Excitation Spectrum of Trapped Bose-Einstein Condensates

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From Many Body Problems to Random Matrices

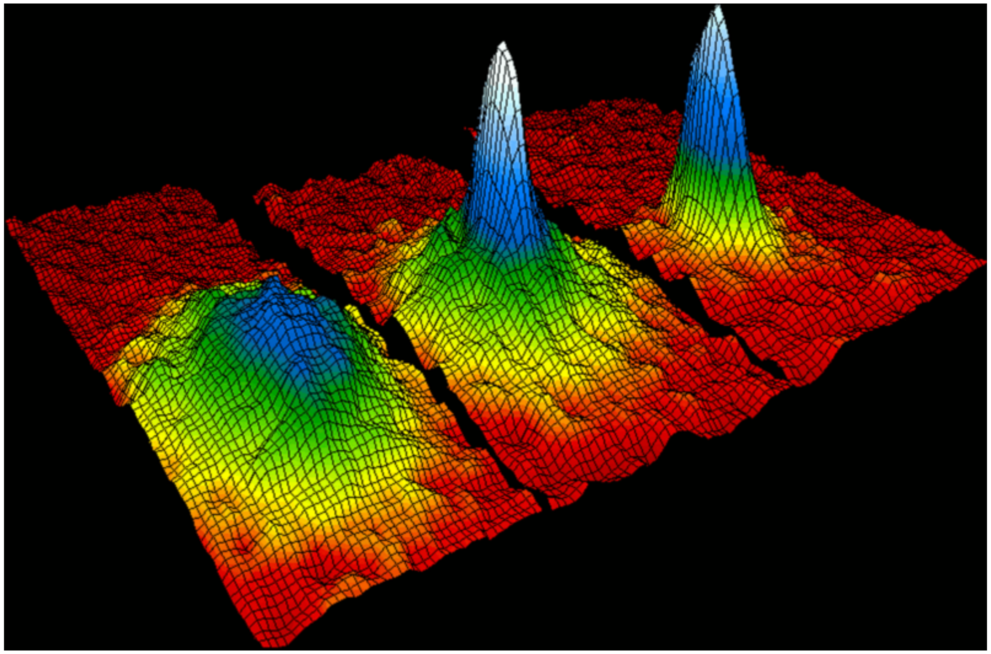
Banff, August 5, 2019

Joint works with Boccato, Brennecke, Cenatiempo, Schraven

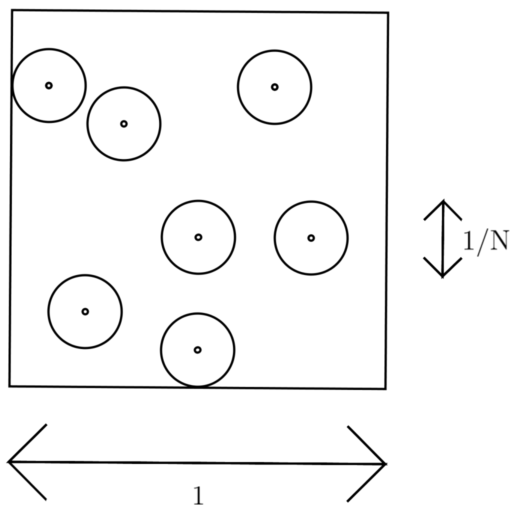
Introduction

Bose-Einstein condensates: in the last two decades, BEC have become accessible to experiments.

Goal: understand low-energy properties of trapped condensates, starting from microscopic description.



Gross-Pitaevskii regime: N bosons in $\Lambda = [0;1]^3$, interacting through potential with effective range of order N^{-1}, as $N \to \infty$.



Range of interaction much shorter than typical distance among particles: collisions rare, **dilute gas**.

Hamilton operator: has form

$$H_N = \sum_{j=1}^{N} -\Delta_{x_j} + \sum_{i<j}^{N} N^2 V(N(x_i - x_j)), \qquad \text{on } L^2_s(\Lambda^N)$$

$V \geq 0$ with **compact support**.

Scattering length: defined by zero-energy **scattering equation**

$$\left[-\Delta + \frac{1}{2}V(x)\right] f(x) = 0, \qquad \text{with} \qquad f(x) \to 1 \quad \text{as } |x| \to \infty$$

$$\Rightarrow \qquad f(x) = 1 - \frac{\mathfrak{a}_0}{|x|}, \qquad \text{for large } |x|$$

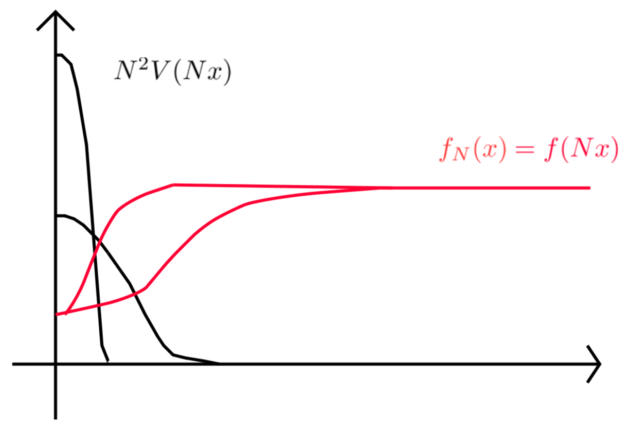
Equivalently,

$$8\pi\mathfrak{a}_0 = \int V(x) f(x) dx$$

By **scaling**,

$$\left[-\Delta + \frac{1}{2}N^2 V(Nx)\right] f(Nx) = 0$$

Rescaled potential has scattering length $\mathfrak{a}_0/N$.



Ground state energy: **[Lieb-Yngvason '98]** proved that

$$E_N = 4\pi \mathfrak{a}_0 N + o(N)$$

BEC: [Lieb-Seiringer '02, '06] showed that $\psi_N \in L^2_s(\Lambda^N)$ with

$$\langle \psi_N, H_N \psi_N \rangle \leq 4\pi \mathfrak{a}_0 N + o(N)$$

exhibits BEC, i.e. reduced density matrix

$$\gamma_N(x;y) = \int dx_2 \dots dx_N \, \psi_N(x, x_2, \dots, x_N) \overline{\psi}_N(y, x_2, \dots, x_N)$$

is such that

$$\lim_{N\to\infty} \langle \varphi_0, \gamma_N \varphi_0 \rangle = 1$$

with $\varphi_0(x) = 1$ for all $x \in \Lambda$.

Warning: this does not mean that $\psi_N \simeq \varphi_0^{\otimes N}$. In fact

$$\langle \varphi_0^{\otimes N}, H_N \, \varphi_0^{\otimes N} \rangle = \frac{(N-1)}{2} \widehat{V}(0) \gg 4\pi \mathfrak{a}_0 N$$

Correlations are important!!

Main results

Theorem [Boccato, Brennecke, Cenatiempo, S., '17]: There exists $C > 0$ such that

$$|E_N - 4\pi\mathfrak{a}_0 N| \leq C$$

uniformly in N.

Furthermore, if $\psi_N \in L^2_s(\Lambda^N)$ such that

$$\langle \psi_N, H_N \psi_N \rangle \leq 4\pi\mathfrak{a}_0 N + \zeta$$

we have

$$1 - \langle \varphi_0, \gamma_N \varphi_0 \rangle \leq \frac{C(\zeta + 1)}{N}$$

Interpretation: in low-energy states, condensation holds with optimal rate, with **bounded** number of excitations.

Question: Is it possible to **resolve** order one contributions to the ground state energy?

Theorem [Boccato, Brennecke, Cenatiempo, S., '18]:
Let $\Lambda_+^* = 2\pi\mathbb{Z}^3\backslash\{0\}$. Then

$$
\begin{aligned}
E_N = &\, 4\pi\mathfrak{a}_0(N-1) + e_\Lambda \mathfrak{a}_0^2 \\
& - \frac{1}{2}\sum_{p\in\Lambda_+^*}\left[p^2 + 8\pi\mathfrak{a}_0 - \sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2} - \frac{(8\pi\mathfrak{a}_0)^2}{2p^2}\right] + \mathcal{O}(N^{-1/4})
\end{aligned}
$$

where

$$
e_\Lambda = 2 - \lim_{M\to\infty} \sum_{\substack{p\in\mathbb{Z}^3\backslash\{0\}: \\ |p_1|,|p_2|,|p_3|\leq M}} \frac{\cos(|p|)}{p^2}
$$

Moreover, for the ground state, we have the **BEC depletion**

$$
1 - \langle\varphi_0, \gamma_N \varphi_0\rangle = \frac{1}{N}\sum_{p\in\Lambda_+^*}\left[\frac{p^2 + 8\pi\mathfrak{a}_0 - \sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2}}{2\sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2}}\right] + \mathcal{O}(N^{-9/8})
$$

Theorem [Boccato, Brennecke, Cenatiempo, S., '18]: The **spectrum** of $H_N - E_N$ below a threshold $\zeta > 0$ consists of eigenvalues

$$\sum_{p \in \Lambda_+^*} n_p \sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2} + \mathcal{O}(N^{-1/4}(1+\zeta^3))$$

where $n_p \in \mathbb{N}$ for all $p \in \Lambda_+^*$.

Interpretation: every excitation with momentum $p \in \Lambda_+^*$ "costs" energy $\varepsilon(p) = \sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2}$.

Remark: excitation spectrum is **crucial** to understand the low-energy properties of Bose gas.

The **linear dependence** of $\varepsilon(p)$ on $|p|$ for small p can be used to explain the emergence of **superfluidity**.

Previous works

Mathematically simpler models described by

$$H_N^\beta = \sum_{j=1}^N -\Delta_{x_j} + \frac{1}{N}\sum_{i<j}^N N^{3\beta} V(N^\beta(x_i - x_j))$$

for $\beta \in [0;1)$.

In **mean field regime**, $\beta = 0$, excitation spectrum determined in [Seiringer, '11], [Grech-Seiringner, '13], [Lewin-Nam-Serfaty-Solovej, '14], [Derezinski-Napiorkowski, '14], [Pizzo, '16].

Dispersion of excitations given by $\varepsilon_{\text{mf}}(p) = \sqrt{|p|^4 + 2\widehat{V}(p)p^2}$.

For **intermediate regimes**, $\beta \in (0;1)$ (and V small enough) excitations spectrum determined in [BBCS, '17].

Dispersion of excitations given by $\varepsilon_\beta(p) = \sqrt{|p|^4 + 2\widehat{V}(0)p^2}$.

For **Gross-Pitaevskii regime**, $\beta = 1$, and V small, excitations spectrum determined in [BBCS, '18].

Extension to BEC in external potentials

Consider N bosons in $\mathbb{R}^3$, with Hamilton operator

$$H_N(V_{\text{ext}}) = \sum_{j=1}^{N} \left[-\Delta_{x_j} + V_{\text{ext}}(x_j)\right] + \sum_{i<j}^{N} N^2 V(N(x_i - x_j))$$

with V_{ext} a trapping potential.

[**Lieb-Seiringer-Yngvason, '00**] proved that

$$\lim_{N\to\infty} \frac{E_N}{N} = \min_{\varphi \in L^2(\mathbb{R}^3): \|\varphi\|=1} \mathcal{E}_{\text{GP}}(\varphi)$$

with the **Gross-Pitaevskii** energy functional

$$\mathcal{E}_{\text{GP}}(\varphi) = \int_{\mathbb{R}^3} \left[|\nabla\varphi|^2 + V_{\text{ext}}|\varphi|^2 + 4\pi\mathfrak{a}_0|\varphi|^4\right] dx$$

[**Lieb-Seiringer, '02**]: ground state exhibits BEC into minimizer φ_{GP} of Gross-Pitaevskii functional, ie.

$$\lim_{N\to\infty} \langle \varphi_{\text{GP}}, \gamma_N \varphi_{\text{GP}} \rangle = 1$$

Theorem [Brennecke-S.-Schraven, in progress]:

Optimal BEC: if $\psi_N \in L^2_s(\mathbb{R}^{3N})$ with

$$\langle \psi_N, H_N(V_{\text{ext}})\psi_N \rangle \le E_N(V_{\text{ext}}) + \zeta$$

then

$$1 - \langle \varphi_{\text{GP}}, \gamma_N \varphi_{\text{GP}} \rangle \le \frac{C(\zeta + 1)}{N}$$

Excitation spectrum: let

$$h_{\text{GP}} = -\Delta + V_{\text{ext}} + 8\pi\mathfrak{a}_0 |\varphi_{\text{GP}}|^2$$

and $\varepsilon_0 = \inf \sigma(h_{\text{GP}})$. Let $D = h_{\text{GP}} - \varepsilon_0$ and

$$E = \left[D^{1/2}(D + 16\pi\mathfrak{a}_0 |\varphi_{\text{GP}}|^2) D^{1/2} \right]^{1/2}$$

Spectrum of $H_N(V_{\text{ext}}) - E_N(V_{\text{ext}})$ below threshold $\zeta > 0$ consists of eigenvalues having the form

$$\sum_{i \in \mathbb{N}} n_i e_i + o(1) \qquad \text{where } e_i \text{ are eigenvalues of } E \text{ and } n_i \in \mathbb{N}.$$

Dynamics generated by change of external fields

First results by [**Erdős-S.-Yau, '06, '08**], and by [**Pickl, '10**].

Theorem [Brennecke-S., '16]: let $\psi_N \in L^2_s(\mathbb{R}^{3N})$ with reduced density matrix γ_N such that

$$
\begin{aligned}
a_N &= 1 - \langle \varphi_{\text{GP}}, \gamma_N \varphi_{\text{GP}} \rangle \to 0 \\
b_N &= \left| N^{-1} \langle \psi_N, H_N(V_{\text{ext}}) \psi_N \rangle - \mathcal{E}_{\text{GP}}(\varphi_{\text{GP}}) \right| \to 0
\end{aligned}
$$

Let

$$
H_N = \sum_{j=1}^N -\Delta_{x_j} + \sum_{i<j}^N N^2 V(N(x_i - x_j)) \qquad \text{on } L^2_s(\mathbb{R}^{3N})
$$

and $\psi_{N,t} = e^{-iH_N t}\psi_N$ solve **many-body Schrödinger equation**. Then

$$
1 - \langle \varphi_t, \gamma_{N,t} \varphi_t \rangle \le C \left[a_N + b_N + N^{-1} \right] \exp(c \exp(c|t|))
$$

where φ_t solves time-dependent **Gross-Pitaevskii** equation

$$
i\partial_t \varphi_t = -\Delta \varphi_t + 8\pi \mathfrak{a}_0 |\varphi_t|^2 \varphi_t, \qquad \text{with } \varphi_{t=0} = \varphi_{\text{GP}}.
$$

Thermodynamic limit

Consider N bosons in $\Lambda_L = [0;L]^3$, with $N, L \to \infty$ but fixed density $\rho = N/L^3$.

As $\rho \to 0$, **Lee-Huang-Yang** predicted

$$\lim_{\substack{N,L\to\infty \\ N/L^3=\rho}} \frac{E_N}{N} = 4\pi\mathfrak{a}_0\rho\left[1 + \frac{128}{15\sqrt{\pi}}(\rho\mathfrak{a}_0^3)^{1/2} + o(\rho^{1/2})\right]$$

Leading order known from [**Lieb-Yngvason, '98**].

Upper bound to second order in [**Erdős-S.-Yau,'08**], [**Yau-Yin,'09**].

[**Fournais-Solovej, '19**] got matching lower bound (next talk!).

Remark: Gross-Pitaevskii regime corresponds to limit $\rho = N^{-2}$.

Still open: prove BEC and determine excitations in thermodynamic limit.

Bogoliubov approximation

Fock space: define $\mathcal{F} = \bigoplus_{n\geq 0} L^2_s(\Lambda^n)$.

Creation and annihilation operators: for $p \in 2\pi\mathbb{Z}^3$, introduce a_p^*, a_p creating and annihilating particle with momentum p.

Canonical commutation relations: for any $p, q \in 2\pi\mathbb{Z}^3$,

$$\left[a_p, a_q^*\right] = \delta_{p,q}, \qquad \left[a_p, a_q\right] = \left[a_p^*, a_q^*\right] = 0$$

Number of particles: $a_p^* a_p$ measures number of particles with momentum p,

$$\mathcal{N} = \sum_{p\in\Lambda^*} a_p^* a_p = \text{total } \textbf{number of particles} \text{ operator}$$

Hamilton operator: we write

$$H_N = \sum_{p \in \Lambda^*} p^2 a_p^* a_p + \frac{1}{N} \sum_{p,q,r \in \Lambda^*} \widehat{V}(r/N) a_{p+r}^* a_q^* a_p a_{q+r}$$

Number substitution: **BEC** implies that

$$a_0, a_0^* \simeq \sqrt{N} \gg 1 = [a_0, a_0^*]$$

Bogoliubov replaced a_0^*, a_0 by factors of $\sqrt{N}$. He found

$$\begin{aligned} H_N \simeq \; & \frac{(N-1)}{2} \widehat{V}(0) + \sum_{p \neq 0} p^2 a_p^* a_p + \widehat{V}(0) \sum_{p \neq 0} a_p^* a_p \\ & + \frac{1}{2} \sum_{p \neq 0} \widehat{V}(p/N) \left[2 a_p^* a_p + a_p^* a_{-p}^* + a_p a_{-p} \right] \\ & + \frac{1}{\sqrt{N}} \sum_{p,q \neq 0} \widehat{V}(p/N) \left[a_{p+q}^* a_{-p}^* a_q + a_q^* a_{-p} a_{p+q} \right] \\ & + \frac{1}{N} \sum_{p,q,r \neq 0} \widehat{V}(r/N) a_{p+r}^* a_q^* a_p a_{q+r} \end{aligned}$$

Diagonalization: neglecting cubic and quartic terms, and using appropriate **Bogoliubov transformation**

$$T = \exp\Big\{ \sum_{p\in\Lambda_+^*} \tau_p \left(a_p^* a_{-p}^* - a_p a_{-p} \right) \Big\}$$

one finds

$$\begin{aligned} T^* H_N T \simeq\; & \frac{(N-1)}{2}\widehat{V}(0) - \frac{1}{2}\sum_{p\neq 0} \frac{\widehat{V}^2(p/N)}{2p^2} \\ & - \frac{1}{2}\sum_{p\neq 0}\left[p^2 + \widehat{V}(0) - \sqrt{|p|^4 + 2\widehat{V}(0)p^2} - \frac{\widehat{V}(0)^2}{2p^2}\right] \\ & + \sum_{p\neq 0} \sqrt{|p|^4 + 2\widehat{V}(0)p^2}\, a_p^* a_p \end{aligned}$$

Born series: for small potentials, scattering length given by

$$\begin{aligned} 8\pi \mathfrak{a}_0 = \;& \widehat{V}(0) \\ & + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n N^n} \sum_{p_1,\dots,p_n\neq 0} \frac{\widehat{V}(p_1/N)}{p_1^2} \prod_{j=1}^{n-1} \frac{\widehat{V}((p_j - p_{j+1})/N)}{p_{j+1}^2} \widehat{V}(p_n/N) \end{aligned}$$

Scattering length: replacing

$$\widehat{V}(0) \to 8\pi\mathfrak{a}_0, \qquad \widehat{V}(0) - \frac{1}{N}\sum_p \frac{\widehat{V}^2(p/N)}{2p^2} \to 8\pi\mathfrak{a}_0$$

Bogoliubov obtained

$$\begin{aligned} T^* H_N T \simeq\ & 4\pi\mathfrak{a}_0(N-1) \\ & - \frac{1}{2}\sum_{p\neq 0}\left[p^2 + 8\pi\mathfrak{a}_0 - \sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2} - \frac{(8\pi\mathfrak{a}_0)^2}{2p^2}\right] \\ & + \sum_{p\neq 0}\sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2}\, a_p^* a_p \end{aligned}$$

Hence

$$E_N = 4\pi\mathfrak{a}_0(N-1) - \frac{1}{2}\sum_{p\neq 0}\left[p^2 + 8\pi\mathfrak{a}_0 - \sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2} - \frac{(8\pi\mathfrak{a}_0)^2}{2p^2}\right]$$

and excitation spectrum consists of

$$\sum_{p\neq 0} n_p \sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2}, \qquad n_p \in \mathbb{N}$$

Final replacement makes up for **missing** cubic and quartic terms!

Factoring out the condensate

Orthogonal excitations: for $\psi_N \in L^2_s(\Lambda^N)$, $\varphi_0 \equiv 1$ on Λ, write

$$\psi_N = \alpha_0 \varphi_0^{\otimes N} + \alpha_1 \otimes_s \varphi_0^{\otimes(N-1)} + \alpha_2 \otimes_s \varphi_0^{\otimes(N-2)} + \cdots + \alpha_N$$

where $\alpha_j \in L^2_{\perp\varphi_0}(\Lambda)^{\otimes_s j}$.

As in [**Lewin-Nam-Serfaty-Solovej, '12**], define unitary map

$$U : L^2_s(\Lambda^N) \to \mathcal{F}_+^{\leq N} := \bigoplus_{j=0}^{N} L^2_{\perp\varphi_0}(\Lambda)^{\otimes_s j}$$
$$\psi_N \to U\psi_N = \{\alpha_0, \alpha_1, \ldots, \alpha_N\}$$

Excitation Hamiltonian: we use unitary map U to define

$$\mathcal{L}_N = U H_N U^* : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$$

For $p, q \in \Lambda_+^* = 2\pi\mathbb{Z}^3 \backslash \{0\}$, we have

$$\begin{aligned} U\, a_p^* a_q\, U^* &= a_p^* a_q, \\ U\, a_0^* a_0\, U^* &= N - \mathcal{N}_+ \\ U\, a_p^* a_0\, U^* &= a_p^* \sqrt{N - \mathcal{N}_+} =: \sqrt{N}\, b_p^*, \\ U a_0^* a_p\, U^* &= \sqrt{N - \mathcal{N}_+}\, a_p =: \sqrt{N}\, b_p \end{aligned}$$

Hence, similarly to **Bogoliubov substitution**,

$$\begin{aligned} \mathcal{L}_N = \frac{(N-1)}{2} \widehat{V}(0) &+ \sum_{p \in \Lambda_+^*} p^2 a_p^* a_p + \sum_{p \in \Lambda_+^*} \widehat{V}(p/N) a_p^* a_p \\ &+ \frac{1}{2} \sum_{p \in \Lambda_+^*} \widehat{V}(p/N) \left[b_p^* b_{-p}^* + b_p b_{-p} \right] \\ &+ \frac{1}{\sqrt{N}} \sum_{p,q \in \Lambda_+^* : p+q \neq 0} \widehat{V}(p/N) \left[b_{p+q}^* a_{-p}^* a_q + a_q^* a_{-p} b_{p+q} \right] \\ &+ \frac{1}{2N} \sum_{p,q \in \Lambda_+^*, r \in \Lambda^* : r \neq -p,-q} \widehat{V}(r/N) a_{p+r}^* a_q^* a_p a_{q+r} \end{aligned}$$

Renormalized excitation Hamiltonian

Problem: in contrast with mean-field regime, after conjugation with U there are still **large contributions** in higher order terms.

Reason: $U^*\Omega = \varphi_0^{\otimes N}$ not good approximation for ground state! We need to take into account **correlations**!

Natural idea: conjugate $\mathcal{L}_N$ with a **Bogoliubov** transformation, ie. a unitary map of the form

$$\widetilde{T} = \exp \left[\frac{1}{2} \sum_{p \in \Lambda_+^*} \eta_p \left(a_p^* a_{-p}^* - a_p a_{-p} \right) \right]$$

generating **correlations**.

Nice feature: action of Bogoliubov transformations is **explicit**:

$$\widetilde{T}^* a_p \widetilde{T} = a_p \cosh(\eta_p) + a_{-p}^* \sinh(\eta_p)$$

Challenge: $\widetilde{T}$ does not preserve excitation space $\mathcal{F}_+^{\leq N}$.

Generalized Bogoliubov transformations: we use

$$T = \exp\left[\frac{1}{2}\sum_{p\in\Lambda_+^*}\eta_p\left(b_p^* b_{-p}^* - b_p b_{-p}\right)\right]$$

where

$$b_p^* = a_p^*\sqrt{\frac{N-\mathcal{N}_+}{N}}, \qquad b_p = \sqrt{\frac{N-\mathcal{N}_+}{N}}\,a_p$$

Recall:

$$U^*\,b_p^*\,U = a_p^*\,\frac{a_0}{\sqrt{N}}, \qquad U^* b_p U = \frac{a_0^*}{\sqrt{N}}a_p$$

Action: on states with **few excitations**, $b_p \simeq a_p$, $b_p^* \simeq a_p^*$. Thus

$$T^* b_p T = \cosh(\eta_p) b_p + \sinh(\eta_p) b_{-p}^* + d_p$$

where

$$\|d_p \xi\| \leq C N^{-1} \|(\mathcal{N}_+ + 1)^{3/2}\xi\|$$

Choice of correlations: consider

$$\left[-\Delta + \frac{1}{2}V\right] f = 0, \qquad \text{with } f(x) \to 1, \text{ as } |x| \to \infty$$

and let $w = 1 - f$. We define

$$\eta_p = -\frac{1}{N^2}\,\widehat{w}(p/N) \qquad \Rightarrow \qquad \eta_p \simeq \frac{C}{p^2}\,e^{-|p|/N}$$

We set

$$T = \exp\left\{\frac{1}{2}\sum_{p\in\Lambda_+^*} \eta_p \left(b_p^* b_{-p}^* - b_p b_{-p}\right)\right\}$$

Observation: recall that

$$T^* a_p T \simeq \cosh(\eta_p) a_p + \sinh(\eta_p) a_{-p}^*$$

Hence

$$\begin{aligned} \langle \Omega, T^* \mathcal{N}_+ T\Omega\rangle &\simeq \sum \sinh^2(\eta_p) \leq C \sum \eta_p^2 \leq C \\ \langle \Omega, T^* \mathcal{K} T\Omega\rangle &\simeq \sum p^2 \sinh^2(\eta_p) \simeq \sum p^2 \eta_p^2 \simeq CN \end{aligned}$$

T generates **finitely many** excitations but **macroscopic** energy.

Renormalized excitation Hamiltonian: define

$$\mathcal{G}_N = T^* \mathcal{L}_N T = T^* U H_N U^* T : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$$

Bounds on $\mathcal{G}_N$: with $\mathcal{H}_N = \mathcal{K} + \mathcal{V}_N$, we find

$$\mathcal{G}_N = 4\pi \mathfrak{a}_0 N + \mathcal{H}_N + \mathcal{E}_N$$

where, for every $\delta > 0$, there exists constant $C > 0$ with

$$\pm \mathcal{E}_N \leq \delta \mathcal{H}_N + C \|V\| \mathcal{N}_+$$

Condensation: for small potential, we can use gap

$$\mathcal{N}_+ \leq (2\pi)^{-2} \mathcal{K} \leq (2\pi)^{-2} \mathcal{H}_N$$

to conclude that

$$\mathcal{G}_N - 4\pi \mathfrak{a}_0 N \geq \frac{1}{2} \mathcal{H}_N - C$$

This implies BEC for low-energy states.

Commutator bounds: we also obtain

$$\pm\left[\mathcal{G}_N,\mathcal{N}_+\right]\leq C(\mathcal{H}_N+1)$$

This is important for **dynamics** and also for **moments** of $\mathcal{N}_+$.

Corollary: Let $\psi_N = \chi(H_N \leq E_N+\zeta)\psi_N$ and $\xi_N = T^*U\psi_N$. Then, for every $k\in\mathbb{N}$, there exists $C>0$ such that

$$\langle\xi_N,(\mathcal{H}_N+1)(\mathcal{N}_++1)^k\xi_N\rangle\leq C(\zeta+1)^{k+1}$$

With these improved bounds, we can go back to $\mathcal{G}_N$.

Theorem: renormalized excitation Hamiltonian is such that

$$\mathcal{G}_N = C_N + \mathcal{Q}_N + \mathcal{C}_N + \mathcal{V}_N + \delta_N$$

where C_N is a **constant**, $\mathcal{Q}_N$ is **quadratic**,

$$\mathcal{C}_N = \frac{1}{\sqrt{N}} \sum_{p,q \in \Lambda_+^*} \widehat{V}(p/N) \left[b^*_{p+q} b^*_{-p} \Big(\gamma_q b_q + \sigma_q b^*_{-q} \Big) + \text{h.c.} \right]$$

$$\mathcal{V}_N = \frac{1}{2N} \sum_{p,q \in \Lambda_+^*} \widehat{V}(r/N) a^*_{p+r} a^*_q a_{q+r} a_p$$

and, where,

$$\pm \delta_N \leq \frac{C}{\sqrt{N}} \left[(\mathcal{H}_N + 1)(\mathcal{N}_+ + 1) + (\mathcal{N}_+ + 1)^3 \right]$$

Problem: $\mathcal{G}_N$ still contains non-negligible **cubic** and **quartic** terms! This is substantial difference compared with case $\beta < 1$!

New cubic phase: we define

$$A = \frac{1}{\sqrt{N}} \sum_{|r|>\sqrt{N}, |v|<\sqrt{N}} \eta_r \Big[\sigma_v b^*_{r+v} b^*_{-r} b^*_{-v} + \gamma_v b^*_{r+v} b^*_{-r} b_v - \text{h.c.} \Big]$$

Set $S = e^A$ and introduce **new excitation Hamiltonian**

$$\mathcal{J}_N = S^* \mathcal{G}_N S = S^* T^* U_N H_N U_N^* T S : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$$

Remark: a similar cubic conjugation was used in [**Yau-Yin, 09**].

Proposition: we can decompose

$$\mathcal{J}_N = \widetilde{C}_N + \widetilde{Q}_N + \mathcal{V}_N + \widetilde{\delta}_N$$

where $\widetilde{C}_N$ is a **constant**, $\widetilde{Q}_N$ is **quadratic** and where

$$\pm\widetilde{\delta}_N \leq \frac{C}{N^{1/4}}\left[(\mathcal{H}_N + 1)(\mathcal{N}_+ + 1) + (\mathcal{N}_+ + 1)^3\right]$$

Mechanism: we have

$$\mathcal{J}_N = e^{-A}\mathcal{G}_N e^{A} \simeq \mathcal{G}_N + [\mathcal{G}_N, A] + \frac{1}{2}[[\mathcal{G}_N, A], A] + \dots$$

where

$$\mathcal{G}_N \simeq C_N + Q_N + \mathcal{C}_N + \mathcal{V}_N$$

Combine $[Q_N, A], [\mathcal{V}_N, A]$ with $\mathcal{C}_N$ (use **scattering equation**).

At same time, $[\mathcal{C}_N, A]$ modifies **constant** and **quadratic** terms.

Diagonalization: with last **Bogoliubov transformation** R, set

$$\mathcal{M}_N = R^* \mathcal{J}_N R = R^* S^* T^* U_N H_N U_N^* T S R : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$$

Then

$$\begin{aligned}\mathcal{M}_N = &\ 4\pi\mathfrak{a}_N (N-1) \\ &- \frac{1}{2} \sum_{p \in \Lambda_+^*} \left[p^2 + 8\pi\mathfrak{a}_0 - \sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2} - \frac{(8\pi\mathfrak{a}_0)^2}{2p^2} \right] \\ &+ \sum_{p \in \Lambda_+^*} \sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2}\, a_p^* a_p + \mathcal{V}_N + \delta_N' \end{aligned}$$

where

$$\pm \delta_N' \leq C N^{-1/4} \left[(\mathcal{H}_N + 1)(\mathcal{N}_+ + 1) + (\mathcal{N}_+ + 1)^3 \right]$$

Main theorem follows from **min-max principle**, because on low-energy states of diagonal quadratic Hamiltonian, we find

$$\mathcal{V}_N \leq C N^{-1} (\zeta + 1)^{7/2}$$