Rate-Induced Tipping:
Beyond Classical Bifurcations in Ecology

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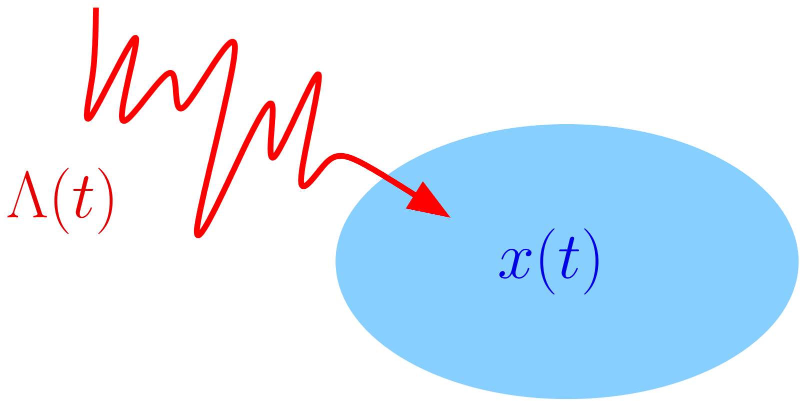
The Setting: Tipping Points in Dynamical Systems

Open System Subject to External Disturbances



$$\dot{x} = f(x, \Lambda(t))$$

$x(t)$ **- state of an open system at time** t

$\Lambda(t)$ **- time-varying external input (external forcing)**

Tipping Point or Critical Transition:

A sudden and large change in the state of the system $x(t)$**,**
trigerred by a slow and small change in the external input $\Lambda(t)$

Outline:

1. R-tipping in Ecology: Failure to Adapt
   (Paul O'Keeffe)

2. R-tipping Definition: Thresholds and Edge States
   (Peter Ashwin, Chun Xie, Chris K.R.T. Jones)

3. Compactification
   (Chun Xie, Chris K.R.T. Jones)

4. Rigorous Testable Criteria for R-tipping
   (Peter Ashwin, Chun Xie, Chris K.R.T. Jones)

Plants (P) and Herbivores (H)

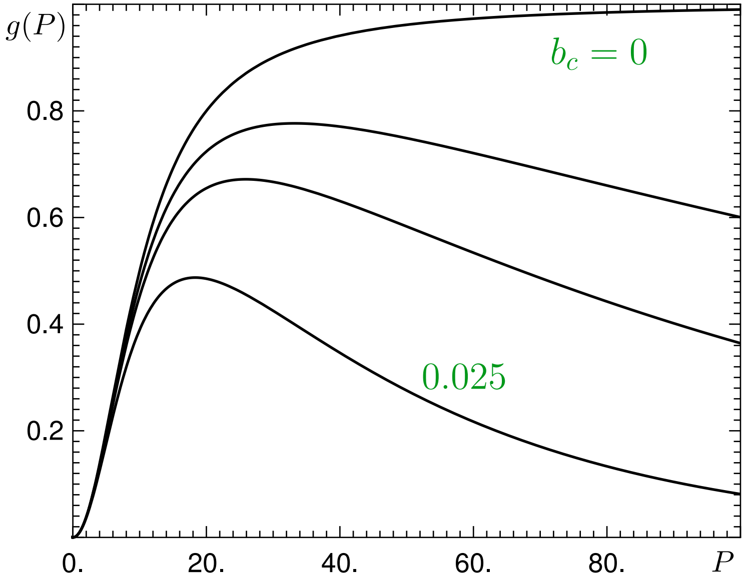
$$\frac{dP}{dt} = r\,P - C\,P^2 - H\,g(P, b_c, a)$$
$$\frac{dH}{dt} = H\,g(P, b_c, a)\,E\,e^{-bP} - m\,H$$

r - maximum plant growth rate

m - mortality rate of herbivores

The Key nonlinearity is in the functional response

$$g(P, b_c, a) = \frac{P^2}{P^2 + a^2}\,e^{-b_c P}$$



[M. Scheffer, E. van Nes, M. Holmgren, T. Hughes, *Ecosystems* 11 (2008) 222]

Equilibrium Solutions

- **Trivial zero population:** $e_1 = (0, 0)$

- **Plant-dominated (potentially stable):** $e_2 = \left(\frac{r}{C}, 0\right)$

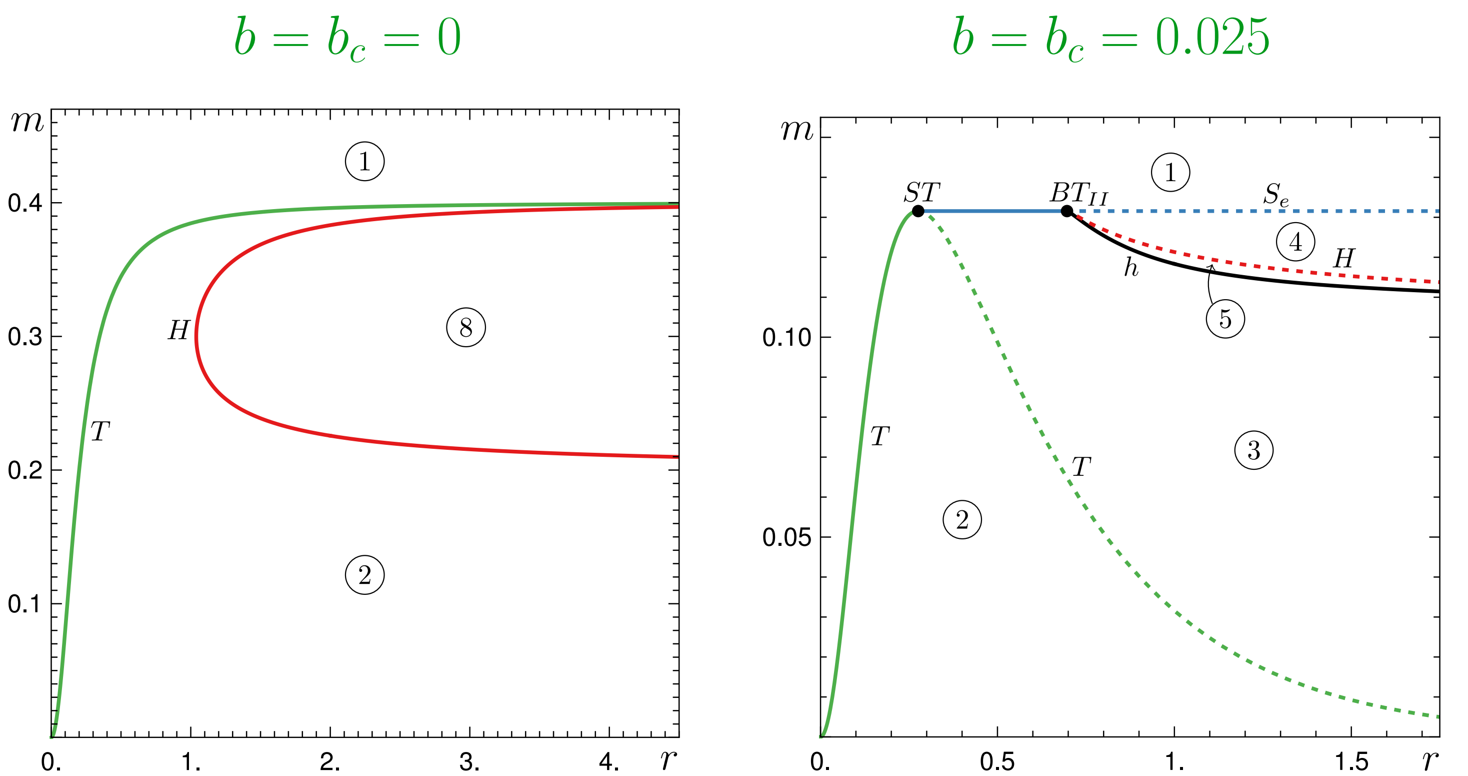
- **Herbivores I (potentially stable):**

$$e_3 = \left(\sqrt{\frac{a^2 m}{Ec_m - m}} + \mathcal{O}(b + b_c),\ \frac{(r - CP)(P^2 + a^2)}{c_m P}\ e^{b_c P} \right)$$

- **Herbivores II (unstable):**

$$e_4 = \left(\frac{1}{b + b_c} \ln\left(\frac{c_m E}{m}\right) + \mathcal{O}(b + b_c),\ \frac{(r - CP)(P^2 + a^2)}{c_m P}\ e^{b_c P} \right)$$

2D Bifurcation Diagrams & Parameter Paths



Region ①: plant-dominated equilibrium

Regions ② and ⑧: plant-dominated + herbivores I equilibria

Regions ③–⑦: plant-dominated + herbivores I + herbivores II equilibria

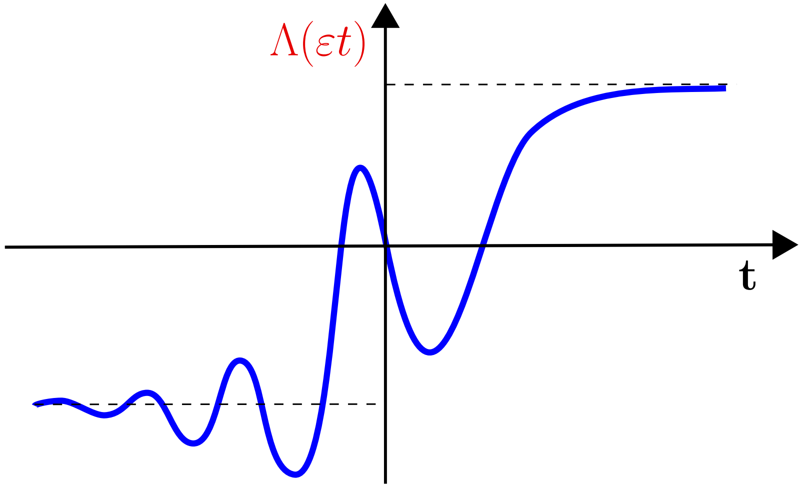
Parameter Shifts along Parameter Paths

$$\frac{dP}{dt} = r(\varepsilon t)\, P - C\, P^2 - H\, g(P, b_c, a)$$
$$\frac{dH}{dt} = H\, g(P, b_c, a)\, E\, e^{-bP} - m(\varepsilon t)\, H$$

To make progress consider:

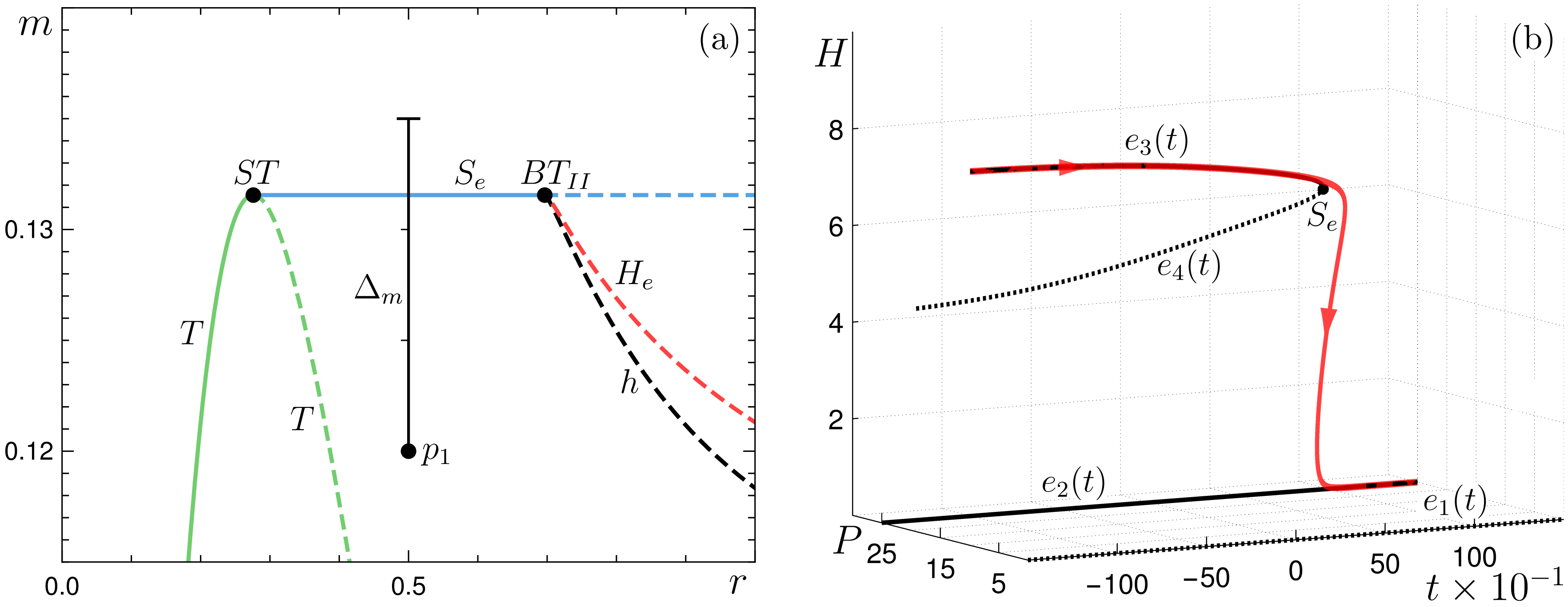
Normal Bi-asymptotic Constant Parameters Shifts.

Smooth $\Lambda(t) \to \lambda^{\pm}$ **and** $\dot{\Lambda}(t) \to 0$ **as** $t \to \pm\infty$. **Rate** ε.



B-tipping

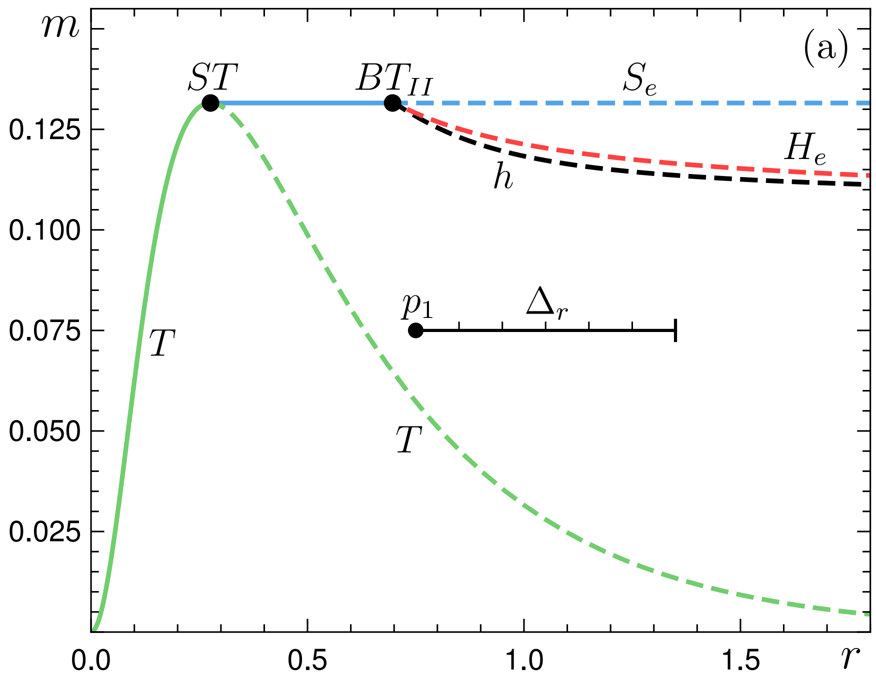
Paths Across Dangerous Bifurcations: Critical Levels

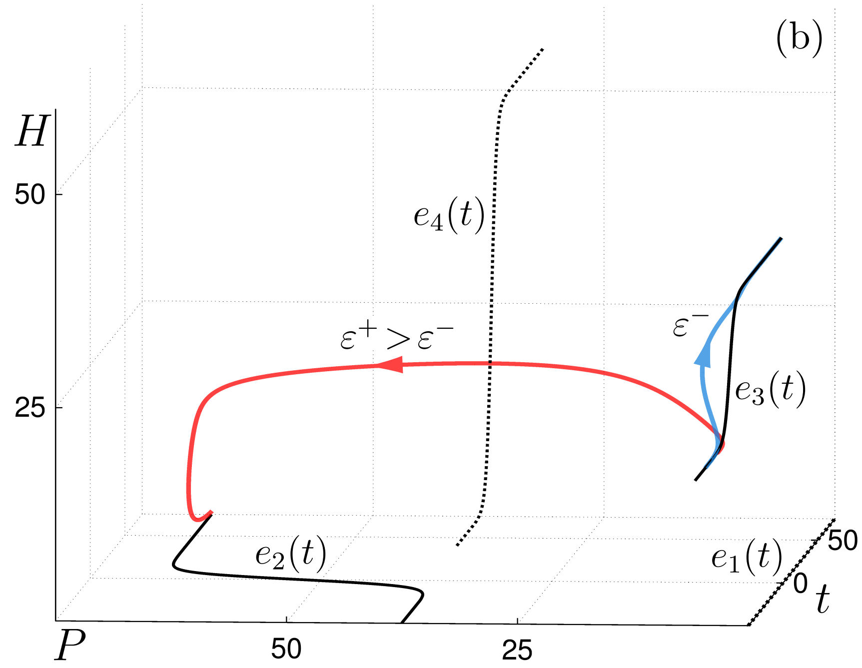


Tipping Point Paradigm

R-tipping

Paths Do Not Cross Any Bifurcations: Critical Rates





Failure to Adapt: Genuine nonautonomous instability

Question: How can we analyse this?

[**P. O'Keeffe and S. Wieczorek arXiv:1902.01796**]

Basin Instability (BI)

Ingredients:

Parameter path in the λ-parameter space: P_λ

Stable equilibrium along the path: $e(\lambda)$

Basin of attraction of e along the path: $B(e, \lambda)$

Definition:

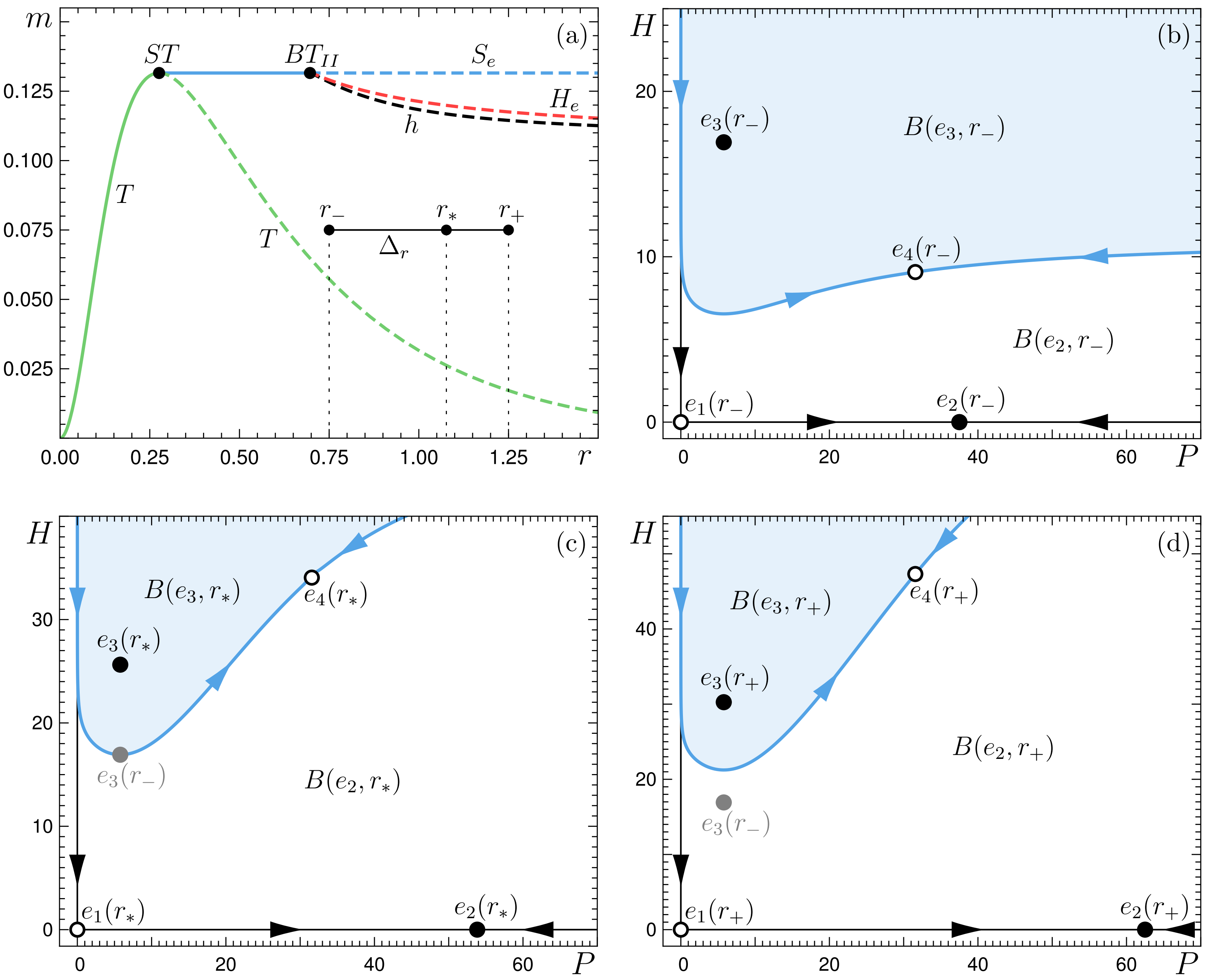
The stable equilibrium is basin unstable on a parameter path P_λ if there are two points on the path, p_1 and p_2, such that $e(p_1)$ is outside the basin of attraction of $e(p_2)$.

Basin Instability Region in the 2D bifurcation diagram

$$BI(e, p_1) = \{p_2 : e(p_1) \notin B(e, p_2)\}$$

[**S.Wieczorek, P. Ashwin, C. Xie, C.K.R.T. Jones,** *in preparation*]

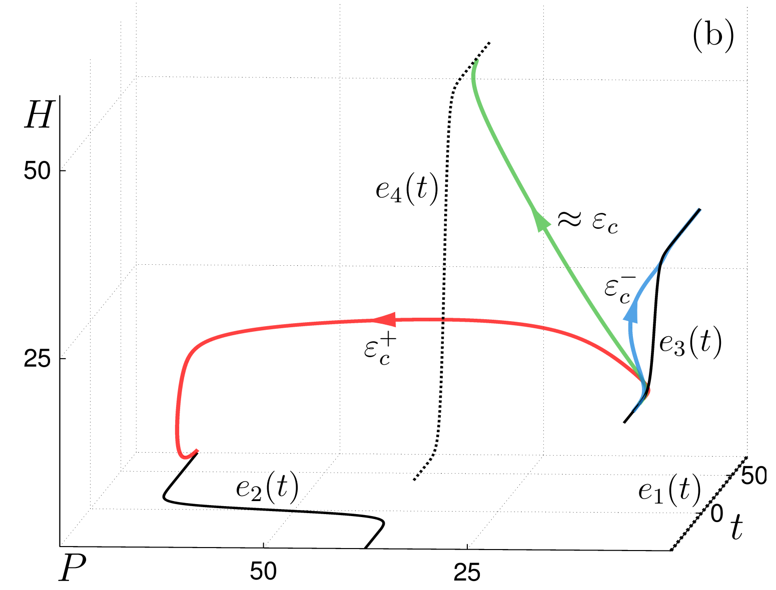
Basin Instability in Region ③



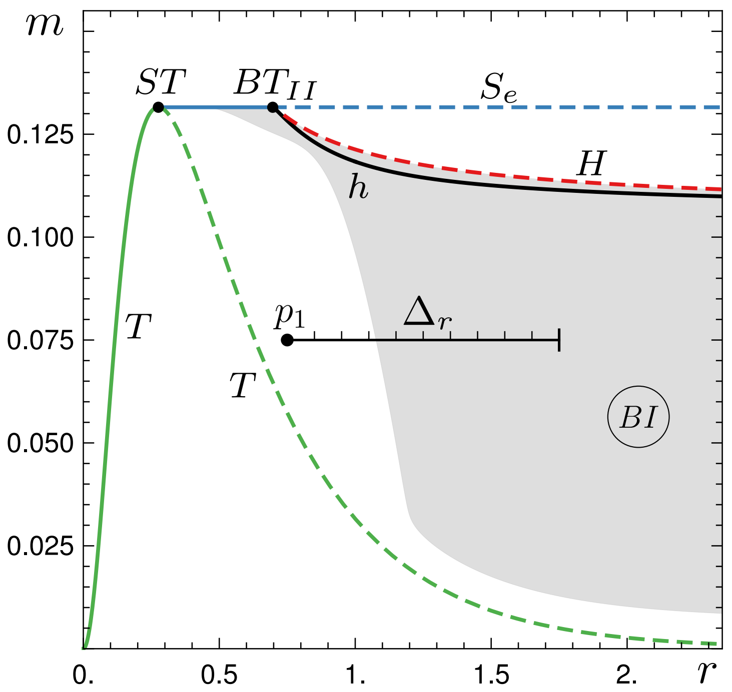
Beyond Classical Bifurcation Diagrams

R-tipping: Maximal Canard, Pullback Attractor

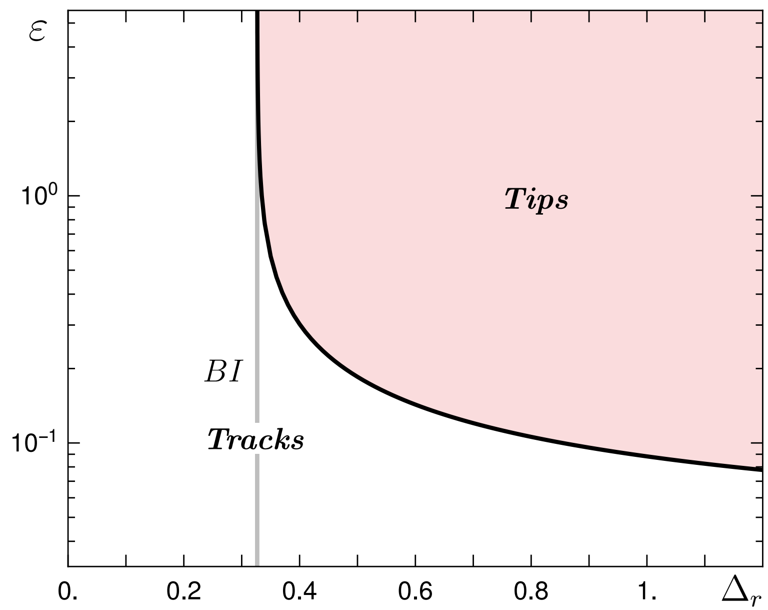
$$r(\varepsilon t) = r_- + \Delta_r\,(\tanh(\varepsilon t) + 1)$$



Clasical Bifurcations + Nonautonomous Instabilities



R-tipping Diagram



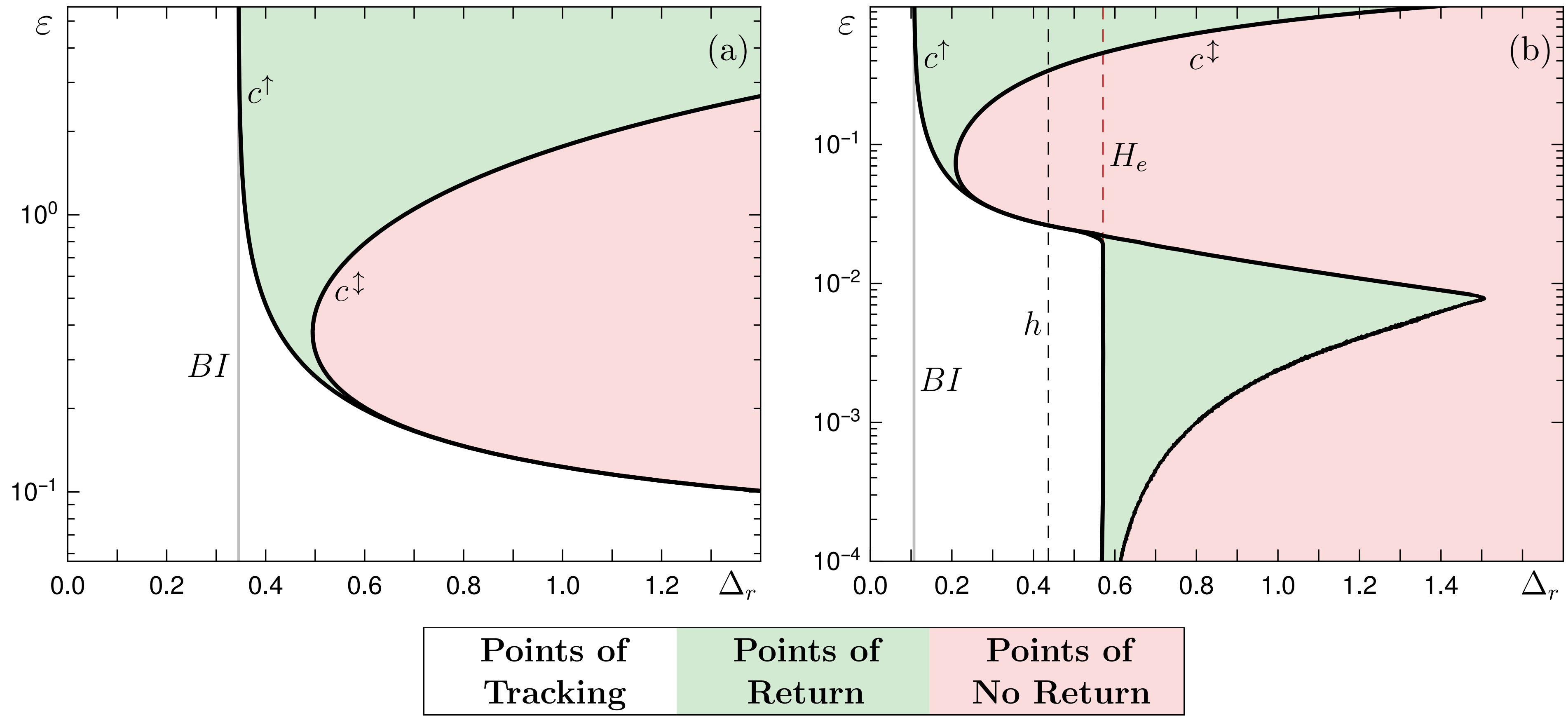
[P. O'Keeffe and S. Wieczorek arXiv:1902.01796]

Points of No Return

Suppose a monotone parameter shift gives tipping.

Question: Can tipping be avoided by reversing the trend in the parameter shift?

Points of No Return: Non-trivial Tipping Diagram



The Key Message from the Ecosystem Example

Classical bifurcations do not capture all tipping phenomena. Need an alternative mathematical framework for R-tipping.

Main Idea

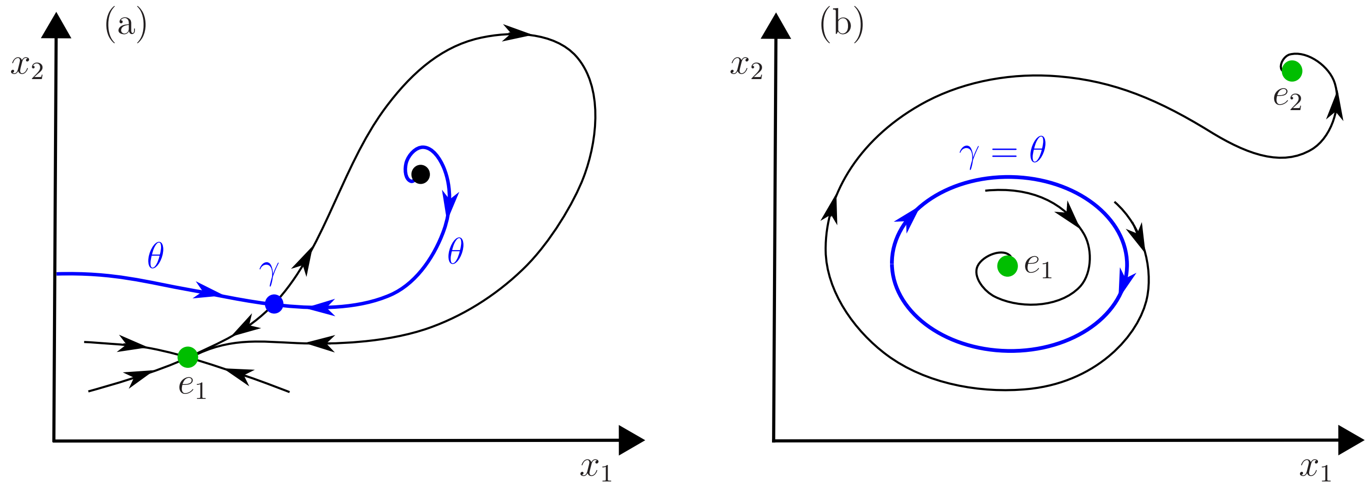
Use the autonomous dynamics and compact invariant sets of

$$\dot{x} = f(x, \lambda) \tag{1}$$

to explain nonautonomous instabilities such as R-tipping in

$$\dot{x} = f(x, \Lambda(t)) \tag{2}$$

Generalise: Basin Instability → Threshold Instability
Thresholds and Edge States of the Frozen System



Definition

For the autonomous system (1), a regular threshold is an orientable codimension-1 forward-invariant embedded manifold $\theta(\lambda)$ in $\mathbb{R}^n$ that is normally hyperbolic and repelling.
We say $\gamma(\lambda)$ is a regular edge state if it is a compact normally hyperbolic invariant set whose stable manifold is a regular threshold.

[**S.Wieczorek, P. Ashwin, C. Xie, C.K.R.T. Jones,** *in preparation*]

Compactification: Bi-asymptotic Autonomous Systems

Usual Approach:

$$
\begin{aligned}
\dot{x} &= f(x, \Lambda(u)) \\
\dot{u} &= 1
\end{aligned}
$$

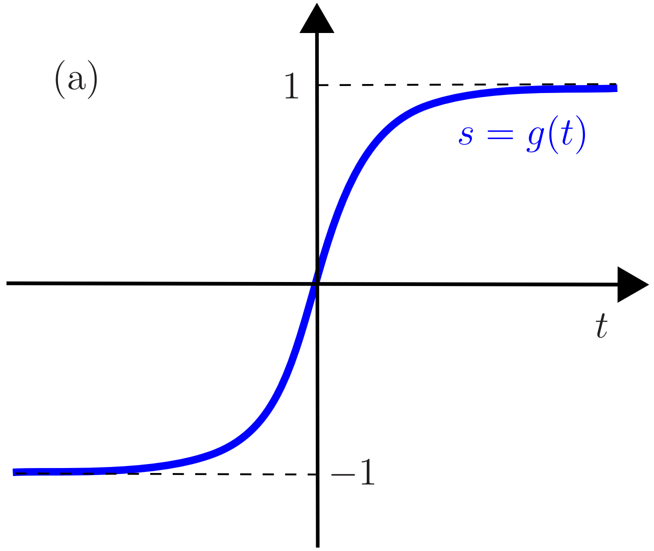
is defined on $\mathbb{R}^n \times \mathbb{R}$ and has unbounded additional dimension $u \in \mathbb{R}$.

Compactification

A process that uses a different dependent variable, s instead of u, to make the additional dimension compact (bounded and closed).

[**S.Wieczorek, C. Xie, C.K.R.T. Jones,** *in preparation*]

Autonomous Compactified System:



Step 1. Nonlinear Coordinate Transformation

Augment the vector field with $s = g(t)$:

$$\begin{aligned} \dot{x} &= f(x, \Lambda(s)) \\ \dot{s} &= \gamma(s) \end{aligned}$$

Step 2. Extend the augmented vector field to $t = \pm\infty$

Bring in $s = \pm 1$ into the phase space: $\Lambda(s) = \lambda^{\pm}$ for $s = \pm 1$.

The system is now defined on $\mathbb{R}^n \times [-1, 1]$.

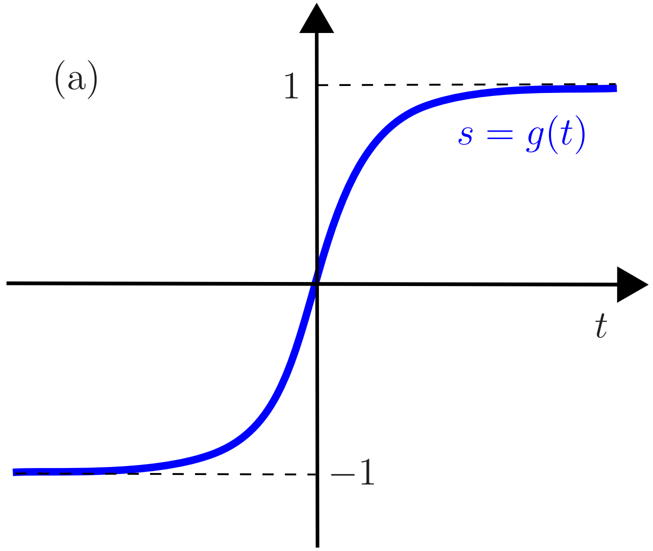
Step 3. Theorem (Compactification Conditions)

The augmented and extended vector field is C^1-smooth at $s = \pm 1$ if

$$\lim_{t \to \pm\infty} \frac{\dot{\Lambda}(t)}{\dot{g}(t)} \quad \textbf{and} \quad \lim_{t \to \pm\infty} \frac{\ddot{g}(t)}{\dot{g}(t)} \quad \textbf{exist.}$$

[**S.Wieczorek, C. Xie, C.K.R.T. Jones,** *in preparation*]

Autonomous Compactified System:



Step 1. Nonlinear Coordinate Transformation

Augment the vector field with $s = g(t)$:

$$
\begin{aligned}
\dot{x} &= f(x, \Lambda(s)) \\
\dot{s} &= \gamma(s) = \alpha\left(1 - s^2\right)
\end{aligned}
$$

Step 2. Extend the augmented vector field to $t = \pm\infty$

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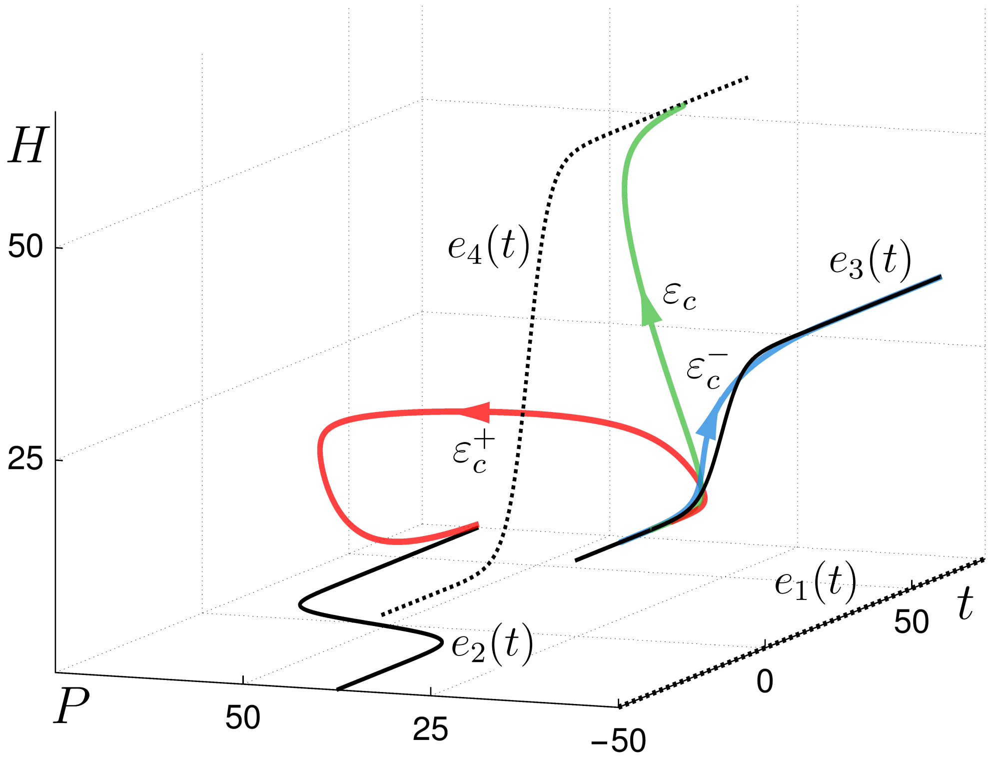
Compactification: The Ecosystem Model

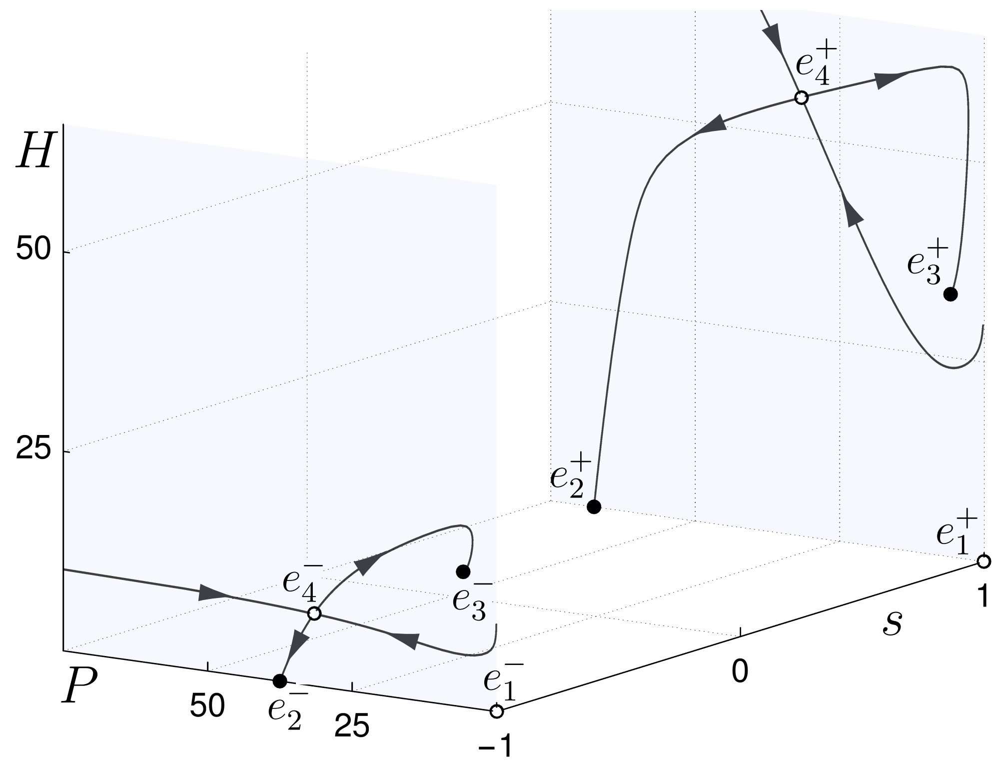
Nonautonomous System on $\mathbb{R}^2 \times \mathbb{R}$



No compact invariant sets.

Nonautonomous Input $\Lambda(t)$

Autonomous Compactified System on $\mathbb{R}^2 \times [-1, 1]$

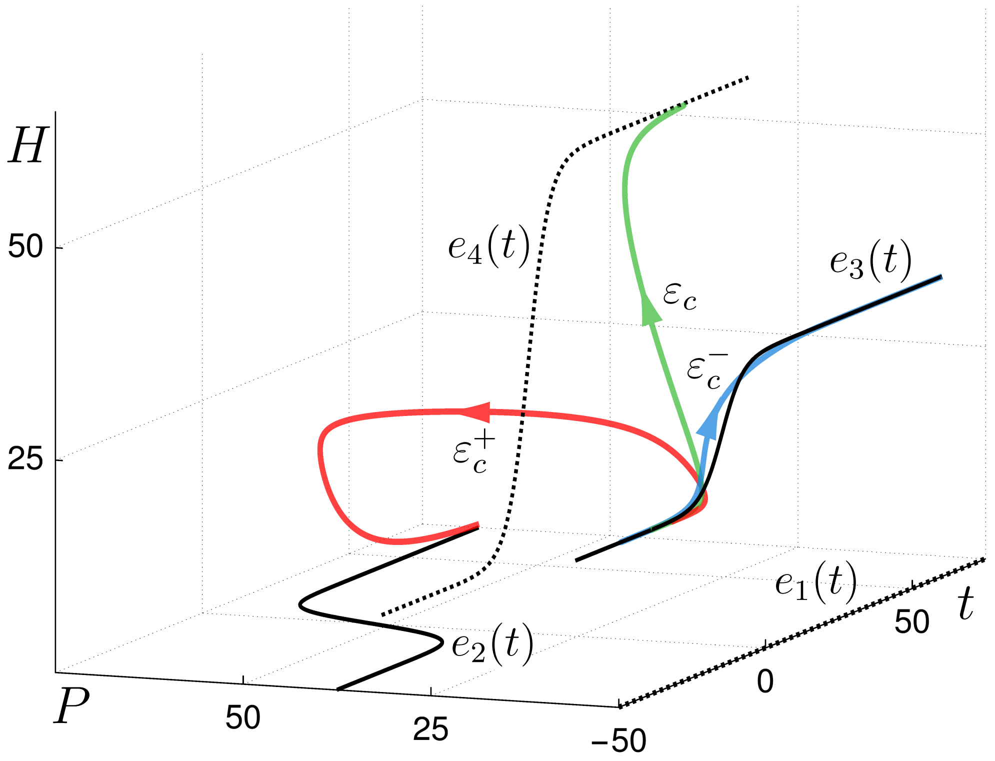


Two attractors: e_2+ and e_3^+, Edge state: e_4+

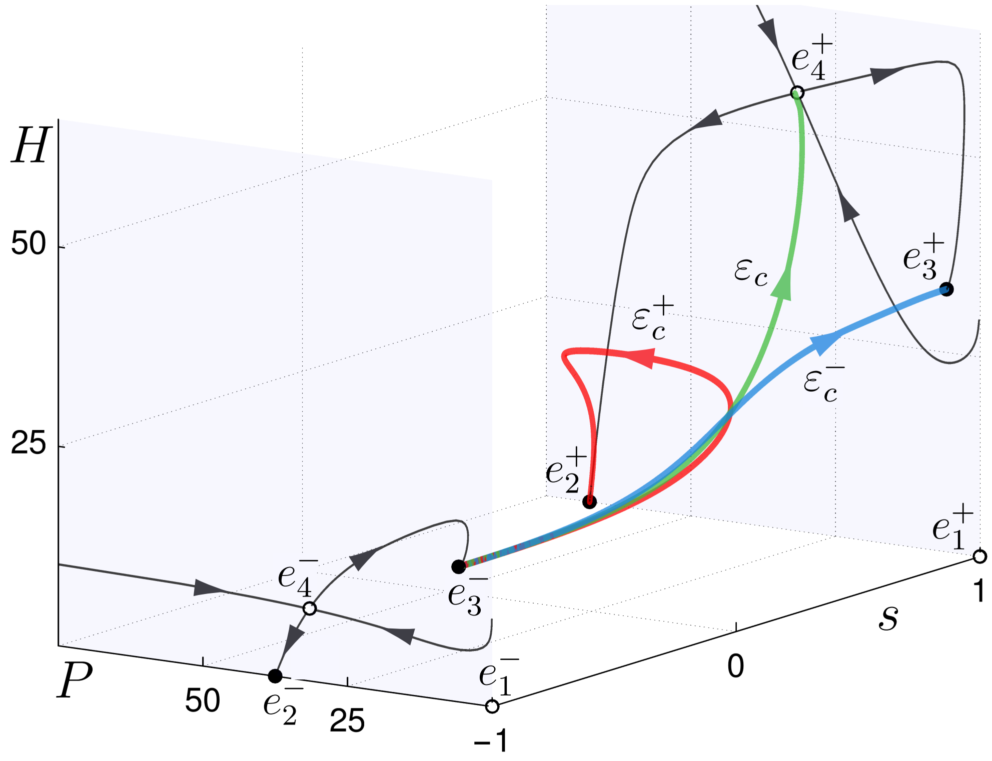
2D R-tipping Threshold encodes $\Lambda(t)$: $W^S(e_4^+)$

Compactification: The Ecosystem Model

Nonautonomous System on $\mathbb{R}^2 \times \mathbb{R}$



Autonomous Compactified System on $\mathbb{R}^2 \times [-1, 1]$



R-tipping = Heteroclinic Orbit

R-tipping: Rigorous and Easily Testable Criteria

Proposition (R-tipping Criteria)
Consider a nonautonomous system with a stable equilibrium $e(\lambda)$ on a path P_λ. Suppose $e(\lambda)$ is threshold unstable on P_λ.
Then, there exists a bi-asymptotically constant input $\Lambda(t)$ that traces out P_λ and gives R-tipping from $e(\lambda)$.

Proof Idea

- **R-tipping is defined for the nonautonomous system: Edge Tails.**
- **Autonomous assumption: threshold instability of $e(\lambda)$ along P_λ.**
- **Compactify.**
- **Construct an input $\Lambda(s)$ that gives a heteroclinic connection from e^- to γ^+ in the compactified system.**
- **Show a generic heteroclinic connection in the compactified system with $\Lambda(s)$ implies R-tipping in the nonautonomous system with $\Lambda(t)$.**

[**S.Wieczorek, P. Ashwin, C. Xie, C.K.R.T. Jones,** *in preparation*]

Summary

1. Classical Bifurcation Theory does not capture all tipping phenomena.

2. Importance of R-tipping in ecology?
   An instability that describes failure to adapt.

3. Compactify to transforms R-tipping problems into connecting heteroclinic orbits problems.

Thank you!