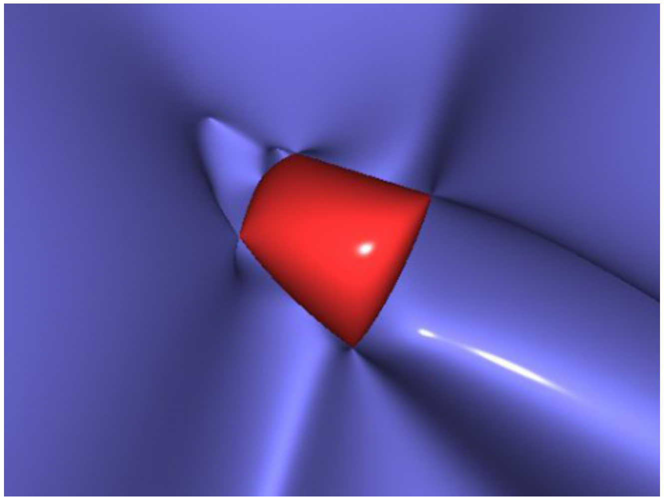
poema

Polynomial Optimization, Efficiency through Moments and Algebra



2 PhD positions with Prof. **Monique Laurent** at CWI, Amsterdam

within an EU Marie Curie-Skłodowska Innovative Training Network

Approximation hierarchies for (non-)commutative polynomial optimization

Approximation hierarchies for graph parameters

poema-network.eu

Solving non-linear PDEs with the Lasserre hierarchy

Didier HENRION

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hal.laas.fr/hal-01830870

Joint work with

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Lasserre

Swann
Marx

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Weisser

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Scalar nonlinear hyperbolic conservation law

$$\frac{\partial y(t,x)}{\partial t} + \frac{\partial f(y(t,x))}{\partial x} = 0 \qquad t \in \mathbf{T},\ x \in \mathbf{X}$$

$$y(0,x) = y_0(x) \qquad x \in \mathbf{X}$$

given f **polynomial**, e.g. $f(y) = y^2$ for Burgers' equation
given y_0, with $y : \mathbf{T} \times \mathbf{X} \to \mathbf{Y} \subset \mathbb{R}$ to be found

Nonlinear partial differential equation (PDE) modelling physical phenomena such as fluid mechanics, traffic flow or acoustics

Even with regular initial data y_0, for t large enough the graph $x \mapsto y(t,x)$ may feature discontinuities = **shocks**

Need for a notion of weak solution...

A **Young measure** or parametrized probability measure is a map

$$\begin{array}{rccc} \mu: & \mathbf{T}\times\mathbf{X} & \to & \mathscr{P}(\mathbf{Y}) \\ & (t,x) & \mapsto & \mu_{t,x} \end{array}$$

It is a measure-valued (mv) solution of the nonlinear PDE

$$\begin{array}{ll} \dfrac{\partial y(t,x)}{\partial t} + \dfrac{\partial f(y(t,x))}{\partial x} = 0 & t \in \mathbf{T},\ x \in \mathbf{X} \\ y(0,x) = y_0(x) & x \in \mathbf{X} \end{array}$$

whenever the linear equation

$$\int_{\mathbf{T}}\int_{\mathbf{X}}\int_{\mathbf{Y}} \left(\frac{\partial \psi(t,x)}{\partial t} y + \frac{\partial \psi(t,x)}{\partial x} f(y)\right) d\mu_{t,x}(y)dx\,dt + \int_{\mathbf{X}}\int_{\mathbf{Y}} \psi(0,x) y\, d\mu_{0,x}(y) dx = 0$$

holds for all test functions $\psi \in \mathscr{C}_c^1(\mathbf{T}\times\mathbf{X})$

Classical solutions $y(t,x)$ are recovered when $\mu_{t,x}(dy) = \delta_{y(t,x)}(dy)$

An mv solution μ is an **entropy mv solution** if

$$\int_{\mathbf{T}}\int_{\mathbf{X}}\int_{\mathbf{Y}}\left(\frac{\partial\psi(t,x)}{\partial t}p(y)+\frac{\partial\psi(t,x)}{\partial x}q(y)\right)d\mu_{t,x}(y)\,dx\,dt$$
$$+\int_{\mathbf{X}}\int_{\mathbf{Y}}\psi(0,x)p(y)d\mu_{0,x}(y)\,dx\geq 0$$

for all $p, q \in \mathscr{C}^1(\mathbf{Y})$ such that p is strictly convex and $q' = f'p'$ and all non-negative test functions $\psi \in \mathscr{C}_c^1(\mathbf{T}\times\mathbf{X})$

The entropy mv solution satisfies **linear** constraints
yet it is **not** a relaxation..

Theorem: Let $C = \text{Lip}(f)$. For all $T \geq 0$ and $r \geq 0$, it holds

$$\int_{|x|\leq r} \int_{\mathbf{Y}} |y - y(T,x)| d\mu_{t,x}(y)\, dx \leq \int_{|x|\leq r+CT} \int_{\mathbf{Y}} |y - y_0(x)| d\mu_{t,x}(y)\, dx$$

In particular, if $\mu_{0,x} = \delta_{y_0(x)}$ then $\mu_{(t,x)} = \delta_{y(t,x)}$ for all $t \in [0,T]$ and all x such that $|x| \leq r$

In words, if the initial condition is concentrated, then the entropy mv solution is concentrated for all time

Solving the linear equation and entropy inequalities is **equivalent** to solving the nonlinear equation

If F_t denotes the flow, i.e. $y(t,x) = F_t(y_0(x))$, then $\mu_{t,x}$ is the image measure of the initial measure $\mu_{0,x}$ through F_t

Young measure $\mu_{t,x}(dy)$ is conditional of **occupation measure**

$$d\nu(t,x,y) = dt\,dx\,\mu(dy \mid t,x)$$

For every set $\mathbf{A}$ in the Borel sigma algebra of $\mathbf{T} \times \mathbf{X} \times \mathbf{Y}$, the value $\nu(\mathbf{A})$ is the time spent in $\mathbf{A}$ by solution $y(t,x)$, averaged wrt initial data $\mu_{0,x}$

www.chaos-math.org/en/chaos-viii-statistics

To summarize, the entropy mv solution μ satisfies the following linear constraints

$$\int \left(\frac{\partial \psi}{\partial t} y + \frac{\partial \psi}{\partial x} f(y)\right) d\mu_{t,x}(y)\, dx\, dt + \int \psi(0,x) y\, d\mu_{0,x}(y) dx = 0$$

and

$$\int \left(\frac{\partial \psi}{\partial t} p(y) + \frac{\partial \psi}{\partial x} q(y)\right) d\mu_{t,x}(y)\, dx\, dt + \int \psi(0,x) p(y)\, d\mu_{0,x}(y) dx \geq 0$$

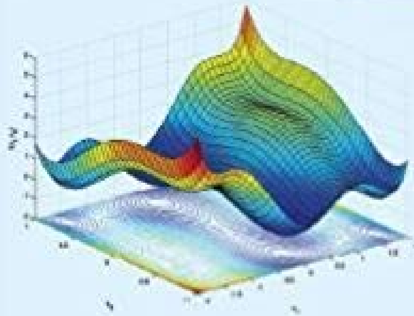
for all smooth test functions ψ and entropy pairs p, q

In practice we solve this problem on (scaled) compact sets $\mathbf{T} = \mathbf{X} = \mathbf{Y} := [0, 1]$ so we can restrict ψ to polynomials

The map f is polynomial, and the entropy pairs p, q are algebraic i.e. they satisfy polynomial inequalities

Therefore we have a **Generalized Moment Problem** (GMP)

Imperial College Press Optimization Series Vol. 1



Moments, Positive Polynomials and Their Applications

Jean Bernard Lasserre

Imperial College Press

Numerical example

For a numerical illustration, we consider the classical Riemann problem for a Burgers equation. In particular, we choose the flux

$$f(y) = \frac{1}{4}y^2$$

Cauchy problem with the following initial condition, piecewise constant with one point of discontinuity:

$$y_0(x) = \begin{cases} l & \text{if } x < 0, \\ r & \text{if } x > 0, \end{cases}$$

where $l, r \in \mathbb{R}$

The solution to the Riemann problem depends strongly on the values of l and r. In particular:

1. If $l > r$, the **shock** at the initial condition spreads along the characteristics

2. If $l < r$, the solution is not necessarily unique. The entropy condition allows to select the right solution, which is known as a **rarefaction wave**

Shock

For $l = 1$ and $r = 0$ the solution is discontinuous, for all $t > 0$

The unique analytical solution is

$$y(t,x) = \begin{cases} 1 & x > \frac{t}{4} \\ \\ 0 & x < \frac{t}{4} \end{cases}$$

Solving the moment problem up to degree 12 with our interface GloptiPoly for Matlab and the semidefinite programming solver MOSEK, we end up with the following moments:

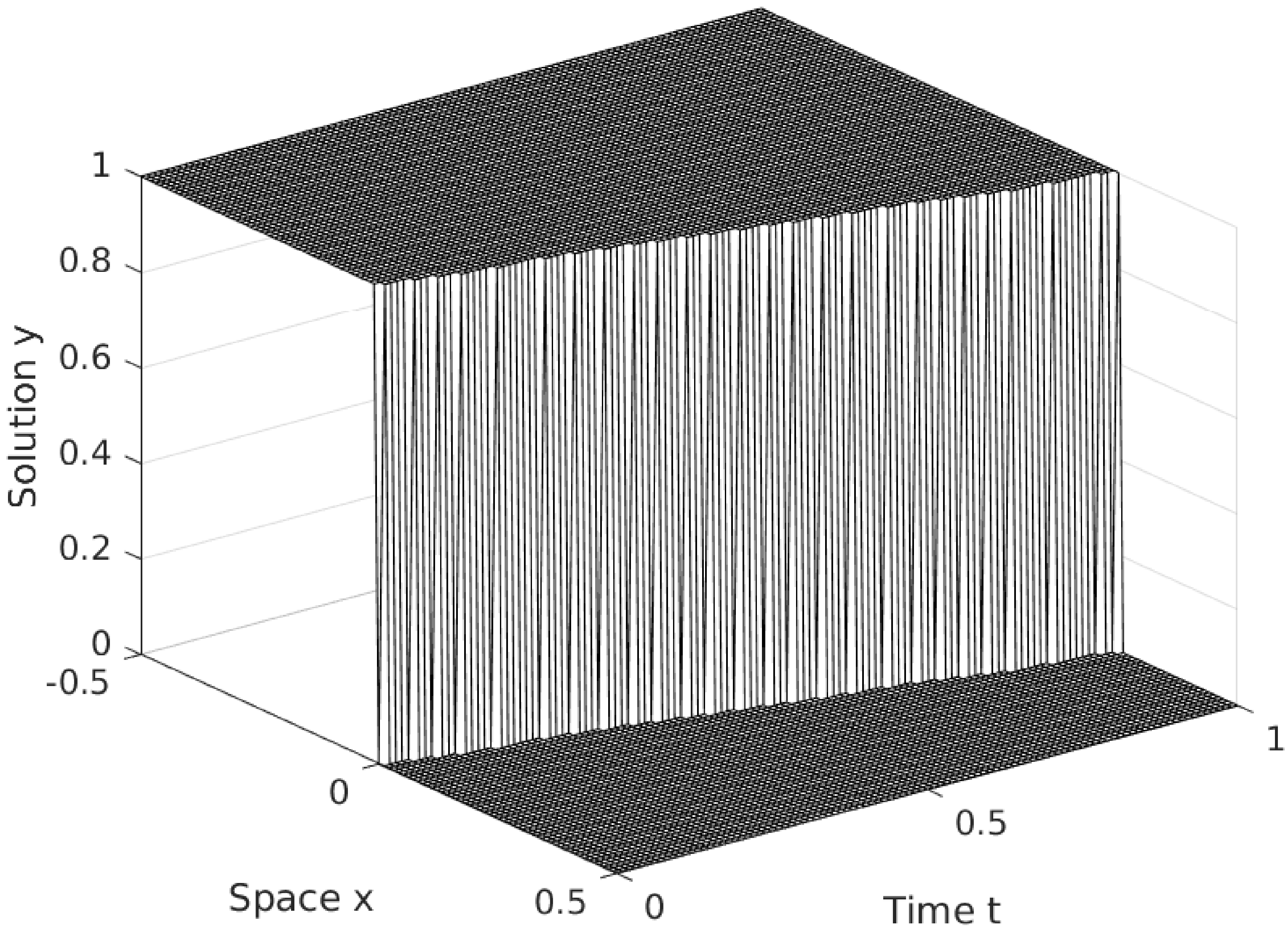
$$\left(\int y^k d\mu_{t,x}(y) dx\, dt\right)_{k=0,1,\ldots} = (1.0000, 0.6250, 0.6250, 0.6250, \ldots)$$

which correspond (up to numerical accuracy) exactly with the moments of the analytic solution $d\mu_{t,x} = \delta_{y(t,x)}$

From moments to graphs

How can we recover the graph of the solution from the (approximate) moments of the Young measure ?

See next talk by Jean Bernard Lasserre !



Solution y
1
0.8
0.6
0.4
0.2
0
-0.5
0
0.5
Space x
0
0.5
1
Time t

Rarefaction wave

For $l = 0$ and $r = 1$ the discontinuity disappears for $t > 0$

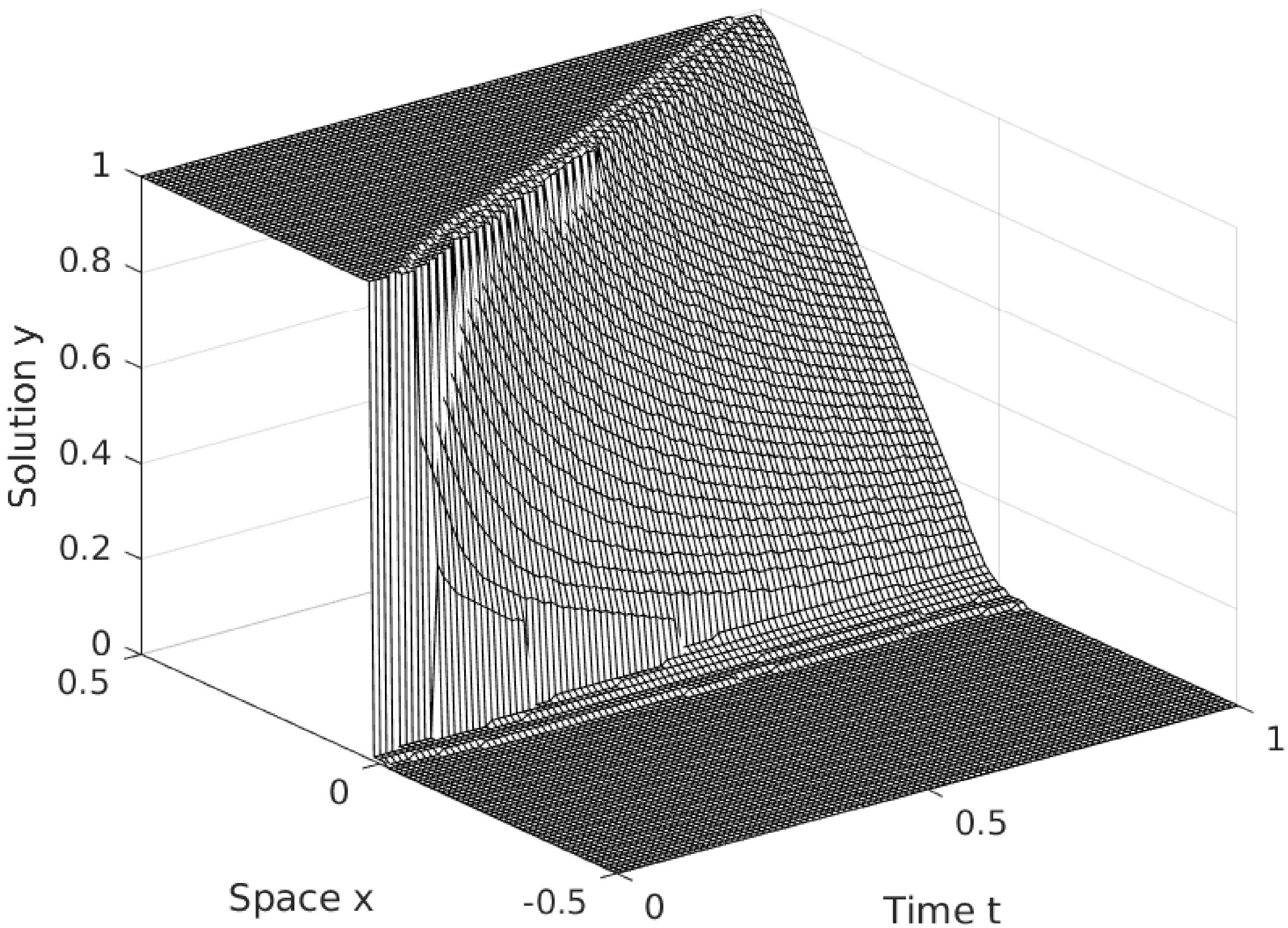
The unique analytic solution is

$$y(t,x) = \begin{cases} 0 & x \leq 0 \\ \frac{2x}{t} & 0 \leq x \leq \frac{t}{2} \\ 1 & x \geq \frac{t}{2} \end{cases} \tag{1}$$

Solving the moment problem up to degree 12 with our interface GloptiPoly for Matlab and the semidefinite programming solver MOSEK, we end up with the following moments:

$$\begin{aligned}(\textstyle\int y^k d\mu_{t,x}(y)dx\,dt)_{k=0,1,\ldots} = \;& (1.0000, 0.3750, 0.3333, 0.3125, \\ & 0.3000, 0.2917, 0.2857, 0.2812, \ldots)\end{aligned}$$

which correspond (up to numerical accuracy) exactly with the moments of the analytic solution $d\mu_{t,x} = \delta_{y(t,x)}$



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Time t

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