Ramana's 1997 paper

Infeasibility certificates

Let $\mathbf{K} \subset V$ be regular, and $L \subset V$. An affine function f on V is called an *infeasibility certificate* of $\mathbf{K} \cap L$ whenever $f(x) \geq 0$ on $\mathbf{K}$ and $f(x) < 0$ on L.



Interesting questions :

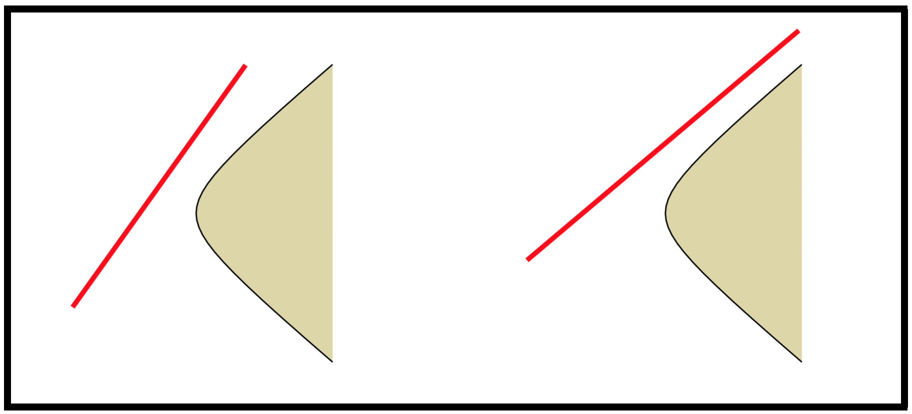
1. Existence of certificates, complexity

2. Rationality

Stable infeasibility

Let $d = \dim L$. We say that $\mathbf{K} \cap L$ is *stably infeasible* if there is an open neighborhood N of L in the Grassmannian of $d-$dimensional spaces in $\mathbb{R}^n$ s.t. $\mathbf{K} \cap L'$ is infeasible for all $L' \in N$.

[*one can perturbe "generically" and stay infeasible*].



stable vs *unstable*

Theorem. $\mathbf{K} \cap L$ is stably infeasible iff one of these is satisfied

1. $\widehat{\mathbf{K}} \cap \widehat{L} = \{0\}$

2. There is $\ell \in Int(\mathbf{K}^*)$ such that $\ell(x) < 0$ for all $x \in L$

Rationality results

Suppose that both $\mathbf{K}$ and L are defined over $\mathbb{Q}$ (*e.g.*, $\mathbf{K}$ is a semialgebraic set defined by inequalities with coefficients in $\mathbb{Q}$) and that $\mathbf{K} \cap L = \emptyset$. Is there a rational certificate ?

Theorem. A stably infeasible program $\mathbf{K} \cap L$ always admits a rational infeasibility certificate.

For LP this condition can be discarded by applying Farkas

Theorem. If $\{x \in \mathbb{R}^n \colon Ax = b\} \cap \mathbb{R}^n_{\geq}$ is infeasible, there exists $y \in \mathbb{Q}^n$ and $\lambda \in \mathbb{Q}$ such that $H = \{x \in \mathbb{R}^n \colon y^T(Ax - b) = \lambda\}$ strongly separates L and $\mathbb{R}^n_{\geq}$.

Irrationality example in SDP

Let $v = \{x^2, y^2, z^2, xy, xz, yz\}$ and let $L' \subset \mathcal{S}^6$ be set of 6×6 symmetric matrices satisfying

$$v^T M v = x^4 + xy^3 + y^4 - 3x^2yz - 4xy^2z + 2x^2z^2 + xz^3 + yz^3 + z^4$$

The set $\mathcal{S}^6_+ \cap L'$ is a 2-dimensional cone with no rational* points.

For $L = (L')^\perp - Id_6$, then $\mathcal{S}^6_+ \cap L$ is strongly infeasible but has no rational certificates, since any such certificate would be a rational point in $\mathcal{S}^6_+ \cap L'$.

*Scheiderer : there are $f \in \mathbb{Q}[x]$ such that $f \in \Sigma(\mathbb{R}[x])^2$ but $f \notin \Sigma(\mathbb{Q}[x])^2$

Preprint on arXiv

Please have a look and give feedback :

"Conic Programming: Infeasibility Certificates and Projective Geometry"

S. Naldi and R. Sinn, arxiv.org/abs/1810.11792